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Necessary and sufficient conditions for generalized convexity

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Abstract. We give some necessary and sufficient conditions for an $n-1$ times differentiable function to be a generalized convex function with respect to an unrestricted n-parameter family.

1. Introduction. A family F of continuous real-valued functions φ defined on an open interval (a,b) is said to be an *n-parameter family on* (a,b) (see [1] and [5]) if for any distinct points $x_1,\ldots,x_n$ in (a,b) and any numbers $y_1,\ldots,y_n$ there exists exactly one $\varphi\in F$ satisfying

$$\varphi(x_i)=y_i,\quad i=1,\ldots,n.$$

Throughout the paper we assume $n\ge 2$.

Let F be an n-parameter family on (a,b). Following [5] we say that a function ψ continuous on (a,b) is *strictly F-convex* (*F-convex*, *strictly F-concave*, *F-concave*) *on* (a,b) if for any points $a<x_1<\ldots<x_n<b$ the unique $\varphi\in F$ determined by

$$\varphi(x_i)=\psi(x_i),\quad i=1,\ldots,n, \tag{1}$$

satisfies the inequalities

$$(-1)^{n+i}\varphi(x)<(\le,>,\ge)\ (-1)^{n+i}\psi(x),\quad x\in(x_i,x_{i+1}),$$

for $i=0,1,\ldots,n$, where $x_0:=a$ and $x_{n+1}:=b$.

The above inequalities can be rewritten as

$$\operatorname{sgn}(\psi(x)-\varphi(x))=\operatorname{sgn}\Big(\prod_{i=1}^{n}(x-x_i)\Big),\quad x\in(a,b), \tag{2}$$

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for strict convexity and

$$\operatorname{sgn}(\psi(x)-\varphi(x)) = -\operatorname{sgn}\Big(\prod_{i=1}^{n}(x-x_i)\Big), \qquad x \in (a,b),$$

for strict concavity.

A family F of C^{n-1} functions on (a,b) is called an *unrestricted n-parameter family* (or briefly an *H_n-family*) on (a,b) (see [3]) if for any distinct $x_1,\ldots,x_k \in (a,b)$, any positive integers $\lambda_1,\ldots,\lambda_k$ such that $\lambda_1+\ldots+\lambda_k = n$, and any numbers $y_i^{\mu_i}$, where $i = 1,\ldots,k$, $\mu_i = 0,\ldots,\lambda_i - 1$, there exists exactly one $\varphi \in F$ satisfying

$$\varphi^{\mu_i}(x_i) = y_i^{\mu_i}, \qquad i = 1,\ldots,k,\ \mu_i = 0,\ldots,\lambda_i - 1, \tag{3}$$

where

$$\varphi^0(x) := \varphi(x), \quad \varphi^l(x) : \frac{d^l\varphi(x)}{dx^l} \quad \text{for } l = 1,2,\ldots$$

This notation will be used throughout the paper.

It is evident that any H_n-family on (a,b) is an n-parameter family on (a,b). Therefore we may consider the generalized convexity with respect to H_n-families. To begin with we introduce the following definitions:

Let F be an H_n-family on (a,b) and let ψ be $n-1$ times differentiable on (a,b). Let $i_1,\ldots,i_k$ be positive integers such that $i_1+\ldots+i_k = n$. The function ψ will be said to *satisfy the condition* $W_n(i_1,\ldots,i_k;F)$ (resp. $\widetilde{W}_n(i_1,\ldots \ldots,i_k;F)$) *on* (a,b) if for any $a < x_1 < \ldots < x_k < b$,

$$\operatorname{sgn}(\psi(x)-\varphi(x)) = \operatorname{sgn}\Big(\prod_{l=1}^{k}(x-x_i)^{i_l}\Big), \qquad x \in (a,b) \text{ (resp. } x \in (x_1,x_k)),$$

where $\varphi \in F$ is determined by

$$\varphi^{j_l}(x_l) = \psi^{j_l}(x_l), \qquad l = 1,\ldots,k,\ j_l = 0,\ldots,i_l - 1. \tag{4}$$

The function ψ will be said to *satisfy the condition* $K_n(i_1,\ldots,i_k;F)$ (resp. $\widetilde{K}_n(i_1,\ \ldots,i_k;F)$) *on* (a,b) if for any $a < x_1 < \ldots < x_k < b$,

$$\operatorname{sgn}(\psi(x)-\varphi(x)) = -\operatorname{sgn}\Big(\prod_{l=1}^{k}(x-x_i)^{i_l}\Big), \qquad x \in (a,b) \text{ (resp. } x \in (x_1,x_k)),$$

where $\varphi \in F$ is determined by (4).

We will use the symbol $\varphi(x_1^{i_1},\ldots,x_k^{i_k};\psi;\cdot)$ to denote the function $\varphi \in F$ satisfying (4).

It is well known (see [3]) that ψ is strictly F-convex on (a,b) iff for any $a < x_1 < \ldots < x_n < b$ the function φ determined by (1) satisfies (2) on (x_1,x_n). This means that ψ satisfies the condition $W_n(1^{(n)};F)$ on (a,b) iff ψ satisfies the condition $\widetilde{W}_n(1^{(n)};F)$ on (a,b). Here $1^{(n)}$ stands for $\underbrace{1,\ldots,1}_{n}$.

It is of interest to know whether the conditions $W_n(i_1,\ldots,i_k;F)$ are equivalent to strict F-convexity.

The case $n=2$ was considered by D. Brydak [2]. He has proved that if F is an H_2-family on (a,b) and ψ is differentiable on (a,b), then ψ is strictly F-convex on (a,b) iff ψ satisfies $W_2(2;F)$ on (a,b).

The case $n=3$ was considered by the author in [4]. The theorem in [4] reads as follows: Let F be an H_3-family on (a,b) and let ψ be twice differentiable on (a,b). Then the conditions

- ψ is strictly F-convex on (a,b);
- ψ satisfies $W_3(1,2;F)$ on (a,b);
- ψ satisfies $W_3(3;F)$ on (a,b);
- ψ satisfies $W_3(2,1;F)$ on (a,b);

are equivalent.

We will prove the following two theorems: Let F be an H_n-family on (a,b) and let ψ be $n-1$ times differentiable on (a,b).

1. If ψ is strictly F-convex on (a,b), then for any positive integers $i_1,\ldots,i_k$ such that $i_1+\ldots+i_k=n$, ψ satisfies $W_n(i_1,\ldots,i_k;F)$ on (a,b).
2. If ψ satisfies $W_n(i_1,\ldots,i_k;F)$ on (a,b) for some $i_1,\ldots,i_k\in\{1,2,3\}$ such that $i_1+\ldots+i_k=n$, then ψ is strictly F-convex on (a,b).

2. Lemmas

LEMMA 1. *Let f and g be defined and k times differentiable in a neighbourhood of a point x_0 and let*

$$f^i(x_0)=g^i(x_0),\qquad i=0,1,\ldots,k-1.$$

(i) *If there exists a sequence $\{x_n\}$ such that $x_n\to x_0^+$ and $f(x_n)\ge g(x_n)$ for $n=1,2,\ldots$, then $f^k(x_0)\ge g^k(x_0)$.*

(ii) *If there exists a sequence $\{x_n\}$ such that $x_n\to x_0^-$ and $f(x_n)\ge g(x_n)$ for $n=1,2,\ldots$, then $(-1)^kf^k(x_0)\ge(-1)^kg^k(x_0)$.*

(iii) *If there exists a sequence $\{x_n\}$ such that $x_n\to x_0^+$ (or $x_n\to x_0^-$) and $f(x_n)\ge g(x_n)\ge 0$ for $n=1,2,\ldots$, and $f^i(x_0)=0$ for $i=0,1,\ldots,k$, then $g^k(x_0)=0$.*

We omit an easy proof.

The two lemmas below are easy consequences of the definitions of $W_n(i_1,\ldots,i_k;F)$ and $K_n(i_1,\ldots,i_k;F)$.

LEMMA 2. *Let $i_1,\ldots,i_k$ be positive integers such that $i_1+\ldots+i_k=n$. Then the following conditions are equivalent*:

- *If G_1 is an H_n-family on (a,b), ψ_1 is $n-1$ times differentiable on (a,b) and ψ_1 is strictly G_1-convex on (a,b), then ψ_1 satisfies $W_n(i_1,\ldots,i_k;G_1)$ on (a,b).*

• *If G_2 is an H_n-family on (a,b), ψ_2 is $n-1$ times differentiable on (a,b) and ψ_2 is strictly G_2-concave on (a,b), then ψ_2 satisfies $K_n(i_1,\dots,i_k;G_2)$ on (a,b).*

LEMMA 3. *Under the assumptions of Lemma 2 the following conditions are equivalent*:

• *If G_1 is an H_n-family on (a,b), ψ_1 is $n-1$ times differentiable on (a,b) and ψ_1 satisfies $W_n(i_1,\dots,i_k;G_1)$ on (a,b), then ψ_1 is strictly G_1-convex on (a,b).*
• *If G_2 is an H_n-family on (a,b), ψ_2 is $n-1$ times differentiable on (a,b) and ψ_2 satisfies $K_n(i_1,\dots,i_k;G_2)$ on (a,b), then ψ_2 is strictly G_2-concave on (a,b).*

The proofs of the next lemmas are not so simple.

LEMMA 4. *Let F be an H_n-family on (a,b) and let ψ be $n-1$ times differentiable on (a,b). If ψ satisfies $\widetilde{W}_n(n-1,1;F)$ and $\widetilde{W}_n(1,n-1;F)$ on (a,b), then ψ satisfies $W_n(n;F)$ on (a,b).*

Proof. We have to show that for any $x_0 \in (a,b)$,

$$\operatorname{sgn}(\psi(x)-\varphi_1(x)) = \operatorname{sgn}((x-x_0)^n), \quad x \in (a,b),$$

where $\varphi_1(x) := \varphi(x_0^n;\psi;x)$, $x \in (a,b)$. We prove this equality on (x_0,b); the proof for (a,x_0) is analogous.

It suffices to show that

$$\psi(x) > \varphi_1(x), \quad x \in (x_0,b).$$

Assume that this inequality does not hold. Then two cases are possible:

1. $\psi(x) \ge \varphi_1(x)$ for $x \in (x_0,b)$ and $\psi(c) = \varphi_1(c)$ for a $c \in (x_0,b)$;
2. $\psi(c) < \varphi_1(c)$ for a $c \in (x_0,b)$.

1. It is easily seen that $\varphi(x_0^{n-1},c^1;\psi;x) = \varphi_1(x)$ for $x \in (a,b)$. Since ψ satisfies $\widetilde{W}_n(n-1,1;F)$ on (a,b), this gives $\psi(x) < \varphi_1(x)$ for $x \in (x_0,c)$, which contradicts 1.

2. Set

$$\varphi_2(x) := \varphi(x_0^{n-1},c^1;\psi;x), \quad x \in (a,b).$$

Since ψ satisfies $\widetilde{W}_n(n-1,1;F)$ on (a,b), we have

$$\psi(x) < \varphi_2(x), \quad x \in (x_0,c). \tag{5}$$

It follows from the definitions of φ_1, φ_2 and from $\psi(c) < \varphi_1(c)$ that

$$\varphi_1^i(x_0) = \varphi_2^i(x_0), \quad i = 0,1,\dots,n-2, \tag{6}$$

$$\varphi_1(c) > \varphi_2(c). \tag{7}$$

We conclude from (6) and (7) that $\varphi_1(x) \ne \varphi_2(x)$ for $x \ne x_0$, because $\varphi_1,\varphi_2 \in F$ and F is an H_n-family on (a,b), whence $\varphi_1(x) > \varphi_2(x)$ for

$x \in (x_0, b)$, and finally, $\psi(x) < \varphi_2(x) < \varphi_1(x)$ for $x \in (x_0, c)$, by (5). We can rewrite the last inequalities as follows:

$$0 < \varphi_1(x) - \varphi_2(x) < \varphi_1(x) - \psi(x), \quad x \in (x_0, c).$$

Applying Lemma 1 for $f := \varphi_1 - \psi$, $g := \varphi_1 - \varphi_2$ and $k := n-1$ we get $g^{n-1}(x_0) = 0$, and consequently, $\varphi_1^{n-1}(x_0) = \varphi_2^{n-1}(x_0)$. Combining this with (6) we obtain

$$\varphi_1^i(x_0) = \varphi_2^i(x_0), \quad i = 0, 1, \ldots, n-1.$$

Since F is an H_n-family on (a, b), $\varphi_1(x) = \varphi_2(x)$ for $x \in (a, b)$, contrary to (7). This proves the lemma.

LEMMA 5. *Let F and φ be as in Lemma 4. If for every $k \in \{2, \ldots, n\}$ and for any positive integers $i_1, \ldots, i_k$ such that $i_1 + \ldots + i_k = n$, φ satisfies $\widetilde{W}_n(i_1, \ldots, i_k; F)$ on (a, b), then for every $k \in \{1, \ldots, n\}$ and for any positive integers $i_1, \ldots, i_k$ such that $i_1 + \ldots + i_k = n$, φ satisfies $W_n(i_1, \ldots, i_k; F)$ on (a, b).*

Proof. It follows from Lemma 4 that φ satisfies $W_n(n; F)$ on (a, b). Since φ satisfies $\widetilde{W}_n(1^{(n)}; F)$ on (a, b), it also satisfies $W_n(1^{(n)}; F)$ on (a, b). This means that we need only consider $k \in \{2, \ldots, n-1\}$.

Fix $k \in \{2, \ldots, n-1\}$ and positive integers $i_1, \ldots, i_k$ such that $i_1 + \ldots + i_k = n$. We now prove that φ satisfies $W_n(i_1, \ldots, i_k; F)$ on (a, b), i.e., for any $a < x_1 < \ldots < x_k < b$,

$$\operatorname{sgn}(\psi(x) - \varphi(x_1^{i_1}, \ldots, x_k^{i_k}; \psi; x)) = \operatorname{sgn}\Big(\prod_{j=1}^{k} (x - x_j)^{i_j}\Big) \tag{8}$$

for $x \in (a, b)$. Since ψ satisfies $\widetilde{W}_n(i_1, \ldots, i_k; F)$ on (a, b), (8) holds on (x_1, x_k). We now show (8) holds on (x_k, b); the proof for (a, x_1) is analogous. It suffices to prove that

$$\psi(x) > \varphi(x_1^{i_1}, \ldots, x_k^{i_k}; \psi; x), \quad x \in (x_k, b). \tag{9}$$

Set $\varphi_1(x) := \varphi(x_1^{i_1}, \ldots, x_k^{i_k}; \psi; x)$, $x \in (a, b)$.

Assume that (9) does not hold and consider, as in the proof of Lemma 4, two cases:

1. $\psi(x) \geq \varphi_1(x)$ for $x \in (x_k, b)$ and $\psi(c) = \varphi_1(c)$ for a $c \in (x_k, b)$;
2. $\psi(c) < \varphi_1(c)$ for a $c \in (x_k, b)$.

1. Let $i_k = 1$ and $\varphi_2(x) := \varphi(x_1^{i_1}, \ldots, x_{k-1}^{i_{k-1}}, c^1; \psi; x)$ for $x \in (a, b)$. Hence

$$\psi(x) < \varphi_2(x), \quad x \in (x_{k-1}, c), \tag{10}$$

because ψ satisfies $\widetilde{W}_n(i_1, \ldots, i_{k-1}, 1; F)$ on (a, b). By the definitions of φ_1, φ_2 and from the equality $\psi(c) = \varphi_1(c)$ we get

$$\begin{aligned}&\varphi_1^{j_l}(x_l)=\varphi_2^{j_l}(x_l), \quad l=1,\ldots,k-1,\ j_l=0,\ldots,i_l-1,\\&\varphi_1(c)=\varphi_2(c).\end{aligned}$$

Therefore $\varphi_1(x)=\varphi_2(x)$ for $x\in(a,b)$. Combining this with (10) we obtain $\psi(x)<\varphi_1(x)$ for $x\in(x_k,c)$, contrary to 1. If $i_k>1$, then considering the function $\varphi(x_1^{i_1},\ldots,x_{k-1}^{i_{k-1}},x_k^{i_k-1},c^1;\psi;x)$ we get the same contradiction as for $i_k=1$.

2. Let $i_k=1$. Then there is a $p\in\{1,\ldots,k-1\}$ such that $i_p>1$. Set

$$\varphi_3(x):=\varphi(x_1^{i_1},\ldots,x_{p-1}^{i_{p-1}},x_p^{i_p-1},x_{p+1}^{i_{p+1}},\ldots,x_k^{i_k},c^1;\psi;x), \quad x\in(a,b).$$

Since ψ satisfies $\widetilde{W}_n(i_1,\ldots,i_k;F)$ and $\widetilde{W}_n(i_1,\ldots,i_{p-1},i_p-1,i_{p+1},\ldots$ $\ldots,i_k,1;F)$ on (a,b), it follows that

$$\psi(x)<\varphi_1(x), \quad x\in(x_{k-1},x_k), \tag{11}$$

$$\psi(x)<\varphi_3(x), \quad x\in(x_k,c). \tag{12}$$

From the definitions of φ_1 and φ_3 and from the inequality $\psi(c)<\varphi_1(c)$, it may be concluded that

$$\begin{aligned}&\varphi_1^{j_l}(x_l)=\varphi_3^{j_l}(x_l), \quad l\in\{1,\ldots,k\}\setminus\{p\},\ j_l=0,\ldots,i_l-1,\\&\varphi_1^{j_p}(x_p)=\varphi_3^{j_p}(x_p), \quad j_p=0,\ldots,i_p-2,\end{aligned} \tag{13}$$

$$\varphi_1(c)>\varphi_3(c). \tag{14}$$

We deduce from (13) and (14) that $\varphi_1(x)\neq\varphi_3(x)$ for $x\in(a,b)\setminus\{x_1,\ldots,x_k\}$; hence and from (14) we have $\varphi_1(x)>\varphi_3(x)$ for $x\in(x_k,b)$. Combining this with (12) we obtain

$$\psi(x)<\varphi_3(x)<\varphi_1(x), \quad x\in(x_k,c). \tag{15}$$

It follows from (11), (15) and from the equality $\psi(x_k)=\varphi_1(x_k)$ that $\psi^1(x_k)=\varphi_1^1(x_k)$. We can rewrite (15) as

$$0<\varphi_1(x)-\varphi_3(x)<\varphi_1(x)-\psi(x), \quad x\in(x_k,c).$$

Applying Lemma 1 for $f:=\varphi_1-\psi,\ g:=\varphi_1-\varphi_3, k:=1$ and for $x_0:=x_k$ we get $g^1(x_k)=0$. Hence $\varphi_1^1(x_k)=\varphi_3^1(x_k)$. From this and from (13), we conclude that $\varphi_1(x)=\varphi_3(x)$ for $x\in(a,b)$, which is impossible by (14).

Let $i_k>1$. Put

$$\varphi_4(x):=\varphi(x_1^{i_1},\ldots,x_{k-1}^{i_{k-1}},x_k^{i_k-1},c^1;\psi;x), \quad x\in(a,b).$$

Analysis similar to that in the case where $i_k=1$ shows that

$$\psi(x)<\varphi_4(x)<\varphi_1(x), \quad x\in(x_k,c)$$

and $\varphi_1^{i_k-1}(x_k)=\varphi_4^{i_k-1}(x_k)$, by Lemma 1. Hence, we have $\varphi_1(x)=\varphi_2(x)$ for $x\in(a,b)$, which gives the same contradiction as for $i_k=1$. This ends the proof.

LEMMA 6. *Let F and ψ be as in Lemma 4. Assume that for every $j \in \{1, \ldots, n-1\}$ and for any points $a < x_1 < \ldots < x_k < b$ $(k := n-j+1)$,*

$$\operatorname{sgn}(\psi(x) - \varphi(x_1^j, x_2^1, \ldots, x_k^1; \psi; x)) = \operatorname{sgn}((x - x_1)^j (x - x_2) \ldots (x - x_k)) \tag{16}$$

for $x \in (a, x_k)$. Then for every $i \in \{1, \ldots, n-1\}$, ψ satisfies $W_n(i, 1^{(n-i)}; F)$ on (a, b).

The proof is similar to the proof of Lemma 5 for $i_k = 1$, so we omit it.

LEMMA 7. *Let F and ψ be as in Lemma 4. If ψ is strictly F-convex on (a, b), then for every $i \in \{1, \ldots, n\}$, ψ satisfies $W_n(i, 1^{(n-i)}; F)$ on (a, b).*

Proof. The proof is by induction on n. It follows from Lemma 4 (cf. [2]) that the lemma holds for $n = 2$. Assume that it holds for $n - 1$ $(n \geq 3)$. Let F be an H_n-family on (a, b), let ψ be $n - 1$ times differentiable on (a, b), and suppose that ψ is strictly F-convex on (a, b).

First we prove that ψ satisfies $W_n(i, 1^{(n-i)}; F)$ on (a, b) for $i = 1, \ldots, n - 1$. To do this, it suffices to show that the assumptions of Lemma 6 hold. If $j = 1$, then $k = n$ and for every $a < x_1 < \ldots < x_n < b$, (16) holds on (a, b), because ψ is strictly F-convex on (a, b). Fix $j \in \{2, \ldots, n-1\}$ and $a < x_1 < \ldots < x_k < b$ $(k = n - j + 1)$. We will prove (16) on (a, x_k). Set

$$G_1 := \{\varphi|_{(a, x_k)} : \varphi \in F,\ \varphi(x_k) = \psi(x_k)\}, \qquad \psi_1 := \psi|_{(a, x_k)}.$$

It is easy to check that G_1 is an H_{n-1}-family on (a, x_k) and ψ_1 is strictly G_1-concave on (a, x_k). Hence, from the inductive assumption and from Lemma 2, we conclude that ψ_1 satisfies $K_{n-1}(j, 1^{(n-j-1)}; G_1)$ on (a, x_k). This implies

$$\operatorname{sgn}(\psi_1(x) - \overline{\varphi}(x)) = -\operatorname{sgn}((x - x_1)^j (x - x_2) \ldots (x - x_{k-1})), \qquad x \in (a, x_k), \tag{17}$$

where $\overline{\varphi} \in G_1$ is determined by the conditions

$$\begin{aligned} \overline{\varphi}^{\,l}(x_1) &= \psi_1^l(x_1), \qquad l = 0, \ldots, j - 1, \\ \overline{\varphi}(x_p) &= \psi_1(x_p), \qquad p = 2, \ldots, k - 1. \end{aligned}$$

It follows from the definitions of G_1, ψ_1, and $\overline{\varphi}$ that

$$\begin{aligned} \overline{\varphi}(x) &= \varphi(x_1^j, x_2^1, \ldots, x_k^1; \psi; x), \qquad x \in (a, x_k), \\ \psi_1(x) &= \psi(x), \qquad x \in (a, x_k). \end{aligned}$$

Therefore, we can rewrite (17) as

$$\operatorname{sgn}(\psi(x) - \varphi(x_1^j, x_2^1, \ldots, x_k^1; \psi; x)) = -\operatorname{sgn}((x - x_1)^j (x - x_2) \ldots (x - x_{k-1})), \qquad x \in (a, x_k).$$

Combining this with $x - x_k < 0$ for $x \in (a, x_k)$ we get (16) on (a, x_k).

The proof will be completed as soon as we can show that ψ satisfies $W_n(n;F)$ on (a,b). To do this, it is sufficient, by Lemma 4 (we have already proved that ψ satisfies $W_n(n-1,1;F)$ on (a,b)), to prove that ψ satisfies $\widetilde{W}_n(1,n-1;F)$ on (a,b). Let $a<x_1<x_2<b$. We have to show that

$$\begin{aligned}(18)\qquad &\operatorname{sgn}(\psi(x)-\varphi(x_1^1,x_2^{n-1};\psi;x))\\ &\qquad\qquad =\operatorname{sgn}((x-x_1)(x-x_2)^{n-1}),\qquad x\in(x_1,x_2).\end{aligned}$$

Define

$$G_2:=\{\varphi|_{(x_1,b)}:\varphi\in F,\ \varphi(x_1)=\psi(x_1)\},\qquad \psi_2:=\psi|_{(x_1,b)}.$$

Obviously, G_2 is an H_{n-1}-family on (x_1,b) and ψ_2 is strictly G_2-convex on (x_1,b). Hence, from the inductive assumption we deduce that ψ_2 satisfies $W_{n-1}(n-1;G_2)$ on (x_1,b). An analysis similar to that used in the first part of the proof shows that

$$\operatorname{sgn}(\psi(x)-\varphi(x_1^1,x_2^{n-1};\psi;x))=\operatorname{sgn}((x-x_2)^{n-1}),\qquad x\in(x_1,b).$$

Combining this with $x-x_1>0$ for $x\in(x_1,b)$ we get (18), which completes the proof of the lemma.

3. Main results. In this section we give necessary and sufficient conditions for strict convexity with the use of the conditions $W_n(i_1,\ldots,i_k;F)$. First we prove that if ψ is strictly F-convex, then ψ satisfies every condition $W_n(i_1,\ldots,i_k;F)$.

THEOREM 1. *Let F be an H_n-family on (a,b) and let ψ be $n-1$ times differentiable on (a,b). If ψ is strictly F-convex on (a,b), then for any positive integers $i_1,\ldots,i_k$ such that $i_1+\ldots+i_k=n$, ψ satisfies $W_n(i_1,\ldots,i_k;F)$ on (a,b).*

Proof. The proof is by induction on n. It follows from Lemma 4 (cf. [2]) that the statement holds for $n=2$. Assume it holds for $2,\ldots,n-1$ $(n\ge 3)$.

Let F and ψ be as in the statement of the theorem. By Lemma 5, it suffices to show that for every $k\in\{2,\ldots,n\}$ and for any positive integers $i_1,\ldots,i_k$ such that $i_1+\ldots+i_k=n$, ψ satisfies $\widetilde{W}_n(i_1,\ldots,i_k;F)$ on (a,b). Since $k\ge 2$, $i_1\le n-1$. If $i_1=n-1$, then $k=2$ and $i_2=1$. By Lemma 7, ψ satisfies $W_n(n-1,1;F)$ on (a,b). Therefore we need only consider the case $i_1\le n-2$.

Fix $k\in\{2,\ldots,n\}$, positive integers $i_1,\ldots,i_k$ such that $i_1+\ldots+i_k=n$ and $i_1\le n-2$, and points $a<x_1<\ldots<x_k<b$. If we prove that

$$(19)\qquad \operatorname{sgn}(\psi(x)-\varphi(x_1^{i_1},\ldots,x_k^{i_k};\psi;x))=\operatorname{sgn}\Big(\prod_{j=1}^{k}(x-x_j)^{i_j}\Big),\qquad x\in(x_1,x_k),$$

the assertion follows. Put

$$G_1 := \{\varphi|_{(x_1,b)} : \varphi \in F,\ \varphi^j(x_1) = \psi^j(x_1),\ j = 0, \ldots, i_1 - 1\},$$
$$\psi_1 := \psi|_{(x_1,b)}.$$

It is easily seen that G_1 is an H_{n-i_1}-family on (x_1, b). By Lemma 7, ψ satisfies $W_n(i_1, 1^{(n-i_1)}; F)$ on (a, b). Consequently, ψ_1 is strictly G_1-convex on (x_1, b). Hence and from the inductive assumption we see that ψ_1 satisfies $W_{n-i_1}(i_2, \ldots, i_k; G_1)$ on (x_1, b). This implies that

$$\operatorname{sgn}(\psi_1(x) - \varphi(x)) = \operatorname{sgn}\Big(\prod_{j=2}^{k}(x - x_j)^{i_j}\Big), \qquad x \in (x_1, b), \tag{20}$$

where $\varphi \in G_1$ is determined by the conditions

$$\varphi^{j_l}(x_l) = \psi^{j_l}(x_l), \qquad l = 2, \ldots, k,\ j_l = 0, \ldots, i_l - 1.$$

It follows from the definitions of G_1, φ and ψ_1 that

$$\begin{aligned} \varphi(x) &= \varphi(x_1^{i_1}, \ldots, x_k^{i_k}; \psi; x), && x \in (x_1, b), \\ \psi_1(x) &= \psi(x), && x \in (x_1, b). \end{aligned}$$

Therefore, we can rewrite (20) as

$$\operatorname{sgn}(\psi(x) - \varphi(x_1^{i_1}, \ldots, x_k^{i_k}; \psi; x)) = \operatorname{sgn}\Big(\prod_{j=2}^{k}(x - x_j)^{i_j}\Big), \qquad x \in (x_1, b).$$

Since $(x - x_1)^{i_1} > 0$ for $x \in (x_1, b)$, we get (19), which completes the proof.

Now we will be concerned with sufficient conditions for strict convexity.

THEOREM 2. *Let F and ψ be as in Theorem* 1. *If ψ satisfies $W_n(i_1, \ldots \ldots, i_k; F)$ on (a, b) for some $i_1, \ldots, i_k \in \{1, 2, 3\}$ such that $i_1 + \ldots + i_k = n$, then ψ is strictly F-convex on (a, b).*

To prove this theorem we need the following

LEMMA 8. *Let G be an H_r-family on (c, d) $(r \geq 4)$ and let ψ be $r - 1$ times differentiable on (c, d). If ψ satisfies $W_r(i_1, \ldots, i_k; G)$ on (c, d), where $i_1, \ldots, i_k \in \{1, 2, 3\}$, $i_1 + \ldots + i_k = r$, $i_k \neq 1$ and Theorem* 2 *holds for $n = n_1 := i_2 + \ldots + i_k$, then ψ satisfies $W_r(i_1, \ldots, i_{k-1}, i_k - 1, 1; G)$ on (c, d).*

Proof. Fix $c < x_1 < \ldots < x_k < x_{k+1} < d$ and set

$$\varphi_1(x) := \varphi(x_1^{i_1}, \ldots, x_{k-1}^{i_{k-1}}, x_k^{i_k - 1}, x_{k+1}^1; \psi; x), \qquad x \in (c, d).$$

If we prove that

$$\operatorname{sgn}(\psi(x) - \varphi_1(x)) = \operatorname{sgn}\Big((x - x_k)^{i_k - 1}(x - x_{k+1})\prod_{l=1}^{k-1}(x - x_l)^{i_l}\Big) \tag{21}$$

for $x \in (c,d)$, the assertion follows. Put

$$F := \{\varphi|_{(x_1,d)} : \varphi \in G,\ \varphi^j(x_1) = \psi^j(x_1),\ j = 0,\dots,i_1-1\},$$
$$\psi_1 := \psi|_{(x_1,d)}.$$

Obviously, F is an H_{n_1}-family on (x_1,d). The function ψ_1 satisfies $W_{n_1}(i_2,\dots,i_k;F)$ on (x_1,d), because ψ satisfies $W_r(i_1,\dots,i_k;G)$ on (c,d). Since Theorem 2 was assumed to hold for $n = n_1$, ψ_1 is strictly F-convex on (x_1,d). By Theorem 1, ψ satisfies $W_{n_1}(i_2,\dots,i_k-1,1;F)$ on (x_1,d). It follows that

$$\begin{aligned}(22)\qquad &\operatorname{sgn}(\psi_1(x)-\overline{\varphi}(x))\\ &= \operatorname{sgn}\Big((x-x_k)^{i_k-1}(x-x_{k+1})\prod_{l=2}^{k-1}(x-x_l)^{i_l}\Big), \qquad x \in (x_1,d),\end{aligned}$$

where $\overline{\varphi} \in F$ is determined by the conditions

$$\begin{aligned}&\overline{\varphi}^{j_l}(x_l) = \psi_1^{j_l}(x_l), \qquad l = 2,\dots,k-1,\ j_l = 0,\dots,i_l-1,\\ &\overline{\varphi}^{j_k}(x_k) = \psi_1^{j_k}(x_k), \qquad j_k = 0,\dots,i_k-2,\\ &\overline{\varphi}(x_{k+1}) = \psi_1(x_{k+1}).\end{aligned}$$

By the definition of $\overline{\varphi}$, ψ_1 and F we have

$$\overline{\varphi}(x) = \varphi_1(x), \quad \psi_1(x) = \psi(x), \qquad x \in (x_1,d).$$

Combining these with (22) we get (21) on (x_1,d), because $(x-x_1)^{i_1} > 0$ for $x \in (x_1,d)$. We only have to show that (21) holds on (c,x_1). To do this, consider

$$\varphi_2(x) := \varphi(x_1^{i_1},\dots,x_{k-1}^{i_{k-1}},x_k^{i_k};\psi;x), \qquad x \in (c,d).$$

Since ψ satisfies $W_r(i_1,\dots,i_k;G)$ on (c,d),

$$(23)\qquad \operatorname{sgn}(\psi(x)-\varphi_2(x)) = \operatorname{sgn}\Big(\prod_{l=1}^{k}(x-x_l)^{i_l}\Big), \qquad x \in (c,d).$$

Hence

$$(24)\qquad (-1)^r\varphi_2(x) < (-1)^r\psi(x), \qquad x \in (c,x_1).$$

By the definition of φ_1 we have $\varphi_1(x_{k+1}) = \psi(x_{k+1})$, and $\psi(x_{k+1}) > \varphi_2(x_{k+1})$ from (23). Therefore

$$(25)\qquad \varphi_1(x_{k+1}) > \varphi_2(x_{k+1}).$$

From the definitions of φ_1 and φ_2 we get

$$\begin{aligned}&\varphi_1^{j_l}(x_l) = \varphi_2^{j_l}(x_l), \quad l = 1,\dots,k-1,\ j_l = 0,\dots,i_l-1,\\ &\varphi_1^{j_k}(x_k) = \varphi_2^{j_k}(x_k), \quad j_k = 0,\dots,i_k-2,\end{aligned}$$

and $i_1 + \ldots + i_{k-1} + (i_k - 1) = r - 1$. Hence, from the definition of an H_r-family ($\varphi_1, \varphi_2 \in G$) and from (25) we obtain

$$(-1)^{r-1}\varphi_1(x) > (-1)^{r-1}\varphi_2(x), \qquad x \in (c, x_1),$$

which gives

$$(-1)^{r}\varphi_1(x) < (-1)^{r}\varphi_2(x), \qquad x \in (c, x_1).$$

Combining this with (24) we see that

$$(-1)^{r}\varphi_1(x) < (-1)^{r}\psi(x), \qquad x \in (c, x_1),$$

which implies (21) on (c, x_1) and the proof is complete.

Proof of Theorem 2. For $n = 2$ and $n = 3$ the theorem is true. Assume that it holds for $2, 3, \ldots, n-1$ ($n \geq 4$).

Let F and ψ be as in the statement of the theorem. By Lemma 8, it suffices to consider the case $i_k = 1$.

If we prove that ψ satisfies $\widetilde{W}_n(1^{(n)}; F)$ on (a, b), the assertion follows. Fix $a < x_1 < \ldots < x_n < b$ and let $\varphi_1(x) := \varphi(x_1^1, \ldots, x_n^1; \psi; x)$ for $x \in (a, b)$. We have to show that

$$\operatorname{sgn}(\psi(x) - \varphi_1(x)) = \operatorname{sgn}\Big(\prod_{l=1}^{n}(x - x_l)\Big), \qquad x \in (x_1, x_n). \tag{26}$$

Set

$$G := \{\varphi|_{(a,x_n)} : \varphi \in F,\ \varphi(x_n) = \psi(x_n)\}, \qquad \psi_1 := \psi|_{(a,x_n)}.$$

Obviously, G is an H_{n-1}-family on (a, x_n). The function ψ_1 satisfies $K_{n-1}(i_1, \ldots, i_{k-1}; G)$ on (a, x_n), because ψ satisfies $W_n(i_1, \ldots, i_k; F)$ on (a, b) and $i_k = 1$. Hence, from the inductive assumption and from Lemma 3 we conclude that ψ_1 is strictly G-concave on (a, x_n). This implies that

$$\operatorname{sgn}(\psi(x) - \varphi_1(x)) = -\operatorname{sgn}\Big(\prod_{l=1}^{n-1}(x - x_l)\Big), \qquad x \in (a, x_n). \tag{27}$$

Since $x - x_n < 0$ for $x \in (a, x_n)$, (27) gives (26) and the proof is complete.

One may ask whether Theorem 2 is true if some $i_j > 3$. We have not been able to settle this question.

Theorems 1 and 2 may be summarized as follows:

THEOREM 3. *Let F and ψ be as in Theorem 1. If ψ satisfies $W_n(i_1, \ldots, i_k; F)$ on (a, b) for some $i_1, \ldots, i_k \in \{1, 2, 3\}$ such that $i_1 + \ldots + i_k = n$, then ψ satisfies $W_n(i_1, \ldots, i_k; F)$ on (a, b) for any positive integers $i_1, \ldots, i_k$ such that $i_1 + \ldots + i_k = n$.*

Similar results can be obtained for strict concavity.

Using an analogous reasoning one can get similar results for convexity and concavity.

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