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Starlikeness of functions satisfying a differential inequality

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Abstract. In a recent paper Fournier and Ruscheweyh established a theorem related to a certain functional. We extend their result differently, and then use it to obtain a precise upper bound on α so that for f analytic in $|z| < 1$, $f(0) = f'(0) - 1 = 0$ and satisfying $\operatorname{Re}\{zf''(z)\} > -\lambda$, the function f is starlike.

1. Introduction and statement of results. Let U be the unit disk $|z| < 1$, and let $\mathcal{H}$ be the space of analytic functions in U with the topology of local uniform convergence. The subclasses A and A_0 of $\mathcal{H}$ consist of functions $f \in \mathcal{H}$ such that $f(0) = f'(0) - 1 = 0$ and $f(0) = 1$ respectively. By S, C, St and K we denote, respectively, the well known subsets of A of univalent, close-to-convex, starlike (with respect to origin) and convex functions. Further, for $\beta < 1$, we introduce

$$\mathcal{P}_\beta = \{f \in A_0 : \operatorname{Re} f(z) > \beta,\ z \in U\}$$

and

$$P_\beta = \{f \in A : \exists \alpha \in \mathbb{R} \text{ such that } \operatorname{Re}[e^{i\alpha}(f'(z) - \beta)] > 0, z \in U\}.$$

If f and g are in $\mathcal{H}$ and have the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} \beta_k z^k,$$

the *convolution* or *Hadamard product* of f and g is defined by

$$h(z) = (f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

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For $V \subset A_0$ the *dual* V^* of V is the set of functions $g \in A_0$ such that $(f * g)(z) \neq 0$ for every $f \in V$, and $V^{**} = (V^*)^*$.

We define functions h_T in A by

$$h_T(z) = \frac{1}{1+iT}\left[iT\frac{z}{1-z} + \frac{z}{(1-z)^2}\right], \quad T \in \mathbb{R},$$

and the subclass V_β of A_0 by

$$V_\beta = \left\{(1-\beta)\frac{1-xz}{1-yz} + \beta : |x| \le 1,\ |y| \le 1,\ \beta < 1\right\}.$$

We refer to [2, 3] for results in duality theory.

For a suitable $\Lambda : [0,1] \to \mathbb{R}$ define

$$L_\Lambda(f) = \inf_{z \in U} \int_0^1 \Lambda(t)\left[\operatorname{Re}\frac{f(tz)}{tz} - \frac{1}{(1+t)^2}\right] dt, \quad f \in C,$$

and

$$L_\Lambda(C) = \inf_{f \in C} L_\Lambda(f).$$

In a recent paper [1] Fournier and Ruscheweyh have established the following

THEOREM A. *Let Λ be integrable on $[0,1]$ and positive on $(0,1)$. If $\Lambda(t)/(1-t^2)$ is decreasing on $(0,1)$ then $L_\Lambda(C) = 0$.*

The functions

$$\Lambda_c(t) = \begin{cases} (1-t^c)/c, & -1 < c \le 2,\ c \neq 0, \\ \log(1/t), & c = 0, \end{cases}$$

satisfy the conditions of Theorem A.

It is clear that Theorem A can be extended to the case of $t\Lambda(t)$ integrable on $[0,1]$, positive on $(0,1)$, and $t\Lambda(t)/(1-t^2)$ decreasing on $(0,1)$. Indeed,

$$\int_0^1 \Lambda(t)\operatorname{Re}\left\{\frac{h_T(tz)}{tz} - \frac{1}{(1+t)^2}\right\} dt$$
$$= \int_0^1 t\Lambda(t)\operatorname{Re}\left\{\frac{1}{1+iT}\left[\frac{iTz}{1-tz} + \frac{z(2-tz)}{(1-tz)^2}\right] + \frac{2+t}{(1+t)^2}\right\} dt,$$

which shows that integrability of $t\Lambda(t)$ is enough for the existence of the integral. Further, if $t\Lambda(t)/(1-t^2)$ is decreasing, so is $\Lambda(t)/(1-t^2)$ and hence the treatment in [1] gives the result. Thus the functions

$$\Lambda_c(t) = (1-t^c)/c, \quad -2 < c \le -1,$$

satisfy the above conditions.

In the present paper we extend Theorem A in the following form.

THEOREM 1. *For Λ not integrable on $[0,1]$, let $t\Lambda(t)$ be integrable on $[0,1]$, positive on $(0,1)$, and suppose*

$$\Lambda(t)/(1-t^2) \text{ is decreasing on } (0,1).$$

Then $L_\Lambda(C)=0$.

We use the theorem to establish the following:

THEOREM 2. *Suppose $\alpha:[0,1]\to\mathbb{R}$ is non-negative with $\int_0^1 \alpha(t)\,dt=1$,*

$$\Lambda(t)=\int_t^1 \frac{\alpha(t)}{t^2}\,dt$$

satisfies the conditions of Theorem 1 and for $\lambda>0$, define

$$V_\alpha(f)=z\int_0^1 \left(1+\frac{\lambda z}{1-tz}\right)\alpha(t)\,dt * f(z), \qquad f\in A. \tag{1}$$

Then for λ given by

$$2\lambda\int_0^1 \frac{\alpha(t)}{1+t}\,dt=1 \tag{2}$$

we have $V_\alpha(P_0)\subset S$, and

$$V_\alpha(P_0)\subset St \Leftrightarrow L_\Lambda(C)=0.$$

For any larger value of λ there exists an $f\in P_0$ with $V_\alpha(f)$ not even locally univalent.

As a special case of the above theorem we obtain a result which is interesting enough to be stated as a theorem.

THEOREM 3. *If $\lambda>0$ and $f\in A$ satisfies the differential inequality*

$$\operatorname{Re} zf''(z) > -\lambda, \tag{3}$$

then $f\in St$ if

$$0<\lambda\le 1/\log 4. \tag{4}$$

For any larger value of λ, a function $f\in A$ satisfying (3) need not even be locally univalent.

THEOREM 4. *Let $\alpha:[0,1]\to\mathbb{R}$ be non-negative with $\int_0^1\alpha(t)\,dt=1$ and suppose $\Lambda(t)=\alpha(t)/t$ satisfies the conditions of Theorem 1. If $V_\alpha(f)$ is defined by (1), then*

$$V_\alpha(P_0)\subset K \Leftrightarrow L_\Lambda(C)=0$$

and λ is given by

$$2\lambda\int_0^1 \frac{2+t}{(1+t)^2}\alpha(t)\,dt=1.$$

2. Proof of Theorem 1. For a fixed $f \in C$ and $z \in U$ let

$$tg(t) = \operatorname{Re} \frac{f(tz)}{tz} - \frac{1}{(1+t)^2}.$$

Then g is analytic in t. Let

$$\Lambda_n(t) = \begin{cases} \Lambda(t), & 1/n \le t \le 1, \\ \dfrac{(1-t^2)\Lambda(1/n)}{1-1/n^2}, & 0 \le t \le 1/n. \end{cases}$$

From Theorem A we get

$$0 \le \frac{n^2}{n^2-1}\Lambda\left(\frac{1}{n}\right) \int_0^{1/n} (1-t^2)tg(t)\,dt + \int_{1/n}^1 t\Lambda(t)g(t)\,dt = H_n + G_n.$$

Now

$$|H_n| \le \frac{\Lambda(1/n)}{2(n^2-1)} M_1 \to 0 \quad \text{as } n \to \infty.$$

Let $\chi_n(t)$ be the characteristic function of $[1/n, 1]$. For each n,

$$|t\Lambda(t)g(t)\chi_n(t)| \le M_2 t\Lambda(t).$$

Since $t\Lambda(t)$ is integrable, it follows that

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty} \int_0^1 t\Lambda(t)g(t)\chi_n(t)\,dt = \int_0^1 t\Lambda(t)g(t)\,dt.$$

Hence $L_\Lambda(f) \ge 0$ for $z \in U$. This completes the proof.

We are thankful to Prof. S. Ruscheweyh for his help with the proof of Theorem 1.

3. Proof of Theorems 2 and 3. For $f \in P_0$ let $F(z) = V_\alpha(f)$. We then have

$$F'(z) = \int_0^1 \left(1 + \frac{\lambda z}{1-zt}\right)\alpha(t)\,dt * f'(z), \qquad f \in P_0.$$

Since $V_0^* = \mathcal{P}_{1/2}$ and $V_0^{**} = \{f' : f \in P_0\}$, $F'(z) \ne 0$ if and only if

$$\frac{1}{2} < \operatorname{Re} \int_0^1 \left(1 + \frac{\lambda z}{1-zt}\right)\alpha(t)\,dt. \tag{5}$$

This gives

$$\lambda \int_0^1 \frac{\alpha(t)}{1+t}\,dt \le \frac{1}{2}.$$

Further, because $\operatorname{Re} e^{i\alpha} f'(z) > 0$, (5) also ensures that $\operatorname{Re} e^{i\alpha} F'(z) > 0$ and hence F is univalent.

For starlikeness we use the easily verifiable property that $F \in A$ is in St if and only if

$$\frac{1}{z}(F * h_T)(z) \neq 0, \qquad T \in \mathbb{R},\ z \in U. \tag{6}$$

This gives

$$\begin{aligned}
0 &\neq \int_0^1 \left(1 + \frac{\lambda z}{1 - tz}\right)\alpha(t)dt * \frac{h_T(z)}{z} * \frac{f(z)}{z} \\
&= \int_0^1 \left[1 + \frac{\lambda}{t}\left\{\frac{1}{z}\int_0^z \left(\frac{h(tw)}{tw} - 1\right) dw\right\}\right]\alpha(t)\, dt * f'(z), \qquad f \in P_0.
\end{aligned}$$

This implies that $F \in St$ if and only if

$$\frac{1}{2} < \operatorname{Re} \int_0^1 \left[1 + \frac{\lambda}{t}\left\{\frac{1}{z}\int_0^z \left(\frac{h(tw)}{tw} - 1\right) dw\right\}\right]\alpha(t)\, dt.$$

On substituting the value of λ from (2) in the above inequality, we obtain

$$0 < \operatorname{Re} \int_0^1 \frac{\alpha(t)}{t^2}\left\{\frac{1}{z}\int_0^z \left(\frac{h(tw)}{w} - \frac{t}{1+t}\right) dw\right\} dt.$$

This is similar to the last equation in [1]. Hence we need

$$\Lambda(t) = \int_t^1 \frac{\alpha(t)}{t^2}\, dt$$

in order to use Theorem A. This completes the proof.

For the proof of Theorem 3 we take $\alpha(t) \equiv 1$. Then $\Lambda(t) = 1/t - 1$ satisfies the conditions of Theorem 1 and F satisfies (3). For $\alpha(t) \equiv 1$ the value of λ obtained from (2) gives (4).

Notice that in (3), $\lambda = 0$ only if $f(z) \equiv z$. Thus functions of the form

$$\varrho z + (1 - \varrho) f(z), \qquad \varrho < 1,$$

where f satisfies (3), are in St for $(1 - \varrho)\lambda \le 1/\log 4$.

Further, if $f \in A$ satisfies (3), then for a non-negative α satisfying $\int_0^1 \alpha(t)\, dt = 1$, the functions

$$\phi(z) = \int_0^1 \frac{\alpha(t)}{t} f(tz)\, dt$$

also satisfy (3) and hence are starlike for the same value of λ.

4. Proof of Theorem 4. We need to prove that $zF'(z) \in St$, $F(z) = V_\alpha(f)$. Hence (6) gives

$$\begin{aligned} 0 &\neq F'(z) * \frac{h_T(z)}{z} \\ &= \int_0^1 \left(1 + \frac{\lambda z}{1 - tz}\right)\alpha(t)\,dt * \frac{h_T(z)}{z} * f'(z) \\ &= \int_0^1 \left[1 + \frac{\lambda}{t}\left(\frac{h(tz)}{tz} - 1\right)\right]\alpha(t)\,dt * f'(z), \qquad f \in P_0. \end{aligned}$$

This holds if and only if

$$\frac{1}{2} < \operatorname{Re} \int_0^1 \left[1 + \frac{\lambda}{t}\left(\frac{h(tz)}{tz} - 1\right)\right]\alpha(t)\,dt.$$

Substitution of the value of λ in the theorem gives

$$0 < \operatorname{Re} \int_0^1 \left[\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right]\frac{\alpha(t)}{t}\,dt.$$

Hence with $\Lambda(t) = \alpha(t)/t$, Theorem 1 gives the result.

The choice of $\alpha(t) = 2(1 - t)$ gives the result of Theorem 3 with λ replaced by 2λ.

References

[1] R. Fournier and S. Ruscheweyh, *On two extremal problems related to univalent functions*, Rocky Mountain J. Math. 24 (1994), 529–538.

[2] S. Ruscheweyh, *Duality for Hadamard products with applications to extremal problems for functions regular in the unit disc*, Trans. Amer. Math. Soc. 210 (1975), 63–74.

[3] —, *Convolution in Geometric Function Theory*, Les Presses de l'Université de Montréal, Montréal, 1982.

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