ANNALES
POLONICI MATHEMATICI
LX.3 (1995)

Convex-like inequality, homogeneity, subadditivity, and a characterization of L^p-norm

by Janusz Matkowski and Marek Pycia (Bielsko-Biała)

Abstract. Let a and b be fixed real numbers such that $0 < \min\{a,b\} < 1 < a+b$. We prove that every function $f : (0,\infty) \to \mathbb{R}$ satisfying $f(as+bt) \le af(s)+bf(t)$, $s,t > 0$, and such that $\limsup_{t\to 0+} f(t) \le 0$ must be of the form $f(t) = f(1)t$, $t > 0$. This improves an earlier result in [5] where, in particular, f is assumed to be nonnegative. Some generalizations for functions defined on cones in linear spaces are given. We apply these results to give a new characterization of the L^p-norm.

Introduction. We deal with the functional inequality

$$f(as+bt) \le af(s)+bf(t),$$

where $a,b \in \mathbb{R}$ are fixed real numbers such that

$$0 < \min\{a,b\} < 1 < a+b \tag{1}$$

and f is a real function defined on $\mathbb{R}_+ := [0,\infty)$ or $(0,\infty)$. Our Theorem 2 says that if $f(0) = 0$, f is bounded above in a neighbourhood of 0, and satisfies this inequality for all $s,t \ge 0$, then f must be a linear function. This improves a result of [6] where f is assumed to be nonnegative. Theorem 1, the main result of the first section, reads as follows: *If* $f : (0,\infty) \to \mathbb{R}$ *satisfies the above inequality for all* $s,t > 0$, *and* $\limsup_{t\to 0+} f(t) \le 0$, *then* $f(t) = f(1)t$, $t > 0$.

In Section 2, using Theorems 1 and 2, we obtain their counterparts for functions defined on convex cones of a linear space. Namely, under some weak regularity conditions an analogue of the above inequality characterizes the Banach functionals.

1991 *Mathematics Subject Classification*: Primary 26D15, 26A51, 39B72; Secondary 46E30.

Key words and phrases: functional inequality, subadditive functions, homogeneous functions, Banach functionals, convex functions, linear space, cones, measure space, integrable step functions, L^p-norm, Minkowski's inequality.

Research supported by the State Committee for Scientific Research (Poland), Grant No. 2 P301 053 03.

Applying these results we give a new characterization of the L^p-norm (cf. Theorem 3).

1. Functions satisfying a convex-like inequality on $(0,\infty)$ **and** $\mathbb{R}_+$**.** The main theorem of this section is a refinement of a relevant result of [6] and reads as follows:

THEOREM 1. *Let $a, b \in \mathbb{R}$ be fixed and such that condition* (1) *holds. If $f : (0,\infty) \to \mathbb{R}$ satisfies*

$$f(as + bt) \le af(s) + bf(t), \qquad s, t > 0, \tag{2}$$

and

$$\limsup_{t\to 0+} f(t) \le 0, \tag{3}$$

then $f(t) = f(1)t$, $t > 0$.

Proof. There is no loss of generality in assuming that $a = \min\{a, b\} < 1$. Moreover, by (2),

$$f(as + b(a+b)^n t) \le af(s) + b(a+b)^n f(t), \qquad s, t > 0,\ n \in \mathbb{N}.$$

Consequently, we may also assume $b > 1$. Now we prove the following

CLAIM. *Under the conditions of Theorem* 1 *and $a < 1 < b$ there exists an $M > 0$ such that*

$$ka^n b^m f(t) + M\delta \ge f(ka^n b^m t + \delta), \tag{4}$$

for all $t, \delta > 0$; $n, m \in \mathbb{N}$, $n + m > 0$; $k = 0, \ldots, \binom{n+m}{m}$.

To show it, take $c > \max\{a + b, a^{-1}\}$. By (3) there exists a $t_0 > 0$ such that f is bounded above on the interval $I := (t_0, ct_0)$. Thus, for some $M > 0$,

$$f(t) \le Mt, \qquad t \in I. \tag{5}$$

From (2), $f((a + b)^n t) \le (a + b)^n f(t)$ for all $n \in \mathbb{N}$ and $t > 0$. Hence

$$f(t) \le Mt, \qquad t \in \bigcup_{n=0}^{\infty} (a+b)^n I.$$

(For $I \subset \mathbb{R}$ and $\lambda \in \mathbb{R}$ we denote by λI the set $\{\lambda x : x \in I\}$.) Since $c > a+b$, the intervals $(a + b)^n I$ and $(a + b)^{n+1} I$ have a nonempty intersection, and, consequently, $\bigcup_{n=0}^{\infty} (a + b)^n I = (t_0, \infty)$. This proves that $f(t) \le Mt$ for all $t \in (t_0, \infty)$.

Assume that for some $n \in \mathbb{N}$,

$$f(t) \le Mt, \qquad t \in a^n I,$$

and take $s \in a^{n+1}I$. There exists an increasing sequence (t_k) such that $t_k \in a^n I$ $(k \in \mathbb{N})$, and $at_k \to s$. From (2) we have

$$\begin{aligned} f(s) &= f(at_k + bb^{-1}(s - at_k)) \le af(t_k) + bf(b^{-1}(s - at_k)) \\ &\le Mat_k + bf(b^{-1}(s - at_k)). \end{aligned}$$

According to (3),

$$f(s) \le Ma(\lim_{k\to\infty} t_k) = Ms, \qquad s \in a^{n+1}I.$$

Hence, by induction,

$$f(s) \le Ms, \qquad s \in \bigcup_{n=0}^{\infty} a^n I.$$

Since the inequality $c > a^{-1}$ implies that $\bigcup_{n=0}^{\infty} a^n I = (0, ct_0)$, it follows that $f(t) \le Mt$, $t \in (0, ct_0)$. Thus we have proved

$$f(t) \le Mt, \qquad t > 0. \tag{6}$$

We now show (4) by induction on $N := n + m$. For $N = 1$, (4) follows immediately from (2) and (6), for $k = 0$ it reduces to (6). Take $N > 1$, $k > 0$, choose k_1, k_2 such that

$$k_1 + k_2 = k, \quad k_1 \le \binom{n+m-1}{m}, \quad k_2 \le \binom{n+m-1}{m-1},$$

and suppose that

$$\begin{aligned} k_1 a^{n-1} b^m f(t) + (2a)^{-1}\delta &\ge f(k_1 a^{n-1} b^m t + (2a)^{-1}\delta), \\ k_2 a^n b^{m-1} f(t) + (2a)^{-1}\delta &\ge f(k_2 a^n b^{m-1} t + (2a)^{-1}\delta). \end{aligned}$$

Hence, in view of (2), we get

$$\begin{aligned} &ka^n b^m f(s) + M\delta \\ &\qquad = a(k_1 a^{n-1} b^m f(s) + M(2a)^{-1}\delta) + b(k_2 a^n b^{m-1} f(s) + M(2a)^{-1}\delta) \\ &\qquad \ge af(k_1 a^{n-1} b^m s + (2a)^{-1}\delta) + bf(k_2 a^n b^{m-1} s + (2a)^{-1}\delta) \\ &\qquad \ge f(ak_1 a^{n-1} b^m s + 2^{-1}\delta + bk_2 a^n b^{m-1} s + 2^{-1}\delta) = f(ka^n b^m s + \delta), \end{aligned}$$

and the induction completes the proof of our claim.

Now note that the set

$$\mathbb{D} := \left\{ ka^n b^m : m, n \in \mathbb{N},\ m + n > 0,\ k = 0, \ldots, \binom{n+m}{m} \right\}$$

is dense in $(0, \infty)$. Indeed, if $\log b / \log a$ is irrational, then, in view of Kronecker's Theorem, its subset $\{a^{n+1} b^m : m, n \in \mathbb{N}\}$ is dense in $(0, \infty)$. In the other case there exist $n, m \in \mathbb{N}$ such that $\log b / \log a = -n/m$, which means that $a^n b^m = 1$. Since for every $k, j \in \mathbb{N}$,

$$ka^j b = ka^{kn+j} b^{km+1} \in \mathbb{D},$$

the set $\mathbb{D}$ contains a dense subset $\{ka^j b : k, j \in \mathbb{N}\}$.

By the definition of $\mathbb{D}$ we can write (4) in the following equivalent form:

$$\lambda f(t) + M\delta \ge f(\lambda t + \delta), \qquad \lambda \in \mathbb{D},\ t, \delta > 0. \tag{7}$$

Now, fix $s, t > 0$ and take a sequence (λ_n) such that $\lambda_n \in \mathbb{D}$, $\lambda_n < s$ ($n \in \mathbb{N}$), $\lim_{n\to\infty} \lambda_n = s$. From (7) we have

$$\lambda_n f(t) + M(s - \lambda_n)t \ge f(\lambda_n t + (s - \lambda_n)t) = f(st), \qquad n \in \mathbb{N}.$$

Letting $n \to \infty$ we obtain $sf(t) \le f(st)$, which obviously implies that $sf(t) = f(st)$. Hence $f(s) = f(1)s$, $s > 0$, which completes the proof.

Remark 1. It is shown in [6] that *every nonnegative function f satisfying* (2) *with a, b such that* (1) *holds must be linear*. Obviously, this result is a consequence of Theorem 1.

Example 1. Take $a, b > 0$ such that $a + b > 1$, and $c > 0$. Then every function $f : (0, \infty) \to \mathbb{R}$ such that $c \le f(t) \le c(a + b)$, $t > 0$, satisfies (2). This shows that the condition (3) in Theorem 1 is essential.

Note that (3) can be considerably weakened if (2) is assumed to hold for all nonnegative s and t. Namely, we have the following

Theorem 2. *Let $a, b \in \mathbb{R}$ satisfy* (1). *If $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies*

$$f(as + bt) \le af(s) + bf(t), \qquad s, t \ge 0,$$

and

(i) $f(0) = 0$;
(ii) *f is bounded above in a right vicinity of* 0,

then $f(t) = f(1)t$, $t > 0$.

This result is an immediate consequence of Theorem 1 and the following

Lemma 1. *Let $a, b \in \mathbb{R}$ satisfy* (1). *Suppose that $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies*

$$f(as + bt) \le af(s) + bf(t), \qquad s, t \ge 0.$$

Then

(i) $f(0) \ge 0$.
(ii) *If, moreover, $f(0) = 0$ and f is bounded above in a right vicinity of* 0, *then condition* (3) *holds.*

Proof. (i) is obvious. To prove (ii) suppose that, say, $a = \min\{a, b\}$ and observe that, by the boundedness above of f to the right of 0, we have

$$c := \limsup_{t\to 0+} f(t) < \infty.$$

Setting in the assumed inequality $s = 0$ and making use of the condition $f(0) = 0$, we get $f(at) \le af(t)$ for all $t \ge 0$. It follows that $c \le ac$. Since $a < 1$ we hence get $c \le 0$, which was to be shown.

EXAMPLE 2. The function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(t) = t^{-1}$, $t > 0$, and $f(0) = 0$ satisfies (2) for all $a, b \in \mathbb{R}$ such that condition (1) holds. This shows that, in Theorem 2, the assumption of f being bounded above in a (right) neighbourhood of 0 is indispensable.

EXAMPLE 3. Let $a, b > 0$ be rational. Then every discontinuous additive function $f : \mathbb{R} \to \mathbb{R}$ satisfies (2). It is well known that the graph of f is a dense subset of the plane (cf. for instance Aczél–Dhombres [1], p. 14). This also shows that the regularity assumptions in Theorems 1 and 2 are necessary.

2. Some generalizations for functions defined on cones. In this section, using Theorems 1 and 2, we prove their more general counterparts.

Let $\boldsymbol{X}$ be a real linear space. A set $\boldsymbol{C} \subset \boldsymbol{X}$ is said to be a *convex cone* in $\boldsymbol{X}$ iff $\boldsymbol{C} + \boldsymbol{C} \subset \boldsymbol{C}$ and $t\boldsymbol{C} \subset \boldsymbol{C}$ for all $t > 0$.

A functional $\boldsymbol{p} : \boldsymbol{C} \to \mathbb{R}$ is called *subadditive* iff

$$\boldsymbol{p}(\boldsymbol{x} + \boldsymbol{y}) \le \boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{p}(\boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{C},$$

and *positively homogeneous* iff

$$\boldsymbol{p}(t\boldsymbol{x}) = t\boldsymbol{p}(\boldsymbol{x}), \qquad t > 0,\ \boldsymbol{x} \in \boldsymbol{C}.$$

In the sequel the functionals satisfying both these conditions (the so-called *Banach functionals*) will appear frequently.

Denote by $\boldsymbol{o}$ the zero vector of $\boldsymbol{X}$. If $\boldsymbol{C}$ is a convex cone in $\boldsymbol{X}$ and $\boldsymbol{o} \in \boldsymbol{C}$, then $t\boldsymbol{C} \subset \boldsymbol{C}$ for all $t \ge 0$.

COROLLARY 1. *Let $\boldsymbol{X}$ be a real linear space and $\boldsymbol{C} \subset \boldsymbol{X}$ a convex cone such that $\boldsymbol{o} \in \boldsymbol{C}$. Suppose that $a, b \in \mathbb{R}$ are fixed and $0 < \min\{a, b\} < 1 < a + b$. Then a function $\boldsymbol{p} : \boldsymbol{C} \to \mathbb{R}$ is subadditive and positively homogeneous if and only if*

(i) $\boldsymbol{p}(\boldsymbol{o}) = 0$;

(ii) *for every $\boldsymbol{x} \in \boldsymbol{C}$, the function $(0, \infty) \ni t \to \boldsymbol{p}(t\boldsymbol{x})$ is bounded above in a right vicinity of* 0; *and*

$$\boldsymbol{p}(a\boldsymbol{x} + b\boldsymbol{y}) \le a\boldsymbol{p}(\boldsymbol{x}) + b\boldsymbol{p}(\boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{C}. \tag{8}$$

Proof. First suppose that $\boldsymbol{p}$ satisfies (i), (ii), and (8). Then for every fixed $\boldsymbol{x} \in \boldsymbol{C}$ the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(t) := \boldsymbol{p}(t\boldsymbol{x})$, $t \ge 0$, satisfies all the assumptions of Theorem 2. Consequently, $\boldsymbol{p}(t\boldsymbol{x}) = f(t) = f(1)t = t\boldsymbol{p}(\boldsymbol{x})$ for all $t \ge 0$, which means that $\boldsymbol{p}$ is positively homogeneous.

Now the subadditivity of $\boldsymbol{p}$ is a consequence of (8). Since the converse is obvious, the proof is complete.

In a similar way, applying Theorem 1, we get

COROLLARY 2. *Let $\boldsymbol{X}$ be a real linear space and $\boldsymbol{C} \subset \boldsymbol{X}$ a convex cone. Suppose that $a, b \in \mathbb{R}$ are fixed and $0 < \min\{a, b\} < 1 < a + b$. Then a function $\boldsymbol{p} : \boldsymbol{C} \to \mathbb{R}$ is subadditive and positively homogeneous if and only if it satisfies* (8) *and*

$$\limsup_{t \to 0+} \boldsymbol{p}(t\boldsymbol{x}) \le 0, \qquad \boldsymbol{x} \in \boldsymbol{C}.$$

Let $\boldsymbol{X}$ be a real linear space, $\boldsymbol{C} \subset \boldsymbol{X}$ a convex cone in $\boldsymbol{X}$ and $\phi : \boldsymbol{C} \to \mathbb{R}$. We say that ϕ is a *linear functional* on $\boldsymbol{C}$ iff $\phi(\boldsymbol{x}+\boldsymbol{y}) = \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{C}$, and $\phi(t\boldsymbol{x}) = t\phi(\boldsymbol{x})$ for all $t > 0$, $\boldsymbol{x} \in \boldsymbol{C}$. Note that if $\phi \not\equiv 0$, then $\phi^{-1}(\{1\}) = \{\boldsymbol{x} \in \boldsymbol{C} : \phi(\boldsymbol{x}) = 1\}$ is a nonempty convex subset of $\boldsymbol{C}$, and put $\mathrm{supp}(\phi) := \{\boldsymbol{x} \in \boldsymbol{C} : \phi(\boldsymbol{x}) \ne 0\}$.

The term "linear functional" is legitimate in view of the following

Remark 2. Let $\phi : \boldsymbol{C} \to \mathbb{R}$ be additive and positively homogeneous on a cone $\boldsymbol{C} \subset \boldsymbol{X}$ such that $\boldsymbol{C} \cap (-\boldsymbol{C}) = \{\boldsymbol{o}\}$. Denote by $\boldsymbol{Y}$ the linear span of $\boldsymbol{C}$. It is easy to check that there exists a unique linear functional $\Phi : \boldsymbol{Y} \to \mathbb{R}$ such that $\Phi|_C = \phi$.

PROPOSITION. *Let $\boldsymbol{X}$ be a real linear space, $\boldsymbol{C} \subset \boldsymbol{X}$ a cone in $\boldsymbol{X}$ such that $\boldsymbol{C} \cap (-\boldsymbol{C}) = \{\boldsymbol{o}\}$, and $\phi : \boldsymbol{C} \to \mathbb{R}$ a linear functional on $\boldsymbol{C}$ such that $\phi \ge 0$ on $\boldsymbol{C}$. Suppose that $a, b \in \mathbb{R}$ are fixed and $0 < \min\{a, b\} < 1 < a + b$. If $H : \mathrm{supp}(\phi) \to \mathbb{R}$ satisfies*

$$H(a\boldsymbol{x} + b\boldsymbol{y}) \le aH(\boldsymbol{x}) + bH(\boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathrm{supp}(\phi),$$

and

$$\limsup_{t \to 0+} H(t\boldsymbol{x}) \le 0, \qquad \boldsymbol{x} \in \mathrm{supp}(\phi),$$

then H is positively homogeneous and subadditive.

Moreover, the function $h : \phi^{-1}(1) \to \mathbb{R}$ defined by

$$h(\boldsymbol{x}) := H(\boldsymbol{x}), \qquad \boldsymbol{x} \in \phi^{-1}(1),$$

is convex,

$$H(\boldsymbol{x}) = \phi(\boldsymbol{x})h(\boldsymbol{x}/\phi(\boldsymbol{x})), \qquad \boldsymbol{x} \in \mathrm{supp}(\phi),$$

and

$$\begin{aligned} \text{(9)} \quad \phi(\boldsymbol{x}+\boldsymbol{y})h\bigg(\frac{\boldsymbol{x}+\boldsymbol{y}}{\phi(\boldsymbol{x}+\boldsymbol{y})}\bigg) \\ \le \phi(\boldsymbol{x})h\bigg(\frac{\boldsymbol{x}}{\phi(\boldsymbol{x})}\bigg) + \phi(\boldsymbol{y})h\bigg(\frac{\boldsymbol{y}}{\phi(\boldsymbol{y})}\bigg), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathrm{supp}(\phi). \end{aligned}$$

Proof. It is easy to check that $\operatorname{supp}(\phi)$ is a convex cone in $\boldsymbol{X}$. Therefore the first conclusion is a consequence of Corollary 2.

To prove the remaining assertion note that $\boldsymbol{z} \in \phi^{-1}(1)$ if and only if there is an $\boldsymbol{x} \in \operatorname{supp}(\Phi)$ such that $\boldsymbol{z} = \boldsymbol{x}/\phi(\boldsymbol{x})$. Take any $\boldsymbol{x} \in \operatorname{supp}(\phi)$. By the positive homogeneity of H and the definition of h we have

$$H(\boldsymbol{x}) = \phi(\boldsymbol{x})H(\boldsymbol{x}/\phi(\boldsymbol{x})) = \phi(\boldsymbol{x})h(\boldsymbol{x}/\phi(\boldsymbol{x})).$$

Hence, the subadditivity of H gives (9). This inequality implies the convexity of h, and the proof is complete.

Remark 3. Taking in the Proposition $\boldsymbol{X} = \mathbb{R}^k$, $\boldsymbol{C} = \mathbb{R}^k_+$, $k \in \mathbb{N}$, and the functional $\phi : \boldsymbol{C} \to \mathbb{R}_+$, $\phi(\boldsymbol{x}) = \phi(x_1, \ldots, x_k) = x_i$, the projection on the x_i-axis, $i \in \{1, \ldots, k\}$, we get the result proved in [5] (cf. also [6]). Moreover, it is shown in [5] that inequality (9) with ϕ being the projection characterizes the convex functions h defined on $(0, \infty)^{k-1}$ and generalizes Minkowski's and Hölder's inequalities. Thus inequality (9) may also be interpreted as a generalization of these two fundamental inequalities.

3. An application to a characterization of the L^p-norm. For a measure space (Ω, Σ, μ) denote by $\boldsymbol{S} = \boldsymbol{S}(\Omega, \Sigma, \mu)$ the linear space of all μ-integrable step functions $\boldsymbol{x} : \Omega \to \mathbb{R}$ and by $\boldsymbol{S}_+ = \boldsymbol{S}_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $\boldsymbol{x} \in \boldsymbol{S}$. If $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are one-to-one, onto and $\varphi(0) = 0$ then the functional $\mathbf{P}_{\varphi,\psi} : \boldsymbol{S} \to \mathbb{R}$ given by the formula

$$\mathbf{P}_{\varphi,\psi}(\boldsymbol{x}) := \psi\Big(\int_{\Omega} \varphi \circ |\boldsymbol{x}|\, d\mu\Big), \qquad \boldsymbol{x} \in \boldsymbol{S},$$

is well defined. The goal of this section is to prove the following

THEOREM 3. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure. Suppose that $a, b \in \mathbb{R}$ are fixed numbers such that*

$$0 < \min\{a, b\} < 1 < a + b,$$

and $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are one-to-one, onto, continuous at 0 and $\varphi(0) = \psi(0) = 0$. If

$$\mathbf{P}_{\varphi,\psi}(a\boldsymbol{x} + b\boldsymbol{y}) \le a\mathbf{P}_{\varphi,\psi}(\boldsymbol{x}) + b\mathbf{P}_{\varphi,\psi}(\boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}_+,$$

then $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^{1/p}$ $(t \ge 0)$ for some $p \ge 1$.

Proof. Take any $\boldsymbol{x} \in \boldsymbol{S}_+$. Then there exist n pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$ of finite measure, and $x_1, \ldots, x_n \in \mathbb{R}_+$ such that $\boldsymbol{x} = \sum_{k=1}^n x_k \chi_{A_k}$. ($\chi_A$ stands for the characteristic function of the set A.) From the definition of $\mathbf{P}_{\varphi,\psi}$ we have

$$\mathbf{P}_{\varphi,\psi}(t\boldsymbol{x}) = \psi\Big(\int_{\Omega} \varphi \circ |t\boldsymbol{x}|\, d\mu\Big) = \psi\Big(\sum_{k=1}^{n} \varphi(tx_k)\mu(A_k)\Big), \qquad t > 0.$$

The continuity of φ and ψ at zero and $\varphi(0) = \psi(0) = 0$ imply that $\lim_{t\to 0+} \mathbf{P}_{\varphi,\psi}(t\boldsymbol{x}) = 0$. By Corollary 2 the functional $\mathbf{P}_{\varphi,\psi}$ is positively homogeneous, i.e.

$$\mathbf{P}_{\varphi,\psi}(t\boldsymbol{x}) = t\mathbf{P}_{\varphi,\psi}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{S}_+,\ t > 0, \tag{10}$$

and subadditive:

$$\mathbf{P}_{\varphi,\psi}(\boldsymbol{x}+\boldsymbol{y}) \le \mathbf{P}_{\varphi,\psi}(\boldsymbol{x}) + \mathbf{P}_{\varphi,\psi}(\boldsymbol{y}), \quad \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{S}_+. \tag{11}$$

By our assumption on the measure space, there are two disjoint sets $A, B \in \Sigma$ of finite positive measure. Put $\alpha := \mu(A)$ and $\beta := \mu(B)$. Taking $\boldsymbol{x} := x_1\chi_A + x_2\chi_B$ with $x_1, x_2 \ge 0$ in (10), we get

$$\psi(\alpha\varphi(tx_1) + \beta\varphi(tx_2)) = t\psi(\alpha\varphi(x_1) + \beta\varphi(x_2)).$$

Since ψ and φ are bijective we can write this equation in the following equivalent form:

$$\begin{aligned}\alpha\varphi(t\varphi^{-1}(x_1)) &+ \beta\varphi(t\varphi^{-1}(x_2)) \\ &= \psi^{-1}(t\psi(\alpha x_1 + \beta x_2)), \quad t > 0,\ x_1, x_2 \ge 0.\end{aligned} \tag{12}$$

Substituting here first $x_2 = 0$, and next $x_1 = 0$ we get

$$\alpha\varphi(t\varphi^{-1}(x_1)) = \psi^{-1}(t\psi(\alpha x_1)), \quad t > 0,\ x_1 \ge 0, \tag{13}$$

$$\beta\varphi(t\varphi^{-1}(x_2)) = \psi^{-1}(t\psi(\beta x_2)), \quad t > 0,\ x_2 \ge 0. \tag{14}$$

The relations (13) and (14) allow us to write (12) in the form

$$\psi^{-1}(t\psi(\alpha x_1)) + \psi^{-1}(t\psi(\beta x_2)) = \psi^{-1}(t\psi(\alpha x_1 + \beta x_2)), \quad t > 0,\ x_1, x_2 \ge 0,$$

or, equivalently,

$$\psi^{-1}(t\psi(x_1)) + \psi^{-1}(t\psi(x_2)) = \psi^{-1}(t\psi(x_1 + x_2)), \quad t > 0,\ x_1, x_2 \ge 0.$$

Thus, for every $t > 0$, the function $\psi^{-1} \circ (t\psi)$ is additive. Since it is nonnegative, it follows that for every $t > 0$ there is an $m(t) > 0$ such that

$$\psi^{-1}(t\psi(u)) = m(t)u, \quad u > 0. \tag{15}$$

Writing an analogous equation for every $s > 0$ we have

$$\psi^{-1}(s\psi(u)) = m(s)u, \quad u > 0.$$

Composing separately the functions on the left- and the right-hand sides of these equations we obtain

$$\psi^{-1}(st\psi(u)) = m(s)m(t)u, \quad u > 0.$$

Replacing t by st in (15) we get

$$\psi^{-1}(st\psi(u)) = m(st)u, \quad u > 0.$$

The last two equations imply that $m(st) = m(s)m(t)$, $s, t > 0$, i.e. $m : (0,\infty) \to (0,\infty)$ is a solution of the multiplicative Cauchy equation. Putting

$u=1$ in (15) we get $m(t)=\psi^{-1}(t\psi(1))$, $t>0$. It follows that m is a bijection of $(0,\infty)$, and, of course, the inverse function to m,

$$m^{-1}(t)=\psi(t)/\psi(1), \qquad t>0,$$

is multiplicative. The continuity of ψ at 0 implies that there exists a $p\in\mathbb{R}$, $p\neq 0$, such that $m^{-1}(t)=t^{1/p}$ for all $t>0$. Hence

$$\psi(t)=\psi(1)t^{1/p}, \qquad t>0.$$

Inserting this into (13) we have $\alpha\varphi(t\varphi^{-1}(x_1))=\alpha x_1 t^p$ for all $t>0$ and $x_1\geq 0$. Taking $x_1:=\varphi^{-1}(1)$ we obtain

$$\varphi(t)=\varphi(1)t^p, \qquad t>0.$$

Now, for the above power functions φ and ψ, (11) reduces to the classical Minkowski inequality. It follows that $p\geq 1$. This completes the proof.

Remark 4. To prove that (13) and (14) imply that φ and ψ are the inverse power functions we could apply some results proved in [4].

A similar result holds if $\mathbf{P}_{\varphi,\psi}$ satisfies the opposite inequality to that of Theorem 3. One should emphasize that, in this case, the regularity assumptions on functions φ and ψ are superfluous. Namely, we have

THEOREM 4. *Let (Ω,Σ,μ) be a measure space with at least two disjoint sets of finite positive measure. Suppose that $a,b\in\mathbb{R}$ are fixed with $0<\min\{a,b\}<1<a+b$, and $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}_+$ are one-to-one, onto, and $\varphi(0)=0$. If*

$$\mathbf{P}_{\varphi,\psi}(a\boldsymbol{x}+b\boldsymbol{y})\geq a\mathbf{P}_{\varphi,\psi}(\boldsymbol{x})+b\mathbf{P}_{\varphi,\psi}(\boldsymbol{y}), \qquad \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{S}_+, \tag{16}$$

then $\varphi(t)=\varphi(1)t^p$ and $\psi(t)=\psi(1)t^{1/p}$ $(t\geq 0)$ for some p, $0<p\leq 1$.

Proof. Since $-\mathbf{P}_{\varphi,\psi}$ satisfies the opposite inequality to (16) and $(-\mathbf{P}_{\varphi,\psi})(\boldsymbol{x})\leq 0$ for all $\boldsymbol{x}\in\boldsymbol{S}_+$, Corollary 2 implies that $\mathbf{P}_{\varphi,\psi}$ is positively homogeneous, and superadditive on $\boldsymbol{S}_+$, i.e.

$$\mathbf{P}_{\varphi,\psi}(\boldsymbol{x}+\boldsymbol{y})\geq \mathbf{P}_{\varphi,\psi}(\boldsymbol{x})+\mathbf{P}_{\varphi,\psi}(\boldsymbol{y}), \qquad \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{S}_+. \tag{17}$$

Arguing in the same way as in the proof of Theorem 3 we show that the function $m:(0,\infty)\to(0,\infty)$, $m(t)=\psi^{-1}[t\psi(1)]$, $t>0$, is multiplicative on $(0,\infty)$.

As in the proof of Theorem 3, take disjoint sets $A,B\in\Sigma$ of finite positive measure, and put $\alpha:=\mu(A)$ and $\beta:=\mu(B)$. Substituting, in (17), $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{S}_+$ such that

$$\boldsymbol{x}:=x_1\chi_A+x_2\chi_B, \qquad \boldsymbol{y}:=y_1\chi_A+y_2\chi_B, \qquad x_1,x_2,y_1,y_2\geq 0,$$

we get

$$\psi(\alpha\varphi(x_1+y_1)+\beta\varphi(x_2+y_2))\geq\psi(\alpha\varphi(x_1)+\beta\varphi(x_2))+\psi(\alpha\varphi(y_1)+\beta\varphi(y_2))$$

for all $x_1, x_2, y_1, y_2 \ge 0$. Take arbitrary $s, t \ge 0$. Putting

$$x_1 = \varphi(s/\alpha)^{-1}, \quad x_2 = y_1 = 0, \quad y_2 = \varphi(t/\beta)^{-1},$$

and making use of the assumption that $\varphi(0) = 0$, we get

$$\psi(s+t) \ge \psi(s) + \psi(t), \qquad s, t \ge 0.$$

Hence ψ is increasing, and, consequently, a homeomorphism of $\mathbb{R}_+$. It follows that the multiplicative function m is a homeomorphism of $(0, \infty)$.

Now, by an argument as in the proof of Theorem 3, we show that there exists a $p \in \mathbb{R}$, $p \ne 0$, such that $\psi(t) = \psi(1)t^{1/p}$ and $\varphi(t) = \varphi(1)t^p$, $t > 0$. Substituting these functions into (16) we obtain the "companion" of the Minkowski inequality which is known to hold only for $p \in (0, 1]$. This concludes the proof.

Remark 5. Theorems 3 and 4 can be interpreted to be converses of the Minkowski inequalities (cf. [7] and [8] where converses of Minkowski's inequality other than Theorem 3 are given).

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia Math. Appl. 31, Cambridge University Press, Cambridge, Sydney, 1989.

[2] E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, Amer. Math. Soc., Providence, R.I., 1957.

[3] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's equation and Jensen's inequality*, Prace Nauk. Uniw. Śl. 489, Polish Scientific Publishers, 1985.

[4] J. Matkowski, *On a characterization of L^p-norm*, Ann. Polon. Math. 50 (1989), 137–144.

[5] —, *A functional inequality characterizing convex functions, conjugacy and a generalization of Hölder's and Minkowski's inequalities*, Aequationes Math. 40 (1990), 168–180.

[6] —, *Functional inequality characterizing nonnegative concave functions in* $(0, \infty)$, ibid. 43 (1992), 219–224.

[7] —, *The converse of the Minkowski's inequality theorem and its generalization*, Proc. Amer. Math. Soc. 109 (1990), 663–675.

[8] —, *L^p-like paranorms*, in: Selected Topics in Functional Equations and Iteration Theory, Proc. Austrian–Polish Seminar, Graz, 1991, Grazer Math. Ber. 316 (1992), 103–135.

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY
WILLOWA 2
43-309 BIELSKO-BIAŁA, POLAND

RAFOWA 21
43-300 BIELSKO-BIAŁA, POLAND

Reçu par la Rédaction le 4.10.1993
Révisé le 16.12.1993