ANNALES
POLONICI MATHEMATICI
LX.3 (1995)

Between the Paley–Wiener theorem and the Bochner tube theorem

by Zofia Szmydt and Bogdan Ziemian (Warszawa)

Abstract. We present the classical Paley–Wiener–Schwartz theorem [1] on the Laplace transform of a compactly supported distribution in a new framework which arises naturally in the study of the Mellin transformation. In particular, sufficient conditions for a function to be the Mellin (Laplace) transform of a compactly supported distribution are given in the form resembling the Bochner tube theorem [2].

1. Notation. We employ the usual notation of set theory. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, and $\mathbb{R}^n_+ = (\mathbb{R}_+)^n$. For $x \in (x_1, \ldots, x_n) \in \mathbb{R}^n$ we set $\langle x\rangle = 1 + |x_1| + \ldots + |x_n|$. We write $x < y$ $(x \le y)$ for $x, y \in \mathbb{R}^n$ to denote $x_j < y_j$ $(x_j \le y_j$ resp.) for $j = 1, \ldots, n$, and we set $I = (0, t] = \{x \in \mathbb{R}^n : 0 < x \le t\}$, where $t \in \mathbb{R}^n_+$. By $\mathbf{1}$ we denote $(1, \ldots, 1)$. $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. We write $|\alpha| = \alpha_1 + \ldots + \alpha_n$ for $\alpha \in \mathbb{N}^n_0$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n_0$ we write $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

We employ the usual notation of distribution theory. $D'(\Omega)$ denotes the space of distributions on an open set $\Omega \subset \mathbb{R}^n$, and $D'_A(\Omega)$ the space of distributions on Ω with support in $A \subset \Omega$. The value of a distribution u on a test function φ is denoted by $u[\varphi]$.

2. Auxiliary theorems

Theorem 1. *Let $u \in D'_K(\mathbb{R}^n)$ where K is a connected compact set in $\mathbb{R}^n$ such that any two points $x, y \in K$ can be joined by a rectifiable curve in K of length $\le \widetilde{C}|x - y|$, $\widetilde{C} < \infty$. Then there exists a constant $C < \infty$ and $k \in \mathbb{N}_0$ such that*

1991 *Mathematics Subject Classification*: Primary 46F12.
Key words and phrases: Mellin distributions, Bochner tube theorem.

$$|u[\psi]| \le C \sum_{\substack{|\alpha|\le k\\ \alpha\in\mathbb{N}_0^n}} \sup_{x\in K} \left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \psi(x)\right| \quad \textit{for } \psi \in C^k(\mathbb{R}^n).$$

The proof of this theorem, based on the Whitney extension theorem, is given in [1].

Now following [3] we recall the spaces of Mellin distributions. Denote by $\mu : \mathbb{R}^n \to \mathbb{R}_+^n$ the diffeomorphism

$$\mu(y) = e^{-y} := (e^{-y_1}, \ldots, e^{-y_n}).$$

We define the space of *Mellin distributions on* $\mathbb{R}_+^n$ for every $\alpha \in \mathbb{R}^n$ as the dual of the space

$$\mathfrak{M}_\alpha = \mathfrak{M}_\alpha(\mathbb{R}_+^n) = \{\sigma \in C^\infty(\mathbb{R}_+^n) : (x^{\alpha+\mathbf{1}}\sigma) \circ \mu \in S(\mathbb{R}^n)\},$$

with the natural topology in $\mathfrak{M}_\alpha$ induced from $S(\mathbb{R}^n)$.

Note that $u \in \mathfrak{M}'_\alpha(\mathbb{R}_+^n)$ if and only if $e^{\alpha y}(u\circ\mu) \in S'(\mathbb{R}^n)$. The M_α *Mellin transform* of $u \in \mathfrak{M}'_\alpha$ is defined by means of the inverse Fourier transform F^{-1}:

$$M_\alpha u = (2\pi)^{n/2} F^{-1}(e^{\alpha y}(u \circ \mu)). \tag{1}$$

Here we assume $F\sigma(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\sigma(x)\,dx$ for $\xi \in \mathbb{R}^n$ and $\sigma \in S(\mathbb{R}^n)$.

For $a \in \mathbb{R}^n$ we introduce the space

$$M_a = M_a(I) = \{\varphi \in C^\infty(I) : \sup_{x\in I} |x^{a+\alpha+\mathbf{1}}(\partial/\partial x)^\alpha \varphi(x)| < \infty,\ \alpha \in \mathbb{N}_0^n\}$$

equipped with the topology defined by the sequence of seminorms

$$\varrho_{a\alpha}(\varphi) = \sup_{x\in I} |x^{a+\alpha+\mathbf{1}}(\partial/\partial x)^\alpha \varphi(x)|, \quad \alpha \in \mathbb{N}_0^n.$$

Note that the space M_a is complete (see [3]) but the set $C^\infty_{(0)}(I)$ (of restrictions to I of functions in C_0^∞) is not dense in $M_a(I)$.

Let $\omega \in (\mathbb{R} \cup \{\infty\})^n$. We define the function space $M_{(\omega)}(I)$ as the inductive limit

$$M_{(\omega)}(I) = \varinjlim_{a<\omega} M_a(I).$$

Now the set $C^\infty_{(0)}(I)$ is dense in $M_{(\omega)}(I)$ and the dual space $M'_{(\omega)} = M'_{(\omega)}(I)$ is a subspace of $D'_I(\mathbb{R}_+^n)$. Therefore the elements of $M'_{(\omega)}$ are called *Mellin distributions on* I. Note that for $a < b < \omega$ and $\omega \in (\mathbb{R} \cup \{\infty\})^n$,

$$M_{(a)}(I) \subset M_a(I) \subset M_b(I) \subset M_{(\omega)}(I),$$
$$M_{(\omega)}(I) = \varinjlim_{a<\omega} M_{(a)}(I),$$
$$M'_{(\omega)}(I) = \bigcap_{a<\omega} M'_a(I) = \bigcap_{a<\omega} M'_{(a)}(I).$$

The totality of Mellin distributions is denoted by

$$M'(I) = \bigcup_{\omega\in(\mathbb{R}\cup\{\infty\})^n} M'_{(\omega)}(I) = \bigcup_{\omega\in\mathbb{R}^n} M'_{(\omega)}(I).$$

$M'(I)$ coincides with the space of restrictions to $\mathbb{R}^n_+$ of distributions on $\mathbb{R}^n$ with support in $\bar{I}$.

Let $u \in M'_{(\omega)}(I)$ for some $\omega \in (\mathbb{R}\cup\{\infty\})^n$. We define the *Mellin transform* of u by

$$Mu(z) = u[x^{-z-\mathbf{1}}] \quad \text{for } \operatorname{Re} z < \omega. \tag{2}$$

This definition differs from the classical one by the change of variable $z \mapsto -z$.

The following theorem gives a relation between the Mellin transformations M and M_α defined by (2) and (1) respectively.

THEOREM 2. *Let $u \in M'_{(\omega)}(I)$. Then Mu is holomorphic for* $\operatorname{Re} z < \omega$ *and $u \in \mathfrak{M}'_\alpha(\mathbb{R}^n_+)$ for every $\alpha < \omega$. The tempered distribution $M_\alpha u$ is a function*:

$$(M_\alpha u)(\beta) = Mu(\alpha + i\beta) = (u\circ\mu)[e^{(\alpha+i\beta)y}] \quad \textit{for } \beta \in \mathbb{R}^n. \tag{3}$$

Moreover, *$M_\alpha : M'_{(\omega)} \to S'$ is continuous for $\alpha < \omega$.*

THEOREM 3 (Paley–Wiener type theorem). *In order that a function $f(z) = f(z_1,\ldots,z_n)$ be the Mellin transform of a unique Mellin distribution $u \in M'_{(\omega)}((0,t])$ it is necessary and sufficient that f be holomorphic in $\{z\in\mathbb{C}^n : \operatorname{Re} z < \omega\}$ and that for every $b < \omega$ and every $\varrho \in \mathbb{R}_+$ there exist $s = s(b) \in \mathbb{N}_0$ and $C = C(b,\varrho) < \infty$ such that*

$$|f(\alpha+i\beta)| \le C\langle\beta\rangle^s (e^\varrho t)^{-\alpha} \quad \textit{for } \alpha \le b. \tag{4}$$

3. The main theorem. Let $t^- = (t^-_1,\ldots,t^-_n)$, $t^+ = (t^+_1,\ldots,t^+_n)$, $0 < t^- < t^+$, write $I = (0,t^+]$ and consider the polyinterval

$$[t^-,t^+] = \{x\in\mathbb{R}^n : t^- \le x \le t^+\}.$$

THEOREM 4. *Let f be a function holomorphic on $\{z\in\mathbb{C}^n : \operatorname{Re} z < 0\} \cup \{z\in\mathbb{C}^n : \operatorname{Re} z > 0\}$ and such that for every $b\in\mathbb{R}^n$ with $b < 0$ and $\varrho\in\mathbb{R}_+$*,

$$|f(\alpha+i\beta)| \le C\langle\beta\rangle^s (e^\varrho t^+)^{-\alpha} \qquad \textit{for } \alpha < b, \tag{5}$$

$$|f(\alpha+i\beta)| \le C\langle\beta\rangle^s (e^{-\varrho} t^-)^{-\alpha} \qquad \textit{for } \alpha > -b, \tag{6}$$

with some $s = s(b) \in \mathbb{N}_0$ and $C = C(b,\varrho) < \infty$. Moreover, *assume that the following limits exist in $S'(\mathbb{R}^n)$ and are equal*:

$$\lim_{\alpha\to 0_-} f(\alpha + i\cdot) = \lim_{\alpha\to 0_+} f(\alpha + i\cdot). \tag{7}$$

Then there exists a unique $u \in D'_{[t^-,t^+]}$ *such that* $Mu = f$. *Furthermore*, f *is an entire function on* $\mathbb{C}^n$ *and for every* $b \in \mathbb{R}^n$ *and* $\varrho \in \mathbb{R}_+$ *there exist* $C = C(b,\varrho) < \infty$ *and* $s = s(b) \in \mathbb{N}_0$ *such that for any* $\sigma \in \{-,+\}^n$,

$$(8)\quad |f(\alpha+i\beta)| \le C\langle\beta\rangle^s (e^{\sigma_1\varrho}t_1^{\sigma_1})^{-\alpha_1}\dots(e^{\sigma_n\varrho}t_n^{\sigma_n})^{-\alpha_n} \quad \text{for } \sigma_j\alpha_j \le \sigma_j b_j,\ j=1,\dots,n.$$

Proof. By assumption (5), which is the sufficient condition in Theorem 3, there exists a unique distribution $u \in M'_{(0)}((0,t^+])$ such that $Mu = f$. Thus $\operatorname{supp} u \subset (0,t^+]$ and $u \in \mathfrak{M}'_\alpha((0,t^+])$ for $\alpha < 0$. Denote by w the tempered distribution defined by (7). Hence

$$\lim_{\alpha\to 0_-} \int_{\mathbb{R}^n} f(\alpha+i\beta)\psi(\beta)\,d\beta = w[\psi] \quad \text{for } \psi \in S(\mathbb{R}^n)$$

and by (3) and (1) we get

$$\begin{aligned} w[\psi] &= \lim_{\alpha\to 0_-} \int_{\mathbb{R}^n} (Mu)(\alpha+i\beta)\psi(\beta)\,d\beta \\ &= (2\pi)^{n/2} \lim_{\alpha\to 0_-} \int_{\mathbb{R}^n} F^{-1}(e^{\alpha y}(u\circ\mu))(\beta)\psi(\beta)\,d\beta \\ &= (2\pi)^{n/2} \lim_{\alpha\to 0_-} F^{-1}(e^{\alpha y}(u\circ\mu))[\psi] \\ &= (2\pi)^{n/2} \lim_{\alpha\to 0_-} (e^{\alpha y}(u\circ\mu))[F^{-1}\psi] \\ &= (2\pi)^{n/2} \lim_{\alpha\to 0_-} (u\circ\mu)[e^{\alpha y}F^{-1}\psi]. \end{aligned}$$

For $\psi = F\varphi$ with $\varphi \in D(\mathbb{R}^n)$ the last formula yields

$$(9)\qquad Fw[\varphi] = (2\pi)^{n/2}(u\circ\mu)[\varphi] \quad \text{for } \varphi \in D(\mathbb{R}^n).$$

Now observe that by assumption (6),

$$|f(-\alpha-i\beta)| < C\langle\beta\rangle^s \left(e^{\varrho}\frac{1}{t^-}\right)^{-\alpha} \quad \text{for } \alpha < b,$$

where $1/x := (1/x_1,\dots,1/x_n)$ for $x \in \mathbb{R}^n_+$. As before, by Theorem 3, there exists a unique distribution $\widetilde{u} \in M'_{(0)}((0,1/t^-])$ such that

$$f(-\alpha-i\beta) = M\widetilde{u}(\alpha+i\beta).$$

Note that $\widetilde{u} \in \mathfrak{M}'_\alpha((0,1/t^-])$ for $\alpha < 0$ and $f(-\alpha-i\beta) = M_\alpha\widetilde{u}(\beta) = (2\pi)^{n/2}F^{-1}(e^{\alpha y}(\widetilde{u}\circ\mu))$.

Since by (7), $w = \lim_{\alpha\to 0_+} f(\alpha+i\cdot)$ we have $\lim_{\alpha\to 0_-} f(-\alpha-i\cdot) = w^\vee$ where $^\vee$ denotes the reflection $\beta \to -\beta$. Take $\varphi \in D(\mathbb{R}^n)$ and let $\psi = F\varphi$.

Then $\psi \in S(\mathbb{R}^n)$ and

$$w^\vee[\psi] = \lim_{\alpha\to 0_-} \int_{\mathbb{R}^n} f(-\alpha - i\beta)\psi(\beta)\,d\beta$$

$$= (2\pi)^{n/2} \lim_{\alpha\to 0_-} \int_{\mathbb{R}^n} F^{-1}(e^{\alpha y}(\widetilde{u}\circ\mu))\psi(\beta)\,d\beta = (2\pi)^{n/2}(\widetilde{u}\circ\mu)[\varphi].$$

Hence

$$Fw^\vee[\varphi] = (2\pi)^{n/2}(\widetilde{u}\circ\mu)[\varphi] \quad \text{for } \varphi \in D(\mathbb{R}^n).$$

This together with (9) yields $(u\circ\mu)^\vee = \widetilde{u}\circ\mu$. Let λ be the mapping $\mathbb{R}^n_+ \ni x \mapsto 1/x$. Since $(u\circ\mu)^\vee = (u\circ\lambda)\circ\mu$ we get $u\circ\lambda = \widetilde{u}$. Hence $u\circ\lambda \in M'_{(0)}((0,1/t^-])$ and by definition of λ we have $\operatorname{supp} u \subset \{x : x \ge t^-\}$, which together with $u \in M'_{(0)}((0,t^+])$ gives the desired assertion $u \in D'_{[t^-,t^+]}$. By Theorem 1, $u \in M'_a$ for every $a \in \mathbb{R}^n$ (i.e. $u \in M'_{(\infty)}$) and hence by Theorem 3 (the necessary condition this time) $f = Mu$ is entire on $\mathbb{C}^n$ and the estimate (5) holds for $\alpha \le b$ for every $b \in \mathbb{R}^n$. Since $u\circ\lambda \in D'_{[1/t^+,1/t^-]}$ and $M(u\circ\lambda)(z) = Mu(-z)$ we get as before, for all $b \in \mathbb{R}^n$ and $\varrho \in \mathbb{R}_+$,

$$|Mu(\alpha + i\beta)| \le C\langle\beta\rangle^s (e^{-\varrho}t^-)^{-\alpha} \quad \text{for } \alpha \ge b$$

with $s = s(b)$ and $C = C(b,\varrho)$. Thus we have proved (8) for $\sigma = (+,\ldots,+)$ and $\sigma = (-,\ldots,-)$. To get the proof for $\sigma^j = (\sigma^j_1,\ldots,\sigma^j_n)$ with $\sigma^j_i = +$ if $i \ne j$, and $\sigma^j_j = -1$ $(j = 1,\ldots,n)$, take the mapping

$$\lambda_j : \mathbb{R}^n_+ \ni x \mapsto (x_1,\ldots,x_{j-1},1/x_j,x_{j+1},\ldots,x_n).$$

Then

$$M(u\circ\lambda_j)(z) = Mu(z_1,\ldots,z_{j-1},-z_j,z_{j+1},\ldots,z_n)$$
$$= f(z_1,\ldots,z_{j-1},-z_j,z_{j+1},\ldots,z_n),$$
$$\operatorname{supp}(u\circ\lambda_j) \subset \{x : t^-_i \le x_i \le t^+_i \text{ for } i \ne j,\ 1/t^+_j \le x_j \le 1/t^-_j\}.$$

Fix arbitrarily $b \in \mathbb{R}^n$, $\varrho \in \mathbb{R}_+$ and j with $1 \le j \le n$. Take $\widetilde{b} = (b_1,\ldots,b_{j-1}, -b_j,b_{j+1},\ldots,b_n)$. By Theorem 3,

$$|M(u\circ\lambda_j)(\alpha + i\beta)| \le C\langle\beta\rangle^s (e^\varrho t^+_1)^{-\alpha_1}\ldots\left(e^\varrho\frac{1}{t^-_j}\right)^{-\alpha_j}\ldots(e^\varrho t^+_n)^{-\alpha_n}$$

for $\alpha \le \widetilde{b}$ and hence

$$|f(\alpha + i\beta)| \le C\langle\beta\rangle^s (e^\varrho t^+_1)^{-\alpha_1}\ldots(e^{-\varrho}t^-_j)^{-\alpha_j}\ldots(e^\varrho t^+_n)^{-\alpha_n}$$

for $\alpha_i \le b_i$ if $i \ne j$, $-\alpha_j \le \widetilde{b}_j = -b_j$, i.e. (8) holds for σ^j $(j = 1,\ldots,n)$.

The remaining cases of σ are left to the reader.

Remark 1. Note that in contrast to the classical Paley–Wiener theorem the sufficiency part of Theorem 4 does not require that the function

f be entire. Instead we assume holomorphy in two wedges with a common edge and identity of the corresponding boundary values. By the necessity result this gives holomorphy in $\mathbb{C}^n$ as well as estimates in the "missing" wedges, which can be regarded as a variant of the Bochner tube theorem.

Remark 2. By applying the techniques of the theory of Fourier hyperfunctions and analytic functionals one can prove a variant of Theorem 4 with $\langle\beta\rangle^s$ in the estimates (5), (6) and (8) replaced by $e^{\theta|\beta|}$ for some $\theta > 0$. Then the identity (7) should be understood as the equivalence of pertinent boundary values in the sense of Fourier hyperfunctions.

References

[1] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1985.

[2] W. Rudin, *Lectures on the Edge of the Wedge Theorem*, Amer. Math. Soc., Providence, 1971.

[3] Z. Szmydt and B. Ziemian, *The Mellin Transformation and Fuchsian Type Partial Differential Equations*, Math. Appl. 56, Kluwer, 1992.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ŚNIADECKICH 8
00-950 WARSZAWA, POLAND

Reçu par la Rédaction le 2.3.1994