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INFINITESIMAL BRUNOVSKÝ FORM FOR NONLINEAR SYSTEMS WITH APPLICATIONS TO DYNAMIC LINEARIZATION

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Abstract. We define, in an infinite-dimensional differential geometric framework, the "infinitesimal Brunovský form" which we previously introduced in another framework and link it with equivalence via diffeomorphism to a linear system, which is the same as linearizability by "endogenous dynamic feedback".

1. Introduction and problem statement. The purpose of this note is to present the constructions made in [1, 22] in the new differential geometric framework introduced in [21]. See [21], published in the same volume, and references

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therein for a presentation and discussion of this topic, almost necessary for a good understanding of the present paper.

The contribution of [1, 22] was to construct a so-called "infinitesimal Brunovský form" ("nonexact Brunovský form" in [22]) for controllable nonlinear systems and to relate it to dynamic linearization; they use the linear algebraic framework introduced in [7]. The point of view on the feedback linearization problem was the one of looking for "linearizing outputs", following the idea of [11, 12, 16]. It is therefore, following the terms of [11, 12, 16], *linearization via endogenous dynamic feedback*. In [22], we relied explicitly upon the notion of *differential flatness* [11, 12, 16], whereas [1] re-defines the notion of linearizing outputs in terms of dynamic decoupling and structure at infinity.

Here, the infinite-dimensional differential geometric framework from [21] is used, and in this context, dynamic linearization is equivalence to a linear system via diffeomorphism on the extended state space manifold; linearizing outputs are functions such that these and all their "time-derivatives" are a set of local coordinates on the generalized state-space manifold. The main interest of this approach over the algebraic ones is that it is possible to give local notions, and singularities are not ignored.

In section 3, we define the infinitesimal Brunovský form and relate it to some work on time-varying linear systems and linearized systems of nonlinear systems [9, 10]. In section 4, we relate this construction to existence of linearizing outputs, and explain why it provides a good framework for searching linearizing outputs.

2. The infinite dimensional geometric framework. The main definitions from [21] are summed up in this section.

2.1. The "*infinite dimensional manifold*" $\mathcal{M}_\infty^{m,n}$ is, for short, $\mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{N}}$. A global system of coordinates is $x_1, \ldots, x_n,\ u_1, \ldots, u_m,\ \dot{u}_1, \ldots, \dot{u}_m,\ \ddot{u}_1, \ldots$ It is endowed with the product topology: an open set may be described by some restrictions on *a finite number* of coordinates, i.e. there is a $\tilde{k}$ such that, considered as an open set of $\mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{N}} = \mathbb{R}^n \times (\mathbb{R}^m)^{\tilde{k}} \times (\mathbb{R}^m)^{\mathbb{N}}$, it can be written $\widetilde{O} \times (\mathbb{R}^m)^{\mathbb{N}}$ with $\widetilde{O}$ an open set of $\mathbb{R}^n \times (\mathbb{R}^m)^{\tilde{k}}$.

2.2. A *smooth function* on $\mathcal{M}_\infty^{m,n}$ is one which depends only on a finite number of coordinates and is smooth as a function of these coordinates. $\mathcal{C}^\infty(U)$ stands for the algebra of smooth functions defined on an open subset U of $\mathcal{M}_\infty^{m,n}$. A *smooth mapping* from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ is a mapping whose composition with any smooth function is a smooth function. A *diffeomorphism from* $U \subset \mathcal{M}_\infty^{m,n}$ *to* $V \subset \mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ is a bijective smooth mapping whose inverse is a smooth mapping.

2.3. A *vector field* is a possibly infinite linear combination $\sum v_i \frac{\partial}{\partial w_i}$ where the v_i's are smooth functions and the w_i's are some of the coordinates $x_1, \ldots, x_n$, $u_1, \ldots, u_m,\ \dot{u}_1, \ldots, \dot{u}_m, \ldots$. A *differential form of degree* 1 (or 1-form) is, with the same conventions, a *finite* linear combination $\sum v_i \mathrm{d}w_i$. $\Lambda^1(U)$ stands for the $\mathcal{C}^\infty(U)$-module of 1-forms defined on U.

2.4. All the "formulas" from finite dimensional differential calculus involving objects like Lie brackets and Lie derivatives are valid. For instance, the Lie derivative of a form $\omega = \sum v_i \mathrm{d}w_i$ along a vector field F may be computed, in coordinates, according to

$$L_F\omega = \sum L_F v_i \mathrm{d}w_i + v_i \mathrm{d}(L_F w_i).$$

Also, a diffeomorphism carries vector fields or differential forms from one manifold to another, we use the usual notation $\varphi_* F$ or $\varphi^*\omega$.

2.5. A smooth *control system*

$$\dot{x} = f(x,u) \tag{1}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ is represented by a single vector field

$$F = f(x,u)\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \dots \tag{2}$$

on $\mathcal{M}_\infty^{m,n}$. We often refer to "system F", confusing system (1) with vector field F.

2.6. The Lie derivative along F defined by (2) is simply the "time-derivative" according to (1): we often write $\dot{\varphi}$ or $\dot{\omega}$ instead of $L_F\varphi$ or $L_F\omega$ for a function φ or a 1-form ω.

2.7. A diffeomorphism from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ given by $(x,u,\dot{u},\ddot{u},\dots) \mapsto (z,v,\dot{v},\ddot{v},\dots)$ is said to be a *static diffeomorphism* if and only if z depends only on x, v depends only on x and u, $\dot{v}$ depends only on x, u and $\dot{u}$ ... A static diffeomorphism is nothing more than a nonsingular static transformation in the usual sense: if F is a system on $\mathcal{M}_\infty^{m,n}$ and $\widetilde{F}$ is a system on $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$, existence of a static diffeomorphism φ such that $\widetilde{F} = \varphi_* F$ is equivalent to $n = \tilde{n}$, $m = \tilde{m}$ and static equivalence of the control systems associated with F and $\widetilde{F}$.

2.8. Of course, $n = 0$ is not ruled out in the above definitions, coordinates on $\mathcal{M}_\infty^{m,0}$ are simply $\{u, \dot{u}, \ddot{u}, \dots\}$, and the only system is the *canonical linear system* with m inputs:

$$C = \sum_0^\infty u^{(j+1)}\frac{\partial}{\partial u^{(j)}}. \tag{3}$$

It has "no state", but one should not worry about this since $n = 0$ is obtained after "cutting all the integrators" in a canonical linear system [3] and arbitrarily renaming some states "inputs". Dynamic linearizability is conjugation via a diffeomorphism to system C:

DEFINITION 1. A system F is *locally dynamic linearizable* at a point $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ if and only if there exists a neighborhood U of $\mathcal{X}$ in $\mathcal{M}_\infty^{m,n}$, an open subset V of $\mathcal{M}_\infty^{m,0}$, and a diffeomorphism φ from U to V such that, on U, $\varphi_* F = C$.

2.9. Consider a $\mathcal{C}^\infty(U)$-module of vector fields $\mathcal{D}$ (resp. of forms $\mathcal{H}$), defined on an open set U. The annihilator of $\mathcal{D}$ is the module of the forms which vanish on all the vector fields of $\mathcal{D}$, and vice-versa:

$$\mathcal{H}^\perp = \{X, \forall \omega \in \mathcal{H}, \langle \omega, X\rangle = 0\}; \quad \mathcal{D}^\perp = \{\omega, \forall X \in \mathcal{D}, \langle \omega, X\rangle = 0\}.$$

$\mathcal{D}(\mathcal{X})$ (resp. $\mathcal{H}(\mathcal{X})$) denotes the subspace of the tangent (resp. cotangent) space to $\mathcal{M}^{m,n}_\infty$ at the point $\mathcal{X} \in U$ made of all the $X(\mathcal{X})$ for $X \in \mathcal{D}$ (resp. $\omega(\mathcal{X})$ for $\omega \in \mathcal{H}$). We call the dimension of $\mathcal{D}(\mathcal{X})$ (resp. $\mathcal{H}(\mathcal{X})$) the *pointwise rank* of $\mathcal{D}$ (resp. $\mathcal{H}$) at $\mathcal{X}$. $\mathcal{D}$ or $\mathcal{H}$ is said to be *nonsingular at* $\mathcal{X}$ if and only if its pointwise rank is finite and constant in a neighborhood of $\mathcal{X}$; it is then equal to the rank of the module over $\mathcal{C}^\infty(U)$.

3. The infinitesimal Brunovský form. Let us define the following sequence of $\mathcal{C}^\infty(\mathcal{M}^{m,n}_\infty)$-modules of vector fields:

$$\begin{aligned}
\mathcal{D}_{-j} &= \operatorname{Span}\left\{\frac{\partial}{\partial u^{(j+1)}}, \frac{\partial}{\partial u^{(j+2)}}, \dots\right\}, \qquad j \geq 0,\\
&\vdots\\
\mathcal{D}_0 &= \operatorname{Span}\left\{\frac{\partial}{\partial \dot{u}}, \frac{\partial}{\partial \ddot{u}}, \frac{\partial}{\partial u^{(3)}}, \dots\right\}\\
\mathcal{D}_1 &= \operatorname{Span}\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial \dot{u}}, \frac{\partial}{\partial \ddot{u}}, \frac{\partial}{\partial u^{(3)}}, \dots\right\}\\
&\vdots\\
\mathcal{D}_{k+1} &= \mathcal{D}_k + [F, \mathcal{D}_k]\\
&\vdots\\
\mathcal{D}_\infty &= \sum_k \mathcal{D}_k
\end{aligned} \tag{4}$$

and, since these are "infinite-dimensional", we define for each $\mathcal{D}_k$ $(k \geq 1)$ its "$\frac{\partial}{\partial x}$ part":

$$\widehat{\mathcal{D}}_k = \mathcal{D}_k \cap \operatorname{Span}\left\{\frac{\partial}{\partial x}\right\}, \qquad k \in [1, \infty] \tag{5}$$

($\operatorname{Span}\{\frac{\partial}{\partial x}\}$ stands for the $\mathcal{C}^\infty(\mathcal{M}^{m,n}_\infty)$-module generated by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$), which makes $\widehat{\mathcal{D}}_k(\mathcal{X})$ (see section 2.9) finite-dimensional for all $\mathcal{X} \in \mathcal{M}^{m,n}_\infty$), and yields

$$\mathcal{D}_k = \widehat{\mathcal{D}}_k \oplus \mathcal{D}_1, \qquad k \in [1, \infty]. \tag{6}$$

Note that (5) and (6) are both valid for $k = \infty$ and that $\widehat{\mathcal{D}}_\infty$ might as well have been defined by $\widehat{\mathcal{D}}_\infty = \sum_k \widehat{\mathcal{D}}_k$. We define also a sequence of $\mathcal{C}^\infty(\mathcal{M}^{m,n}_\infty)$-modules of forms:

$$
\begin{aligned}
\mathcal{H}_{-j} &= \mathrm{Span}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{(j)}\}, \qquad j \geq 0,\\
&\vdots\\
\mathcal{H}_0 &= \mathrm{Span}\{\mathrm{d}x, \mathrm{d}u\}\\
\mathcal{H}_1 &= \mathrm{Span}\{\mathrm{d}x\}\\
&\vdots\\
\mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k, \dot{\omega} = L_F\omega \in \mathcal{H}_k\}\\
&\vdots\\
\mathcal{H}_\infty &= \bigcap_k \mathcal{H}_k.
\end{aligned}
\tag{7}
$$

See sections 2.4 and 2.6 for a definition of $\dot{\omega}$ or $L_F\omega$. We have the following relation between the $\mathcal{D}_k$'s and the $\mathcal{H}_k$'s:

PROPOSITION 1. *All the modules $\mathcal{D}_k$ and $\mathcal{H}_k$ are invariant by static feedback, i.e. by static diffeomorphism of $\mathcal{M}^{m,n}_\infty$ (see section 2.7), and, for all k,*

$$
\mathcal{H}_\infty \subset \mathcal{H}_{k+1} \subset \mathcal{H}_k, \quad \mathcal{D}_k \subset \mathcal{D}_{k+1} \subset \mathcal{D}_\infty, \quad \mathcal{H}_k = \mathcal{D}_k^\perp, \quad \mathcal{D}_k \subset \mathcal{H}_k^\perp, \tag{8}
$$

with $\mathcal{H}_k^\perp = \mathcal{D}_k$ at points where $\widehat{\mathcal{D}}_k$ is nonsingular (see section 2.9).

Proof. From (4) and [21, proposition 1], a static diffeomorphism φ does not change $\mathcal{D}_k$ for $k \leq 1$; since the recursive definition of $\mathcal{D}_k$ for larger k only uses Lie brackets, it is then clear that the modules built according to (4) from $\varphi_* F$ are exactly $\varphi_*\mathcal{D}_k$. The two first relations in (8) are obvious from (4) and (7) and the fourth one is a consequence of the third one because $\mathcal{D}_k \subset \left(\mathcal{D}_k^\perp\right)^\perp$, with an equality at nonsingular points. Let us prove the first one by induction. It is obvious for $k \leq 1$. Let us suppose that it is true for $k \geq 1$. From the fact that if $\langle \omega, X\rangle = 0$ then $\langle L_F\omega, X\rangle = -\langle \omega, [F, X]\rangle$, we have:

$$
\begin{aligned}
\omega \in \mathcal{H}_{k+1} &\Leftrightarrow \omega \in \mathcal{H}_k \text{ and } L_\varphi\omega \in \mathcal{H}_k\\
&\Leftrightarrow \forall X \in \mathcal{D}_k, \langle \omega, X\rangle = \langle L_F\omega, X\rangle = 0\\
&\Leftrightarrow \forall X \in \mathcal{D}_k, \langle \omega, X\rangle = \langle \omega, [F, X]\rangle = 0\\
&\Leftrightarrow \omega \in \mathcal{D}_{k+1}^\perp. \ \blacksquare
\end{aligned}
$$

We shall now relate this construction to accessibility. The following Lie algebra is defined in [24], and often called the *strong accessibility Lie algebra*: this Lie algebra of vector fields on $\mathbb{R}^n$ is the Lie ideal generated by all the vector fields $f(u, .) - f(v, .)$ for all possible values of u and v in the Lie algebra generated by the vector fields $f(u, .)$ for all possible values of u. The main result on strong accessibility in [24] (see the definition there) is that it is equivalent to the strong accessibility Lie algebra having rank n. In [6], the *strong jet accessibility Lie algebra* is defined; it differs from the strong accessibility Lie algebra in that the differences $f(u, .) - f(v, .)$ are replaced by derivatives of all orders with respect to all the components of u. It is easy to see (this is actually its definition in [6]) that it is the Lie algebra generated by all the vector fields

$$\mathrm{ad}^j_{f(.,u)} g^K_u, \quad j \in \mathbb{N}, \qquad K = (k_1, \ldots, k_m) \in \mathbb{N}^m, \quad g^K_u = \frac{\partial^{k_1+\ldots+k_m} f}{\partial u_1^{k_1} \ldots \partial u_m^{k_m}}. \tag{9}$$

It a priori depends on u. In the analytic case, it does not depend on u and is equal, for all values of u, to the strong accessibility Lie algebra. Of course, in the general (smooth) case, full rank for this Lie algebra is sufficient, but not necessary, for strong accessibility. A vector field on $\mathbb{R}^n$ depending on u, like these defined in (9) and all their iterated Lie brackets, clearly defines a vector field on $\mathcal{M}^{m,n}_\infty$ (which belongs to $\mathrm{Span}\{\frac{\partial}{\partial x}\}$ and commutes with all the $\frac{\partial}{\partial u_k^{(j)}}$ for $j \geq 1$ but not a priori with the $\frac{\partial}{\partial u_k}$'s). Here, we call $\widehat{\mathcal{L}}$ the Lie algebra composed of the vector fields on $\mathcal{M}^{m,n}_\infty$ associated to these in the strong jet accessibility Lie algebra as defined by (9) (or in [6]), and we define $\mathcal{L}$ by

$$\mathcal{L} = \widehat{\mathcal{L}} \oplus \mathcal{D}_1 = \widehat{\mathcal{L}} \oplus \mathrm{Span}\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial \dot{u}}, \frac{\partial}{\partial \ddot{u}}, \frac{\partial}{\partial u^{(3)}}, \ldots\right\}. \tag{10}$$

$\mathcal{L}$ is obviously a Lie algebra because $[\frac{\partial}{\partial u_k}, \widehat{\mathcal{L}}] \subset \widehat{\mathcal{L}}$ and $[\frac{\partial}{\partial u_k^{(j)}}, \widehat{\mathcal{L}}] = \{0\}$ for $j \geq 1$. The phrase "strong jet accessibility Lie algebra" will further refer to $\mathcal{L}$ rather than to a Lie algebra of vector fields on $\mathbb{R}^n$, and $\widehat{\mathcal{L}}$ is its $\frac{\partial}{\partial x}$-component. We have:

PROPOSITION 1. *For any open subset U of $\mathcal{M}^{m,n}_\infty$,*

1. *$\mathcal{L}|_U$ (restriction to U of the strong jet accessibility Lie algebra) is the Lie algebra generated by (i.e. the involutive closure of) $\mathcal{D}_\infty|_U$ (the restriction of $\mathcal{D}_\infty$ to U).*

2. *If the $\mathcal{C}^\infty$-module $\widehat{\mathcal{D}}_\infty|_U$ is finitely generated, then it is a Lie algebra, and so is $\mathcal{D}_\infty|_U$, and hence:*

$$\mathcal{D}_\infty|_U = \mathcal{L}|_U \quad i.e. \quad \widehat{\mathcal{D}}_\infty|_U = \widehat{\mathcal{L}}|_U. \tag{11}$$

Proof. Call $G^{j,K}$ the vector field on $\mathcal{M}^{m,n}_\infty$ associated with $\mathrm{ad}^j_{f(.,u)} g^K_u$ defined in (9). A computation shows that $G^{j,K}$ is equal to $\mathrm{ad}^j_F \mathrm{ad}^{k_1}_{\partial/\partial u_1} \mathrm{ad}^{k_2}_{\partial/\partial u_2} \ldots \mathrm{ad}^{k_m}_{\partial/\partial u_m} F$ plus a linear combination of vector fields $G^{j',K'}$ with $j' < j$, and iterated Lie brackets of such vector fields. This proves by induction that all the fields $G^{j,K}$ are in the Lie algebra generated by $\mathcal{D}_\infty|_U$, which therefore contains $\mathcal{L}|_U$. The converse is clear because $\mathcal{L}$ is a Lie algebra, as noticed above, and contains $\mathcal{D}_\infty$ from (9), (10) and (4). This proves point 1. To prove point 2, let us prove that if U is such that $\widehat{\mathcal{D}}_\infty|_U$ is finitely generated, then the module of vector fields

$$\mathcal{M} = \{X \in \mathcal{D}_\infty|_U, [X, \mathcal{D}_\infty|_U] \subset \mathcal{D}_\infty|_U\}$$

is equal to $\mathcal{D}_\infty|_U$. By assumption, $\mathcal{D}_\infty|_U$ is generated by the vector fields $\frac{\partial}{\partial u_k^{(j)}}$, $1 \leq k \leq m$, $j \geq 0$, plus a finite number of vector fields of $\mathrm{Span}\{\mathrm{d}x\}$ whose expressions involve only a finite number, say J, of time-derivatives of u; $\mathcal{D}_\infty|_U$

is therefore invariant by Lie bracket by the vector fields $\frac{\partial}{\partial u_k^{(j)}}$ for $j \geq J$, which span $\mathcal{D}_{-(J-1)}$. $\mathcal{M}$ therefore contains $\mathcal{D}_{-(J-1)}$; furthermore, it is a submodule of $\mathcal{D}_\infty|_U$, invariant by F from Jacobi identity. Since it is clear that, for all k, and in particular $k = -(J-1)$, $\mathcal{D}_\infty|_U$ is the smallest module of vector fields which contains $\mathcal{D}_k$ and is invariant by Lie brackets by F, $\mathcal{M} = \mathcal{D}_\infty|_U$. ■

For further considerations, we will avoid "singular" points in the sense of the following definition where $\mathcal{H}_k + \dot{\mathcal{H}}_k$ stands for the module over smooth functions spanned by all the forms ω and $\dot{\omega}$ with $\omega \in \mathcal{H}_k$. "Nonsingular" was defined in section 2.9.

DEFINITION 2. A point $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ is called a *Brunovský-regular point* for system F if and only if one of the two following (equivalent) conditions is satisfied:

(i) All the modules $\widehat{\mathcal{D}}_k$ $(k \geq 2)$ are nonsingular at $\mathcal{X}$.
(ii) All the modules $\mathcal{H}_k + \dot{\mathcal{H}}_k$ $(k \geq 2)$ are nonsingular at $\mathcal{X}$.

These properties are true for all $k \geq 0$ if and only if they are true for $k = 2, \ldots, n+1$. We call $\boldsymbol{\rho}_k$ the locally constant rank of $\mathcal{H}_k$. Around a Brunovský-regular point, there exists an integer $\boldsymbol{k}^*$ such that, for all $k \leq k^*$, $\rho_{k+1} \leq \rho_k - 1$ and $\mathcal{H}_k = \mathcal{H}_{k+1} = \mathcal{H}_\infty$ for $k > k^*$.

Proof of i⇔ii. Suppose that all the $\widehat{\mathcal{D}}_k$'s, and thus all the $\mathcal{H}_k$'s, are nonsingular at $\mathcal{X}$. For a certain k, let $\{\eta_1, \ldots, \eta_{p+q}\}$ be a basis of $\mathcal{H}_k$ with $\{\eta_1, \ldots, \eta_p\}$ a basis of $\mathcal{H}_{k+1}$. The forms $\eta_1, \ldots, \eta_{p+q}, \dot{\eta}_{p+1}, \ldots, \dot{\eta}_{p+q}$ span $\mathcal{H}_k + \dot{\mathcal{H}}_k$. On the other hand, if a linear combination $\sum_{i=1}^{p+q} \mu_i \eta_i + \sum_{i=1}^{q} \lambda_i \dot{\eta}_{p+i}$ vanishes at $\mathcal{X}$ then, for all vector field $X \in \mathcal{D}_k$, $\langle \sum_{i=1}^{q} \lambda_i \dot{\eta}_{p+i}, X \rangle$, which is equal to $\langle \sum_{i=1}^{q} \lambda_i \eta_{p+i}, [F, X] \rangle$, vanishes at $\mathcal{X}$, hence $\langle \sum_{i=1}^{q} \lambda_i \eta_{p+i}, Y \rangle(\mathcal{X}) = 0$ for all $Y \in \mathcal{D}_{k+1}$; since $\{\eta_1(\mathcal{X}), \ldots$ $\ldots, \eta_p(\mathcal{X})\}$ is a basis of the annihilator of $\mathcal{D}_{k+1}(\mathcal{X})$ and $\{\eta_1(\mathcal{X}), \ldots, \eta_{p+q}(\mathcal{X})\}$ are independent, all the λ_i's vanish at $\mathcal{X}$; hence $\sum_{i=1}^{p+q} \mu_i \eta_i$ vanishes at $\mathcal{X}$, hence all the μ_i's also vanish at $\mathcal{X}$. Hence $\{\eta_1(\mathcal{X}), \ldots, \eta_{p+q}(\mathcal{X}), \dot{\eta}_{p+1}(\mathcal{X}), \ldots, \dot{\eta}_{p+q}(\mathcal{X})\}$ is a basis of $\mathcal{H}_k(\mathcal{X}) + \dot{\mathcal{H}}_k(\mathcal{X})$ and $\mathcal{H}_k + \dot{\mathcal{H}}_k$ is nonsingular at $\mathcal{X}$.

Conversely suppose that all the modules $\mathcal{H}_k + \dot{\mathcal{H}}_k$ are nonsingular at $\mathcal{X}$. Let $\mathcal{C}_k = \{X \in \mathcal{D}_k, [F, X] \in \mathcal{D}_k\}$ and $\widehat{\mathcal{C}}_k = \mathcal{C}_k \cap \operatorname{Span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\}$. Clearly, $\mathcal{C}_k = \widehat{\mathcal{C}}_k \oplus \mathcal{D}_0$. Arguments similar to these of the end of the proof of proposition 1 show that $(\mathcal{H}_k + \dot{\mathcal{H}}_k)^\perp = \mathcal{C}_k$ (equality between modules). All the $\widehat{\mathcal{C}}_k$'s are therefore nonsingular at $\mathcal{X}$. Let us prove by induction that all the modules $\widehat{\mathcal{D}}_k$ are nonsingular too. This is true for $k = 1$ $(\widehat{\mathcal{D}}_1 = \{0\})$. Suppose that it is true for $k \geq 1$, and let $\{\ldots, \frac{\partial}{\partial \ddot{u}}, \frac{\partial}{\partial \dot{u}}, X_1, \ldots, X_{p+q}\}$ be a basis of $\mathcal{D}_k$ with $\{\ldots, \frac{\partial}{\partial \ddot{u}}, \frac{\partial}{\partial \dot{u}}, X_1, \ldots, X_p\}$ a basis of $\mathcal{C}_k$. Then the same arguments as in the first part of this proof show that $\{X_1(\mathcal{X}), \ldots, X_{p+q}(\mathcal{X}), [F, X_{p+1}](\mathcal{X}), \ldots, [F, X_{p+q}](\mathcal{X})\}$ is a basis of $\operatorname{Span}\{\frac{\partial}{\partial u}\} \oplus \mathcal{D}_{k+1}(\mathcal{X})$ and $\widehat{\mathcal{D}}_{k+1}$ is nonsingular at $\mathcal{X}$. ■

THEOREM 2 [Infinitesimal Brunovský form]. *Around a Brunovský-regular point there exists ρ_∞ functions of x only $\chi_1, \ldots, \chi_{\rho_\infty}$, and m 1-forms $\omega_1, \ldots, \omega_m$,* *and m non-negative integers $r_1 \ldots, r_m$ such that*

$$\{\mathrm{d}\chi_1, \ldots, \mathrm{d}\chi_{\rho_\infty}\} \text{ is a basis of } \mathcal{H}_\infty = \mathcal{H}_l, \text{ for } l \geq k^* + 1 \tag{12}$$

$$\{\mathrm{d}\chi_1, \ldots, \mathrm{d}\chi_{\rho_\infty}\} \cup \{\omega_k^{(j)}, r_k \geq l, 0 \leq j \leq r_k - l\} \text{ is a basis of } \mathcal{H}_l, \quad \text{for all } l \leq k^*. \tag{13}$$

Furthermore all the ω_k's are in $\mathcal{H}_1 = \mathrm{Span}\{\mathrm{d}x\}$—i.e. $r_k \geq 1$ for all k—if and only if, at the point (x, u) under consideration,

$$\operatorname*{rank}_{\mathbb{R}} \left\{ \frac{\partial f}{\partial u_1}(x,u), \ldots, \frac{\partial f}{\partial u_m}(x,u) \right\} = m. \tag{14}$$

At a Brunovský-regular point, $\mathcal{D}_\infty$ is equal to $\mathcal{D}_{n+1}$ and is hence nonsingular and hence locally finitely generated. Hence strong accessibility implies, from Theorem 1, that $\rho_\infty = 0$. In that case and if (14) is met, (13) implies

$$\begin{aligned} &\{\omega_k^{(j)}, 0 \leq k \leq m, 0 \leq j \leq r_k - 1\} \text{ is a basis of } \mathcal{H}_1 = \mathrm{Span}\{\mathrm{d}x\}, \\ &\{\omega_k^{(j)}, 0 \leq k \leq m, 0 \leq j \leq r_k\} \text{ is a basis of } \mathcal{H}_0 = \mathrm{Span}\{\mathrm{d}x, \mathrm{d}u\}. \end{aligned} \tag{15}$$

Hence, with $\omega_{k,j} = \omega_k^{(j)}$, and with the $a_{i,j}$'s and $b_{i,j}$'s some functions such that the matrix $[b_{i,j}]_{i,j}$ is invertible at $\mathcal{X}$,

$$\begin{aligned} \dot{\chi}_1 &= \gamma_1(\chi_1, \ldots, \chi_{\rho_\infty}) \\ &\vdots \\ \dot{\chi}_{\rho_\infty} &= \gamma_{\rho_\infty}(\chi_1, \ldots, \chi_{\rho_\infty}) \end{aligned}$$

$$\left.\begin{aligned} \dot{\omega}_{i,1} &= \omega_{i,2} \\ \dot{\omega}_{i,2} &= \omega_{i,3} \\ &\vdots \\ \dot{\omega}_{i,r_i-1} &= \omega_{i,r_1} \\ \dot{\omega}_{i,r_1} &= \sum_{j=1}^{n} a_{i,j}\mathrm{d}x_j + \sum_{j=1}^{m} b_{i,j}\mathrm{d}u_j \end{aligned}\right\} \quad 1 \leq i \leq m. \tag{16}$$

We call this "infinitesimal Brunovský form" because it looks like the canonical Brunovský form [3] for linear system; it is not a "canonical form" for any equivalence relation: the data of the forms $\omega_1, \ldots, \omega_m$ and of (16) does not give a unique system.

P r o o f. The proof goes along the lines of [1] or [22]. Since we are at a Brunovský-regular point, $\mathcal{H}_\infty$ is nonsingular and locally spanned by exactly ρ_∞ forms. These forms depend on a finite number of variables $x, u, \ldots, u^{(K)}$. One may then project these forms, and hence $\mathcal{H}_\infty$, on the finite dimensional manifold

$\mathcal{M}_K^{m,n}$ (see [21]) and use the finite dimensional Frobenius theorem: from Theorem 1, $\mathcal{H}_\infty$ is completely integrable and therefore is spanned by ρ_∞ exact forms $\mathrm{d}\chi_1 \ldots \mathrm{d}\chi_{\rho_\infty}$ with $\chi_1 \ldots \chi_{\rho_\infty}$ some functions, which depend only on x because $\mathrm{d}\chi_i \in \mathcal{D}_\infty \subset \mathcal{D}_1$. Then the forms ω_k may be constructed recursively such that (13) holds:

• it holds for $l \geq k^* + 1$ provided all the r_k's are no larger than k^* (it will be the case).

• chose $\omega_1, \ldots, \omega_{\rho_{k^*}}$ so that $\{\mathrm{d}\chi_1, \ldots, \mathrm{d}\chi_{\rho_\infty}, \omega_1, \ldots, \omega_{\rho_{k^*}}\}$ is a basis of $\mathcal{H}_{k^*}$, and set $r_1 = \ldots = r_{\rho_{k^*}} = k^*$, (13) is then satisfied for $l \geq k^*$ provided all the remaining r_k's are no larger than $k^* - 1$ (it will be the case).

• Induction on ℓ, downward from $\ell = k^*$ to $\ell = 0$: for $0 \leq \ell \leq k^* - 1$, let us suppose that (13) is true for $l \geq \ell + 1$ (assuming that all the r_k's corresponding to ω_k's which have not yet been built are no larger than ℓ), and build some ω_k's with $r_k = \ell$ so that (13) is true for $l \geq \ell$. It is not difficult to prove (see [1, proof of Th. 3.5], really similar because by assumption $\mathcal{H}_{\ell+1} + \dot{\mathcal{H}}_{\ell+1}$ is nonsingular here) that $\{\mathrm{d}\chi_1, \ldots, \mathrm{d}\chi_{\rho_\infty}\} \cup \{\omega_k^{(j)}, r_k \geq \ell+1, 0 \leq j \leq r_k - \ell\}$ is a set of linearly independent elements of $\mathcal{H}_\ell$, actually a basis of $\mathcal{H}_{\ell+1} + \dot{\mathcal{H}}_{\ell+1} \subset \mathcal{H}_\ell$. Add, if they do not form a basis of $\mathcal{H}_\ell$, some new ω_k's with the corresponding r_k's equal to ℓ.

After $l = 0$, no new ω_k's are needed because if there is a certain number of ω_k's such that (13) holds for $l = 0$ (we have not yet proved there are exactly m of them), then $\mathrm{d}u_1, \ldots, \mathrm{d}u_m$ are linear combinations of the $\mathrm{d}\chi_i$'s and the $\omega_k^{(j)}$'s for $r_k \geq 0$ and $0 \leq j \leq r_k$, which immediately implies that, for $q > 0$, $\mathrm{d}u_1^{(q)}, \ldots, \mathrm{d}u_m^{(q)}$ are linear combinations of the $\mathrm{d}\chi_i$'s and the $\omega_k^{(j)}$'s for $r_k \geq 0$ and $0 \leq j \leq r_k + q$, i.e. (13) is met for $l = -q < 0$ without any additional ω_k's; this ends the construction of the ω_k's and proves $r_k \geq 0$ for all k. There are exactly m ω_k's because an obvious consequence of (13) is that $\rho_l - \rho_{l+1}$ is equal to the number of r_k's larger or equal to l; in particular, since $\rho_l - \rho_{l+1} = m$ for $l \leq 0$ (see (7)), the total number of ω_k's is m. To prove the very last part of the theorem, one therefore has to prove that $\rho_1 - \rho_2 = m$ if and only if (14) holds, which is obvious because, from (4), $\mathcal{D}_2 = \mathcal{D}_1 \oplus \mathrm{Span}\{\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_m}\}$ and because of Brunovský-regularity. ∎

The reason for defining this "Brunovský form" in [1, 22] was to suggest a way to look for "linearizing outputs" (see theorem 3 below for definition and comments). For this, we defined the following infinitesimal version of linearizing outputs:

Definition 3 ([1, 22, 21]). A Pfaffian system $(\omega_1, \ldots, \omega_m)$ is called a *linearizing Pfaffian system* at a point $\mathcal{X}$ if and only if, for a certain neighborhood U of $\mathcal{X}$, the restrictions to U of the forms $L_F^j \omega_k$, $j \geq 0$, $1 \leq k \leq m$ form a basis of the $\mathcal{C}^\infty(U)$-module $\Lambda^1(U)$ of all differential forms on U.

One should not be misled by the terminology: a linearizing Pfaffian system, contrary to a linearizing output, does not linearize anything unless it has more properties (integrability, see Theorem 3). An immediate consequence of Theorems 1 and 2 is:

COROLLARY 2. *If a system F is locally strongly accessible around a point $\mathcal{X}$, which is Brunovský-regular for F, then F admits, locally around $\mathcal{X}$ a linearizing Pfaffian system $(\omega_1, \ldots, \omega_m)$. A possible choice is the forms $\omega_1, \ldots, \omega_m$ constructed in Theorem 2. If (14) holds, $\omega_1, \ldots, \omega_m$ are in $\mathcal{H}_1 = \mathrm{Span}\{\mathrm{d}x\}$.*

Comments on this "Brunovský form". Let us indicate the similarity between the content of this section and the algebraic framework for "time-varying" linear systems developed in [8, 9] for example.

For U an open subset of $\mathcal{M}_\infty^{m,n}$, let $\mathcal{C}^\infty(U)[L_F]$ be the algebra of differential operators which are polynomials in the Lie derivative with respect to F with coefficients in $\mathcal{C}^\infty(U)$. This is a noncommutative algebra since $(aL_F)(bL_F) = abL_F^2 + a(L_F b)L_F$. It plays the same role as the non-commutative ring $k[\mathrm{d}/\mathrm{dt}]$ (k is a differential field) introduced in [8] to define linear time-varying systems: a linear system is a module over this ring and it is controllable if and only if it is a free $k[\mathrm{d}/\mathrm{dt}]$-module (which is also a k vector space).

In the nonlinear case, in [8, 10] a system is represented by a differential field k and, via Kähler differentials, one may define the linearized system as a $k[\mathrm{d}/\mathrm{dt}]$-module, whose equivalent here is the $\mathcal{C}^\infty(U)[L_F]$-module $\Lambda^1(U)$.

Relying upon results from [23, 6] which state that a nonlinear system satisfying the strong accessibility condition has a controllable linear approximation along "almost any" trajectory, a nonlinear system is said to be controllable in [10] if and only if the $k[\mathrm{d}/\mathrm{dt}]$-module associated to the differential field k is free.

Note that the assertion "$(\omega_1, \ldots, \omega_m)$ is a linearizing Pfaffian system" (or (13) with $\rho_\infty = 0$) is equivalent to "$(\omega_1, \ldots, \omega_m)$ is a basis of the $\mathcal{C}^\infty(U)[L_F]$-module $\Lambda^1(U)$"; hence Corollary 2 constructs a basis of this module, and hence establishes that it is free. We have *proved* (theorem 1), that, at a Brunovský-regular point (and even at a point where $\widehat{\mathcal{D}}_\infty$ is locally finitely generated), the strong accessibility rank condition implies that the module is free, or that the linearized system is controllable in the sense of [8, 10]. This is not exactly a consequence of [23, 6]. Technically, the result is contained in the fact that $\mathcal{D}_\infty$ is (around a regular point) closed under Lie bracket, which may be interpreted as: the torsion submodule of the $\mathcal{C}^\infty(U)[L_F]$-module $\Lambda^1(U)$ is "integrable".

An algebraic construction of the "canonical Brunovský form" (or of a basis of the module) for controllable time-varying linear systems, based on some filtrations, is proposed in [9]. The sequence of the $\mathcal{H}_k$'s *is* a filtration of $\Lambda^1(U)$. It does not coincide with these introduced in [10], but might certainly be interpreted in the same terms. The "well-formedness" assumption in [10] corresponds to (14) at the end of Theorem 2.

4. Dynamic linearization as an integrability problem. Dynamic linearizability from definition 1 is actually *linearizability by endogenous dynamic* *feedback* as defined in [16, 11, 12]. It is proved there that this is equivalent to *flatness*, i.e. to existence of *linearizing outputs* or *flat outputs*. In the present framework, these are defined below. They are given an interpretation in terms of dynamic decoupling and structure at infinity in [1] and in [16], and they are defined as the free generators of the differential algebra $\mathcal{C}^\infty(\mathcal{M}_\infty^{m,n})$ in [14, 15].

THEOREM 3 ([21]). *Let $\mathcal{X}$ be a point of $\mathcal{M}_\infty^{m,n}$. The following assertions are equivalent*:

1. *The system F is locally dynamic linearizable at point $\mathcal{X}$.*

2. *There exist m smooth functions $h_1, \ldots, h_m$ from a neighborhood of $\mathcal{X}$ in $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that $(L_F^j h_k)_{1\le k\le m, 0\le j}$ is a local system of coordinates at $\mathcal{X}$. Such m functions are called linearizing outputs* (*or simply one linearizing output*) [11, 12, 16].

3. *F admits, on a neighborhood of $\mathcal{X}$, a linearizing Pfaffian system $(\eta_1, \ldots, \eta_m)$ which is completely integrable, i.e. such that $\mathrm{d}\eta_k \wedge \eta_1 \wedge \ldots \wedge \eta_m = 0$, $k = 1 \ldots m$.*

We saw in the previous section that all strongly accessible systems admit, at Brunovský-regular points, a linearizing Pfaffian system, which, of course, may not be integrable. We therefore have to investigate what *all* linearizing Pfaffian systems are, and we may say that a system is dynamic linearizable if and only if there exists one among all these which is integrable.

For an open subset U of $\mathcal{M}_\infty^{m,n}$, let $\mathcal{A}(U)$ be the algebra of $m \times m$ matrices with entries in the algebra of differential operators $\mathcal{C}^\infty(U)[L_F]$:

$$\mathcal{A}(U) \overset{\Delta}{=} \mathcal{M}_{m\times m}\left(\mathcal{C}^\infty(U)[L_F]\right). \tag{17}$$

A matrix in $\mathcal{A}(U)$ defines an operator on m-uples of 1-forms in a straightforward manner, and we have:

PROPOSITION 3. *Let $(\omega_1, \ldots, \omega_m)$ be a linearizing Pfaffian system and let $\eta_1, \ldots, \eta_m$ be m 1-forms defined on an open set U of $\mathcal{M}_\infty^{m,n}$. $(\eta_1, \ldots, \eta_m)$ is a linearizing Pfaffian system if and only if there exists $P(L_F)$ in $\mathcal{A}(U)$ which is invertible in $\mathcal{A}(U)$ and is such that*

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix} = P(L_F) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix}. \tag{18}$$

Proof. There always exists $P(L_F) \in \mathcal{A}(U)$ such that (18) holds because $(\omega_1, \ldots, \omega_m)$ is a linearizing Pfaffian system. If $(\eta_1, \ldots, \eta_m)$ is also a linearizing Pfaffian system, there exists $Q(L_F) \in \mathcal{A}(U)$ such that

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} = Q(L_F) \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix}.$$

Hence $Q(L_F)P(L_F)$ and $P(L_F)Q(L_F)$ transform respectively $(\omega_1,\ldots,\omega_m)$ and $(\eta_1,\ldots,\eta_m)$ into themselves. Hence $Q(L_F)P(L_F) = P(L_F)Q(L_F) = I$ because the forms $\omega_k^{(j)}$ (resp. $\eta_k^{(j)}$), $1 \le k \le m$, $j \ge 0$, are linearly independent. Conversely, it is obvious that (18) with $P(L_F)$ invertible implies that $(\eta_k^{(j)})_{1\le k\le m, j\ge 0}$ is a basis of the $\mathcal{C}^\infty(U)$-module $\Lambda^1(U)$. ∎

A straightforward consequence of Theorem 2 and proposition 3 is:

THEOREM 4. *Let* $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ *be a Brunovský-regular point for a system* F, *and let* $\omega_1,\ldots,\omega_m$ *be the* 1*-forms constructed in Theorem* 2, *defined on a certain neighborhood* U *of* $\mathcal{X}$. *The system* F *is locally dynamic linearizable at a point* $\mathcal{X}$ *if and only if there exists an invertible matrix* $P(L_F) \in \mathcal{A}(U)$ *such that*

$$\begin{pmatrix} \overline{\omega}_1 \\ \vdots \\ \overline{\omega}_m \end{pmatrix} = P(L_F) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} \tag{19}$$

is a locally completely integrable Pfaffian system, i.e. $\mathrm{d}\overline{\omega}_k \wedge \overline{\omega}_1 \wedge \ldots \wedge \overline{\omega}_m = 0$ *for* $k = 1,\ldots,m$.

Of course, this is not per se a solution to the dynamic feedback linearization problem; it is rather a convenient way to pose the problem of deciding whether or not linearizing outputs exist. The main difficulty comes from the fact that the degree of P may be arbitrarily large because the linearizing outputs may depend on an arbitrary number of time-derivatives of u. Let us make this number artificially finite:

DEFINITION 4. A system F is said to be $(x,u,\ldots,u^{(K)})$-*linearizable* (for $K = -1$, this reads x-linearizable) at point $\mathcal{X}$ if and only if there exists some linearizing outputs function of $(x,u,\ldots,u^{(K)})$ only (on x only for $K = -1$).

Of course, a system is dynamic feedback linearizable (in the sense of definition 1, i.e. linearizable by endogenous dynamic feedback according to [11, 12, 16], or dynamic linearizable according to [14, 15]) if and only if it is $(x,u,\ldots,u^{(K)})$-linearizable for a certain K. We have the following theorem which precises Theorem 4.

THEOREM 5. *Let* $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ *be a Brunovský-regular point for system* F, *and let* $\omega_1,\ldots,\omega_m$, *and* $r_1,\ldots,r_m$ *be, respectively, the 1-forms and integers constructed in Theorem* 2. *System* F *is* $(x,u,\ldots,u^{(K)})$*-linearizable at point* $\mathcal{X}$ *if and only if there exists an invertible matrix* $P(L_F) \in \mathcal{A}(U)$ *satisfying the conditions of Theorem 4 and such that the degree of the entries of the* k*-th column is at most* $K + r_k$.

Proof. The condition is necessary for $(x,u,\ldots,u^{(K)})$-linearizability because if $h_1,\ldots,h_m$ are some linearizing outputs function of $x,u,\ldots,u^{(K)}$ only, (19) holds with $\overline{\omega}_k = \mathrm{d}\varphi_k$ and, from (13), the columns of P have to satisfy the degree inequalities. Conversely, suppose that (19) holds with the degree of the kth column of P being at most $K + r_k$ and the system $(\overline{\omega}_1, \ldots, \overline{\omega}_m)$ completely integrable, then $(\overline{\omega}_1, \ldots, \overline{\omega}_m)$ is spanned by some exact forms $(\mathrm{d}h_1, \ldots, \mathrm{d}h_m)$; the functions h_k are linearizing outputs; the degree inequalities imply that all the $\overline{\omega}_k$'s are in $\mathcal{H}_{-K} = \mathrm{Span}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{(K)}\}$, and hence that the h_k's are functions of $x, u, \ldots, u^{(u)}$ only. ■

One of the reasons why our results provide a rather convenient framework is that, outside some singular points, it is not difficult to describe invertible matrices of a prescribed degree. As noticed in [9, 10, 13], the polynomial ring $\mathcal{C}^\infty(U)[L_F]$ enjoys many interesting properties. Namely, it is possible to perform right and left Euclidian division by a polynomial whose leading coefficient does not vanish. It is well known (see for example [25]) that, in the constant coefficient case, all invertible polynomial matrices are finite products of "elementary matrices", i.e. either diagonal invertible matrices or permutation matrices or matrices whose diagonal entries are all equal to 1 while only one of the non-diagonal entries is nonzero, and it is an arbitrary polynomial. Since the tool to get such a decomposition is only Euclidian division, this remains true in the case of coefficients in $\mathcal{C}^\infty(U)$ as long as one does not have to perform Euclidian division by a polynomial whose leading coefficient vanishes. This does not happen often, although it is not very easy in general to say which singularities the original matrix should not have for this not to happen; in the meromorphic case ([1, 22]), this never happens since the coefficient of the polynomials then belong to a field and are therefore invertible, even if they "vanish" at a point, if they are not zero. Now, if one bounds a priori the degree of the columns of P (say one wishes to decide whether $(x, u, \ldots, u^{(K)})$-linearizability holds), then all invertible matrices satisfying these bounds may be sorted into a finite number of types of finite products of elementary matrices, each type involving a finite number of functions. In each case,

$$\mathrm{d}\left(P(L_F) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} \right) = 0$$

(with d acting on each entry) is a set of partial differential equations in these functions. The solubility of these PDE's is equivalent to the existence of a system of linearizing outputs depending only on a fixed finite number of time-derivatives of u.

5. Conclusion. We have developed a framework for looking for linearizing outputs which gives a convenient way for writing down a system of equations whose solubility is equivalent to the existence of a system of linearizing outputs. Some work has already been done in the direction of characterizing the cases where linearizing outputs exist. These results give either sufficient conditions or necessary and sufficient conditions for existence of linearizing outputs for some particular cases. For example, $(x, u, \ldots, u^{(K)})$-linearizability (in most cases,

$K = -1$) or a prescribed "structure at infinity" (see [17, 18, 19]). A criterion for existence of a matrix P of degree zero for general two-inputs systems is given in [22]. The "sufficiency" part of the result contained in [19] is re-derived in [2] in a way that simplifies, to our opinion, the argument partly due to E. Cartan. Finally, a characterization of (x, u)-linearizability for affine systems with 4 states and 2 inputs is given in [20]. These last results seem to demonstrate that "infinitesimal Brunovský form" is a convenient way to tackle the problem of looking for linearizing outputs.

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