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BOUNDARIES AND THE FATOU THEOREM FOR SUBELLIPTIC SECOND ORDER OPERATORS ON SOLVABLE LIE GROUPS

BY

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1. Introduction. This paper is a continuation of our paper [DH]. We are going to study the behavior of the Poisson integrals on the Furstenberg–Guivarc'h–Raugi boundaries for bounded functions harmonic with respect to a second order, left-invariant, nonnegative, subelliptic differential operator L on a solvable Lie group $S = NA$, which is a semidirect product of a nilpotent Lie group N and an Abelian Lie group A acting diagonally on N.

In [DH] we have identified all such boundaries. A boundary is an S-space $X \simeq \mathbb{R}^\chi$ equipped with a probability measure ν (see Section 2 for a precise definition of S) such that the Poisson integral

$$F(s) = \int_X f(sx)\, d\nu(x) = Pf(s)$$

of a function f in $L^p(\mathbb{R}^\chi)$, $1 \le p \le \infty$, is an L-harmonic function on S.

The main point of the present paper is to prove the almost everywhere admissible convergence of the Poisson integrals of functions $f \in L^p$, $p > 1$, on an arbitrary boundary X, which is an analog of the Fatou theorem. The admissible approach to the boundary in the general case is defined in very much the same way as in the case when X is a group ([K], [St], [Sj] and [D]).

Our theorem will be proved under the additional assumption that A acts rationally on N. In the case when the boundary X can be identified with a subgroup or a factor group of N, this theorem has already been proved in [D]. In the general case, however, a much more refined technique seems to be necessary and only the methods developed by M. Christ [Chr] combined with the older ones by P. Sjögren [Sj] have allowed us to obtain the result.

The rationality assumption, satisfied automatically in the case of symmetric spaces, i.e. when NA is the solvable part of the Iwasawa decomposition of a semisimple Lie group, was necessary in [D]. It is also crucial here. It is a challenging problem to establish whether the Fatou theorem is valid without it.

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The paper is organized as follows. After the preliminary Section 2 we discuss the admissible convergence to the boundary in Section 3, where we also formulate the main theorem and make some comments. The proof of the main theorem consists of several steps. These are put in separate sections.

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2. Preliminaries. Let $\mathbf{s}$ be a solvable Lie algebra. We assume that $\mathbf{s}$ is the direct sum of two subalgebras, $\mathbf{s} = \mathbf{n} \oplus \mathbf{a}$, where $\mathbf{n}$ is nilpotent and $\mathbf{a}$ Abelian. We assume that there exists a basis $E_1, \ldots, E_n$ of $\mathbf{n}$ such that for every H in $\mathbf{a}$,

$$[H, E_j] = \langle \lambda_j, H \rangle E_j, \qquad \lambda_j \in \mathbf{a}^*,\ j = 1, \ldots, n.$$

We write $\{\lambda_1, \ldots, \lambda_n\} = \Delta$. For λ in Δ let

$$\mathbf{n}^\lambda = \{Y \in \mathbf{n} : \mathrm{ad}_H Y = \langle \lambda, H \rangle Y \text{ for all } H \text{ in } \mathbf{a}\}.$$

We say that a subspace $\mathbf{n}'$ of $\mathbf{n}$ is *homogeneous* if $\mathrm{ad}_H \mathbf{n}' \subset \mathbf{n}'$ for every H in $\mathbf{a}$.

Let

$$S = \exp \mathbf{s}, \quad N = \exp \mathbf{n} \quad \text{and} \quad A = \exp \mathbf{a}.$$

Then $S = NA$ is a semidirect product of the groups N and A, with A acting on N by

$$a \exp\Big\{\sum_j x_j E_j\Big\} a^{-1} = \exp\Big\{\sum_j x_j e^{\langle \lambda_j, \log a \rangle} E_j\Big\}. \tag{2.1}$$

Let L be a second order, left-invariant, degenerate elliptic operator without a constant term:

$$L = X_1^2 + \ldots + X_m^2 + X_0.$$

We shall assume that $X_0, X_1, \ldots, X_m$ satisfy the *Hörmander condition*, i.e. the smallest Lie subalgebra which contains $X_0, \ldots, X_m$ is equal to $\mathbf{s}$. We write

$$X_0 = Y_0 + Z_0, \qquad Y_0 \in \mathbf{n},\ Z_0 \in \mathbf{a}. \tag{2.2}$$

Now let

$$\Delta_0 = \{\lambda \in \Delta : \langle \lambda, Z_0 \rangle \geq 0\}.$$

We define the subalgebra

$$\mathbf{n}_0(L) = \bigoplus_{\lambda \in \Delta_0} \mathbf{n}^\lambda$$

and the corresponding subgroup $N_0(L) = \exp \mathbf{n}_0(L)$. Let $\mathbf{n}_0$ be a homogeneous subalgebra of $\mathbf{n}$ containing $\mathbf{n}_0(L)$ and let $N_0 = \exp \mathbf{n}_0$.

In [DH] we have shown that the boundaries of the pair S, L are precisely the S-spaces $X = S/N_0A = N/N_0$. To be more precise, we write $S \times X \ni (s,u) \mapsto su \in X$ for the natural action of S on X. We select a point e in X and we define the map $\mathbf{p} : S \ni s \mapsto se \in X$. For a measure ν on X and a bounded measure or a distribution with compact support μ on S we write $\mu * \nu$ for the natural convolution corresponding to this action. We say that (X, ν) is a *boundary* for the pair S, L if X is an S-space, ν a probability measure on X and

(2.3) $\quad \check{L} * \nu = 0, \quad$ or equivalently, $\quad \check{\mu}_t * \nu = \nu \quad$ for each $t > 0$,

where $\{\check{\mu}_t\}_{t>0}$ is the semigroup of probability measures on S whose infinitesimal generator is $\check{L} = X_1^2 + \ldots + X_m^2 - X_0$, and

(2.4) $s_t\nu$ tends weak* to a point mass on X as $t \to \infty$, for almost all trajectories s_t of the diffusion process on S generated by L.

Conversely, any locally compact Hausdorff S-space for which there exists a probability measure ν such that (2.3) and (2.4) hold is of the form S/N_0A, for some homogeneous subalgebra $\mathbf{n}_0 = \log N_0$ of $\mathbf{n}$ containing $\mathbf{n}_0(L)$ [DH].

Let f be a function on X and suppose $f \in L^p(\mathbb{R}^X)$ for some p, $1 \le p \le \infty$. Then (2.3) implies that the function

(2.5)
$$F(s) = \int_X f(sx)\, d\nu(x)$$

on S is L-harmonic. We call (2.5) the *Poisson integral* of f. As is proved in [DH], ν is the weak* limit of $\mathbf{p}(\mu_t)$ as $t \to \infty$. Let us list some properties of ν proved in [DH].

(2.6) $\quad \nu$ has a smooth density $d\nu(x) = P(x)\, dx$.

The function P is called the *Poisson kernel* for the boundary X. Let $\|\cdot\|$ be a norm in X.

(2.7) $\quad$ There exists $\eta > 0$ such that $\int_X \|y\|^\eta P(x)\, dx < \infty$.

Consequently,

(2.8) $\quad P \in L^\beta \quad$ for some $\beta < 1$.

(2.9) $\quad$ For every multiindex I there are constants c, M such that
$$|\partial^I P(y)| \le c(1 + \|y\|)^M.$$

(2.10) $\quad$ There exist $c, \varepsilon > 0$ such that $P(y) \le c(1 + \|y\|)^{-\varepsilon}$.

3. Almost everywhere admissible convergence. Now we fix a boundary $X = S/N_0A$ with $N_0 = \exp \mathbf{n}_0$. Let $\mathbf{n}_1$ be a homogeneous subspace of $\mathbf{n}_1$ such that $\mathbf{n} = \mathbf{n}_1 \oplus \mathbf{n}_0$. Without loss of generality we may assume that $E_1, \ldots, E_\chi$ is a linear basis of $\mathbf{n}_1$. Let

$$\Delta_1 = \{\lambda_1, \ldots, \lambda_\chi\}.$$

For a given compact subset K of S, and $y \in N$, let $\Gamma_y^K = \{yaz : a \in A,\ z \in K\}$. We say that *s tends admissibly to the boundary* X, $s \in S$, and we write $s \to X$, if $s \in \Gamma_y^K$ and

$$\lim \langle \lambda, \log a(s) \rangle = -\infty \quad \text{for every } \lambda \in \Delta_1,$$

where $a(s)$ is the image of s under the canonical homomorphism of S onto $A = S/N$. A simple verification shows that (2.1) implies

$$\lim_{s \to X} sz \cdot x = \mathbf{p}(s)$$

uniformly for x in a compact subset in X and z in a compact subset of S. Consequently, for $f \in C_c(X)$ and every compact subset K of S we have

$$\lim_{ya \to X} \int_X f(yazx) P(x)\, dx = f(\mathbf{p}(y)) \tag{3.1}$$

uniformly in $z \in K$. We shall use the abbreviation

$$Pf(s) = \int_X f(sx) P(x)\, dx.$$

A natural generalization of (3.1) to the almost everywhere convergence of $Pf(s)$ to $f(\mathbf{p}(s))$ for f in $L^p(X)$ could be the following: For every function f in $L^p(X)$, $1 < p \le \infty$, there is a set $X_0 \subset X$ such that $|X \backslash X_0| = 0$ and

$$\text{if } \mathbf{p}(y) \in X_0, \text{ then } \lim_{ya \to X} \int_X f(yazx) P(x)\, dx = f(\mathbf{p}(y)).$$

This is true if $\mathbf{p}$ is one-to-one on N (see [D]). Then the maximal function

$$Mf(y) = \sup_{a \in A,\, z \in K} \int_X f(yazx) P(x)\, dx$$

is bounded on $L^p(N)$. If, however, $N_0 \ne e$, then Mf has no chance of being in $L^p(N)$, since if e.g. N_0 is a normal subgroup of N, then Mf is constant on cosets of N_0. To formulate our almost everywhere convergence theorem for the admissible convergence as defined above, we consider a selector from the cosets and redefine the maximal function appropriately.

Let as above $\mathbf{n}_1$ be a homogeneous subspace of $\mathbf{n}$ such that $\mathbf{n} = \mathbf{n}_1 \oplus \mathbf{n}_0$. In view of the easy Proposition (1.25) of [DH], if $N_1 = \exp \mathbf{n}_1$, then

$$N_1 \times N_0 \ni (y, z) \mapsto yz \in N_1 N_0 = N \tag{3.2}$$

is a diffeomorphism such that if $x = yz$, then y and z depend polynomially on x in the coordinates given by exp on N_1, N_0 and N, respectively. Clearly

$X \simeq N_1 \simeq \mathbb{R}^\chi$ for some natural number χ. Thus, N_1 is a selector from the cosets $N/N_0 = X$. We transfer the action of S on X to the action on N_1: let $\pi : S \to N_1$ be defined as follows. For s in S we write s uniquely as $s = yza$ with $y \in N_1$, $z \in N_0$, $a \in A$ and we put $\pi(s) = y$. Then

$$\pi(s_1\pi(s_2y)) = \pi(s_1s_2y) \quad \text{for} \ \ s_1, s_2 \in S,\ y \in N_1. \tag{3.3}$$

This defines an action of S on $N_1 : S \times N_1 \ni (s, y) \mapsto \pi(sy) \in N_1$, and, of course, $\mathbf{p}_{|N_1}$ is an isomorphism between the S-spaces N_1 and X.

We shall also consider a group of transformations of N_1 "from the right" generated by the mappings $N_1 \ni x \mapsto \pi(xu) \in N_1$ for u in N_1, and we shall prove that this is a (finite-dimensional) nilpotent group. Of course this group is equal to N_1 if N_1 is a subgroup of N, but the latter does not hold in general.

Our main theorem will be proved under the following

(3.4) Rationality assumption. There exists a basis $E_1, \ldots, E_\chi$ of $\mathbf{n}_1$ and a basis $H_1, \ldots, H_k$ of $\mathbf{a}$ such that the corresponding functionals $\lambda_1, \ldots, \lambda_\chi$ take integral values on $H_1, \ldots, H_k$.

Let $y_0 \in N_0$ and K be a compact subset of S. We consider the maximal function

$$M^K_{y_0} f(y_1) = \sup_{a \in A, z \in K} \int\limits_X |f|(y_1y_0azx)P(x)\,dx.$$

We are going to prove the following

(3.5) Theorem. *Under the rationality assumption, for $p > 1$ for a constant $C = C_{K,y_0,p}$ we have*

$$\|M^K_{y_0} f\|_{L^p(N_1)} \le C\|f\|_{L^p(X)}.$$

Theorem (3.5) has an immediate consequence:

(3.6) Main Theorem. *Let $f \in L^p(X)$ for some $p > 1$. For every y_0 in N_0 there is a subset X_{y_0} in X such that the Lebesgue measure of $X \setminus X_{y_0}$ is 0 and such that for every compact subset K of S, if $y = y_1y_0$ and $\mathbf{p}(y) \in X_{y_0}$ we have*

$$\lim_{\langle \lambda, \log a\rangle \to -\infty,\, \lambda \in \Delta_1} Pf(y_1y_0az) = f(\mathbf{p}(y))$$

uniformly for $z \in K$.

Remarks.

(3.7) If $y_0 = e$ and N_1 is a subgroup of N this is precisely the "almost every admissible convergence theorem" of [D] and if also S is the NA part of the Iwasawa decomposition of a semisimple Lie group with L being the Laplace–Beltrami operator on the symmetric space S, it is the main theorem of [Sj].

(3.8) It is an open question whether the set X_{y_0} can be selected independently of y_0. Certainly the existence of such a universal set would be implied by a more general version of the approach to the boundary. Indeed, let us say that s in S tends to the boundary X *strongly admissibly* $s \to X$ if for a compact subset K_0 of N and a compact subset K of S,

$$s \in \bigcup_{\mathbf{p}(y)=x,\, y\in K_0} yAK$$

and $\lim\langle\lambda, \log a\rangle = -\infty$ for every $\lambda \in \varDelta_1$. Then for every compact subset C of X, $sx \to \mathbf{p}(s)$ uniformly in $x \in C$. Hence

$$\lim_{ya\to X} \int_X f(yazx)P(x)\,dx = f(\mathbf{p}(y)) \tag{3.9}$$

for $f \in C_0(X)$. It is not true, however, that (3.9) holds for almost all $\mathbf{p}(y) \in X$, even for f in $L^\infty(X)$ (see [S]).

4. Reduction to lacunary dilations. By the rationality assumption we see that the set

$$\varGamma = \Big\{H \in \mathbf{a} : H = \sum a_j H_j,\ a_j \in \mathbb{Z}\Big\}$$

has the property that

$$\langle\lambda, \gamma\rangle \in \mathbb{Z} \quad \text{for } \gamma \in \varGamma,\ \lambda \in \varDelta_1. \tag{4.1}$$

Let U be a subset of $\mathbf{a}$ with compact closure such that every H in $\mathbf{a}$ can be written uniquely in the form

$$H = u + \gamma, \quad u \in U,\ \gamma \in \varGamma.$$

For a in A we write $[a]$ for the unique γ in $\varGamma$ such that $\log a = u + \gamma$ with γ in $\varGamma$, u in U.

For a compact subset K of S let $K' = \exp(UK)$. By the Harnack inequality, there is a constant c such that

$$\max_{s\in K'} F(s) \le cF(e)$$

for every nonnegative harmonic function F. Consequently, since L is left-invariant,

$$P|f|(xas) \le cP|f|(x[a]) \quad \text{for } s \in K',\ x \in N.$$

Therefore, for a fixed compact subset K of S and $y_0 \in N_0$,

$$\begin{aligned} M^K_{y_0} f(y_1) &\le c \sup_{a\in A} P|f|(y_1 y_0 [a]) \\ &\le c \sup_{\log a\in\varGamma} P|f|(y_1 y_0 a) = c\overline{M} f(y_1), \quad y_1 \in N_1. \end{aligned}$$

As $\mathbf{p}$ establishes an isomorphism of the S-spaces X and N_1 ((3.3)), the maximal function $\overline{M}$ can be considered as a function from $L^p(N_1)$ into $L^p(N_1)$. For $u \in N_1$ we write $u = \exp\{\sum_{j=1}^{\chi} u_j E_j\}$. Consequently,

$$\overline{M}f(y_1) = \sup_{\gamma\in\Gamma} \int_{N_1} |f|\Big(\pi\Big(y_1 y_0 \exp\Big\{\sum_{j=1}^{\chi} u_j e^{\langle\gamma,\lambda_j\rangle} E_j\Big\}\Big)\Big) P(u)\, du. \tag{4.2}$$

Now we proceed as in [Sj]. In view of (2.10), there exist two constants c and ξ such that

$$P(u) \le c \min\{1, |u_j|^{-\xi} \ : j = 1, \ldots, \chi\}. \tag{4.3}$$

Let $\mathcal{E}_m = \{u : P(u) > 2^{-m}\}$, $m = 0, 1, \ldots$ By (4.3), for some c_1, c_2 and all $m \ge 0$,

$$\mathcal{E}_m \subset \{u : |u_j| \le c_1 2^{c_2 m},\ j = 1, \ldots, \chi\}. \tag{4.4}$$

Moreover, since ∇P grows at most polynomially (see (2.9)), there is a $\varrho > 0$ such that

$$\operatorname{dist}(\mathcal{E}_m, \mathcal{E}_{m+1}^{\mathrm{c}}) \ge 2^{-\varrho m}. \tag{4.5}$$

We divide N_1 into disjoint cubes of size $2^{-\varrho m}$. Let $Q_{m,j}$, $j = 1, \ldots, j_m$, be those cubes whose intersection with $\mathcal{E}_m$ is not empty. By (4.5), we have

$$Q_{m,j} \subset \mathcal{E}_{m+1}. \tag{4.6}$$

Hence $j_m \le 2^{\varrho m \chi} |\mathcal{E}_{m+1}|$, where $\chi = \dim N_1$. But, since $P \in L^\beta$ for some $\beta < 1$ (see (2.8)), by the Chebyshev inequality we have $|\mathcal{E}_{m+1}| \le c 2^{m(1-\beta)}$ and so

$$j_m \le 2^{\varrho m \chi + m(1-\beta)}.$$

Let now

$$M_{m,j} f(y_1) = \sup_{\gamma\in\Gamma} \int_{Q_{m,j}} |f|\Big(\pi\Big(y_1 y_0 \exp\Big\{\sum_{k=1}^{\chi} u_k e^{\langle\gamma,\lambda_k\rangle} E_k\Big\}\Big)\Big) P(u)\, du.$$

Then

$$\overline{M}f(y_1) \le c \sum_{m=1}^{\infty} \sum_{j=1}^{j_m} M_{m,j} f(y_1).$$

Thus the estimate

$$\|M_{m,j} f\|_{L^p} \le c 2^{-\varrho m \chi} m^p \|f\|_{L^p}, \qquad j = 1, \ldots, j_m, \tag{4.7}$$

to be proved below, implies

$$\|\overline{M}f\|_{L^p} \le c \sum_{m=1}^{\infty} 2^{-m} j_m 2^{-\varrho m \chi} m^p \|f\|_{L^p} = c \sum_{m=1}^{\infty} 2^{-m\beta} m^p \|f\|_{L^p}.$$

The rest of the paper is devoted to the proof of (4.7). We are going to prove the following

(4.8) THEOREM. *Let $o_1, \ldots, o_\chi \in \mathbb{R}$ and $r_1, \ldots, r_\chi$ be positive real numbers and*

$$(4.9)\quad Mf(y_1) = \sup_{\gamma\in\Gamma} (r_1 \ldots r_\chi)^{-1} \int\limits_{|u_j-o_j|<r_j, j=1,\ldots,\chi} |f|(\pi(y_1 y_0 \delta_\gamma(u)))\, du,$$

where for $u = \exp\{\sum_{j=1}^{\chi} u_j E_j\}$,

$$(4.10)\qquad \delta_\gamma(u) = \exp\Big\{\sum_{j=1}^{\chi} u_j e^{\langle\gamma,\lambda_j\rangle} E_j\Big\}.$$

Then for every $p > 1$ there exist constants c_p and C independent of $o_1, \ldots, o_\chi$ and $r_1, \ldots, r_\chi$ such that

$$(4.11)\qquad \|Mf\|_{L^p} \le c_p \bigg(1 + \log^+ \frac{\max|o_j|}{\min r_j}\bigg)^C \|f\|_{L^p}.$$

The center $(o_1, \ldots, o_\chi)$ of the cube $Q_{m,j}$ belongs to $\mathcal{E}_{m+1}$, so, by (4.4), $\max|o_k| \le c_1 2^{c_2(m+1)}$, while $r_k = 2^{-\varrho m}$, $k = 1, \ldots, \chi$. Thus Theorem (4.8) implies (4.7). Because of homogeneity of the right hand side of (4.11), it is sufficient to prove Theorem (4.8) for $r_1 = \ldots = r_\chi = 1$ and arbitrary $o_1, \ldots, o_\chi$.

5. A nilpotent group of transformations. The aim of this section is to show that the transformations

$$(5.1)\qquad N_1 \ni x \mapsto \pi(xy) \in N_1, \quad y \in N,$$

generate a nilpotent group of transformations acting transitively on N_1. Let $E_1, \ldots, E_\chi$ be a basis of $\mathbf{n}_1$ such that $[H, E_j] = \langle\lambda_j, H\rangle E_j$, $\lambda_j \in \Delta_1$. We define a natural family of dilations $\{\delta_r\}_{r>0}$ on N_1 by

$$\delta_r x = \exp\Big\{\sum_{j=1}^{\chi} r^{\langle Z_0,\lambda_j\rangle} x_j E_j\Big\},$$

where $x = \exp\{\sum_{j=1}^{\chi} x_j E_j\}$ and Z_0 is as in (2.2). We order the basis $E_1, \ldots, E_\chi$ in such a way that if $\langle\lambda_j, Z_0\rangle = d_j$, then $d_1 \le \ldots \le d_\chi$. Of course we may assume $d_1 = 1$.

For a polynomial in the variables $x_1, \ldots, x_\chi$ we define a degree by putting

$$(5.2)\qquad \deg x_j = d_j,$$

$$(5.3)\qquad \text{if } I = (i_1, \ldots, i_\chi) \text{ is a multiindex, then } \deg x^I = \sum i_j d_j,$$

$$(5.4)\qquad \deg \sum c_I x^I = \max\{\deg x^I\}.$$

A mapping ϕ from N_1 into a nilpotent Lie group G is called a *polynomial* if for a basis $X_1, \ldots, X_M$ of the Lie algebra of G we have

$$\phi\Big(\exp\Big\{\sum_{j=1}^{\chi} x_j E_j\Big\}\Big) = \exp\Big\{\sum_j W_j(x) X_j\Big\},$$

where the W_j are polynomials.

(5.5) THEOREM. *The mappings* (5.1) *generate a subgroup G of a homogeneous group $\mathcal{G}$ such that for a fixed y_0 in N_0 the mapping $\phi : N_1 \to G$ defined by*

$$\pi(xy_0u) = \phi(u)x \tag{5.6}$$

is a polynomial and $\phi(e) = e$.

Proof. In view of Proposition (1.22) in [DH] there are polynomials $P_1, \ldots, P_\chi$ such that if $u \in N_1$, then

$$(\pi(xu))_i = x_i + u_i + P_i(x, u), \tag{5.7}$$

$$P_i \text{ depends only on } x_1, \ldots, x_{i-1}, u_1, \ldots, u_{i-1}, \tag{5.8}$$

$$P_i(x, 0) = P_i(0, u) = 0, \tag{5.9}$$

$$\deg_x P_i < d_i, \tag{5.10}$$

$$P_i(\delta_r x, \delta_r u) = r^{d_i} P_i(x, u). \tag{5.11}$$

Therefore for $x \mapsto \pi(xy) = \pi(x\pi(y)),\ y \in N$, we have

$$\pi(xy)_i = x_i + P_i'(x, y), \tag{5.12}$$

where $\deg_x P_i' < d_i$, P_i' depends only on $x_1, \ldots, x_{i-1}$ and $P_i'(x, y_0) = 0$ for $y_0 \in N_0$.

Let V_i be the linear span of the polynomials in $x_1, \ldots, x_{i-1}$ of degree at most $d_i - 1$. We form a group $\mathcal{G}$ with underlying set $V_1 \oplus \ldots \oplus V_\chi$. Let $P = (P_1, \ldots, P_\chi)$ be a generic element of $\mathcal{G}$. Then P acts on N_1 as a transformation T_P defined by

$$(T_P x)_i = x_i + P_i(x_1, \ldots, x_{i-1}). \tag{5.13}$$

The mapping $P \mapsto T_P$ is injective. We have to show that for P and R in $\mathcal{G}$ there is an element PR in $\mathcal{G}$ such that $T_{PR} = T_P \odot T_R$, and $T_P^{-1} = T_{P^{-1}}$ for some P^{-1} in $\mathcal{G}$. In fact, since

$$(T_P T_R x)_i = (T_R x)_i + P_i((T_R x)_1, \ldots, (T_R x)_{i-1})$$

and

$$(T_R x)_j = x_j + R_j(x_1, \ldots, x_{j-1})$$

with $R_j \in V_j$, $j = 1, \ldots, i$, $P_i \in V_i$, there are $W_i \in V_i$, $i = 1, \ldots, \chi$, such that $(T_P T_R x)_i = x_i + W_i(x_1, \ldots, x_{i-1})$. Similarly, if

$$y_i = x_i + P_i(x_1, \ldots, x_{i-1}), \tag{5.14}$$

we solve (5.14) for x_i and obtain

$$(T_P^{-1}y)_i = y_i + P_i^{-1}(y),$$

where P_i^{-1} is a polynomial in $y_1, \ldots, y_{i-1}$ and

$$P_i^{-1}(y) = P_i(y_1 + P_1^{-1}(y), \ldots, y_{i-1} + P_{i-1}^{-1}(y)). \tag{5.15}$$

From (5.15) we prove by induction that $\deg P_i^{-1} < d_i$.

Putting for $r > 0$, $\delta_r = \delta_{(\log r)Z_0}$ and

$$\delta_r P(x) = (r^{d_1} P_1(\delta_{r^{-1}}x), \ldots, r^{d_\chi} P_\chi(\delta_{r^{-1}}x)) \tag{5.16}$$

we easily verify that

$$T_{\delta_r(PR)} = T_{\delta_r P} \odot T_{\delta_r R},$$

so $\{\delta_r\}_{r>0}$ is a group of automorphic dilations of $\mathcal{G}$.

To complete the proof of Theorem (5.5) we take the natural basis of monomials, $x^{\alpha,i} = (\ldots, x^\alpha, \ldots)$, $i = 1, \ldots, \chi$, $|\alpha| < d_i$, in $V_1 \oplus \ldots \oplus V_\chi$. We order it in the following way. We place $x^{\alpha,i}$ before $x^{\beta,j}$ whenever $i < j$ or if $i = j$ and $|\alpha| > |\beta|$. If $i = j$ and $|\alpha| = |\beta|$ the order is irrelevant. Let $(y_1, \ldots, y_M)$ be the coordinates in $\mathcal{G}$ with respect to this basis. In these coordinates the multiplication in $\mathcal{G}$ is given by

$$(yy')_i = y_i + y_i' + W_i(y, y'), \tag{5.17}$$

where W_i is a polynomial which depends on $y_1, \ldots, y_{i-1}$, $y_1', \ldots, y_{i-1}'$ and such that $W_i(0, y') = W_i(y, 0) = 0$. Now we identify $V_1 \oplus \ldots \oplus V_\chi$ with the Lie algebra of $\mathcal{G}$ and the ordered basis $X_1, \ldots, X_M$ of monomials becomes a basis of the Lie algebra. By (5.17), the transformation of coordinates $y = (y_1, \ldots, y_M) \mapsto (z_1, \ldots, z_M)$, where $y = \exp\{\sum_{i=1}^M z_i X_i\}$, is triangular, i.e. $z_i = y_i + R_i(y)$, where the polynomial R_i depends only on $y_1, \ldots, y_{i-1}$.

Now, by (5.12) we have

$$\pi(xy)_i = x_i + P_i'(x, y) = x_i + \sum_\alpha \Big(\sum_\beta a^i_{\alpha,\beta} y^\beta\Big) x^{\alpha,i}.$$

Hence, if $y = y_0 u$, $u \in N_1$ and

$$\phi(u) = \Big(\sum_\alpha \Big(\sum_\beta a^1_{\alpha,\beta} y^\beta\Big) x^{\alpha,1}, \ldots, \sum_\alpha \Big(\sum_\beta a^\chi_{\alpha,\beta} y^\beta\Big) x^{\alpha,\chi}\Big) \in \mathcal{G},$$

then in view of (5.17), $\phi(u) = \exp\{\sum_{j=1}^M W_j(u) X_j\}$, where W_j are polynomials. Moreover, we see that $\pi(xy_0u) = T_{\phi(u)}x$ and the proof is complete.

Now we transfer our maximal function (4.9) to the group G and use the transference principle (see [CW]). This means that we define a maximal function on $L^p(G)$ by

$$Mf(x) = \sup_{\gamma\in\Gamma} \int_{|u_k-o_k|\le 1,\, k=1,\dots,\chi} |f|\Big(x \exp\Big\{\sum_{j=1}^{M} W_j(\delta_\gamma u)X_j\Big\}\Big)\, du,$$

and we are going to prove

$$\|M\|_{L^p\to L^p} \le c_p(1+\log^+\{\max|o_k|\})^C.$$

Expanding $W_j(u)$ as sums of monomials u^α and rearranging the basis X_j, possibly multiplying by constants, we rewrite the maximal function M as

$$Mf(x) = \sup_{\gamma\in\Gamma} \int_{|u_k-o_k|\le 1,\, k=1,\dots,\chi} |f|\Big(x \exp\Big\{\sum_{\alpha\in\mathcal{A}} (\delta_\gamma u)^\alpha X_\alpha\Big\}\Big)\, du,$$

where $\mathcal{A}\subset\mathbb{N}^\chi$ is a finite set of multiindices $\alpha=(\alpha_1,\dots,\alpha_\chi)$, which does not contain the multiindex $(0,\dots,0)$. Thus it suffices to prove the following

(5.18) Theorem. *Let G be a connected, simply connected nilpotent Lie group and $\mathcal{A}$ a finite subset of $\mathbb{N}^\chi\backslash\{(0,\dots,0)\}$. For each $\alpha\in\mathcal{A}$ let X_α be an element of the Lie algebra of G. Consider a maximal function on $L^p(G)$ defined by*

$$Mf(x) = \sup_{\gamma\in\Gamma} \int_{|u_k-o_k|\le 1,\, k=1,\dots,\chi} |f|\Big(x \exp\Big\{\sum_{\alpha\in\mathcal{A}} (\delta_\gamma u)^\alpha X_\alpha\Big\}\Big)\, du.$$

Then there exist constants c_p and C independent of $o_1,\dots,o_\chi$ such that

$$\|M\|_{L^p\to L^p} \le C_p(1+\log^+\{\max|o_k|\})^C. \tag{5.19}$$

6. Maximal function after M. Christ [Chr]. The maximal function Mf in Theorem (5.18) is bounded on $L^p(G)$ as proved by M. Christ [Chr]. What we need here is the estimate (5.19) of its norm. This is attained by a careful examination of the proof given in [Chr]. We introduce appropriate dilations both in N_1 identified with $\mathbb{R}^\chi$ and in the free nilpotent group G whose algebra is freely generated by X_α, $\alpha\in\mathcal{A}$. To put these two things together we rewrite the main steps of Christ's arguments here adapted to our situation.

As in [Chr], we begin by recalling the transference principle again to replace the group G by the nilpotent free group $\mathbf{G}$ whose Lie algebra is generated by X_α, $\alpha\in\mathcal{A}$. For every sequence $J=\{J_\alpha\}_{\alpha\in\mathcal{A}}$ with $J_\alpha\in\mathbb{Z}$, i.e. $J\in\mathbb{Z}^{\mathcal{A}}=\mathbf{P}$, we define a unique dilation on $\mathbf{G}$ by

$$\mathbf{d}_J X_\alpha = e^{J_\alpha}X_\alpha.$$

Let $\psi\in C_c^\infty(\mathbb{R}^\chi)$ with $\psi(u)=1$ for u in $\{u:\forall_{k=1,\dots,\chi}\ |u_k-o_k|\le 1\}$ and

$\operatorname{supp}\psi \subset \{u : \forall_{k=1,\dots,\chi}\ |u_k - o_k| < 2\}$. We define a measure μ on $\mathbf{G}$ by

$$\int_{\mathbf{G}} f\,d\mu = \int_{\mathbb{R}^\chi} f\Big(\exp\Big\{\sum_{\alpha\in\mathcal{A}} u^\alpha X_\alpha\Big\}\Big)\psi(u)\,du$$

and the dilated measures μ_J by

$$\int_{\mathbf{G}} f\,d\mu_J = \int_{\mathbb{R}^\chi} f\Big(\exp\Big\{\sum_{\alpha\in\mathcal{A}} e^{J_\alpha} u^\alpha X_\alpha\Big\}\Big)\psi(u)\,du.$$

As in Christ [Chr], we deduce Theorem (5.18) from the following theorem which we are going to prove now.

(6.1) Theorem. *There is a constant c_p independent of* $\max|o_k|$ *such that the maximal function*

$$Mf(y) = \sup_{J\in\mathbf{P}} |f| * \mu_J(y)$$

is bounded on $L^p(\mathbf{G})$ with

$$\|M\|_{L^p\to L^p} \le C_p(1+\log^+\{\max|o_k|\})^q,$$

where q is a constant depending only on $\mathbf{G}$ (see Proposition (6.4)).

The proof follows closely the proof of the main theorem of [Chr]. We recall here the main steps to show how we obtain the required estimate $C_p(1+\log^+\max|o_k|)^q$. Obviously we may assume that $\max|o_k| \ge 1$.

Let $\{\delta_r\}_{r>0}$ be the unique family of automorphic dilations of $\mathbf{G}$ such that

$$\delta_r X_\alpha = r^{|\alpha|} X_\alpha.$$

For a measure ν on $\mathbf{G}$ we define $\delta_r\nu$ by $\langle f, \delta_r\nu\rangle = \langle f\circ\delta_r, \nu\rangle$. Now we put

$$r = \max|o_k|.$$

For each I in $\mathbf{P}^+ = (\mathbb{Z}^+)^{|\mathcal{A}|}\setminus\{0\}$ we let

$$\mathbf{g}_I = \{Y\in\mathbf{g} : \mathbf{d}_J Y = e^{\langle I,J\rangle}Y \text{ for all } J\in\mathbf{P}\},$$

where $\langle I,J\rangle = \sum_{\alpha\in\mathcal{A}} I_\alpha J_\alpha$. For each K in $\mathbf{P}^+$ we define $\max K = \max\{K_\alpha : \alpha\in\mathcal{A}\}$ and $|K| = \sum_\alpha K_\alpha$.

Let $\mathbf{g}^\alpha$ be the sum of the $\mathbf{g}_I$ such that $I_\alpha \ne 0$. Then $\mathbf{g}^\alpha$ is the ideal in $\mathbf{g}$ spanned by X_α. Let $d_\alpha = \dim\mathbf{g}^\alpha$. We fix a b_α in $C_c^\infty(\mathbf{g}^\alpha)$ with $\int b_\alpha = 1$. For k in $\mathbb{N}^+$ let

$$b_{\alpha,k}(u_1,\dots,u_{d_\alpha}) = e^{kd_\alpha} b_\alpha(e^k u_1,\dots,e^k u_{d_\alpha}).$$

We define measures $\sigma_{\alpha,k}$ and $\lambda_{\alpha,k}$ by

$$\sigma_{\alpha,k} = \exp_*(b_{\alpha,k}(u)du), \quad \lambda_{\alpha,k} = \sigma_{\alpha,k} - \sigma_{\alpha,k-1}, \quad k\ge 1,$$

where $\exp_*$ denotes the push-forward of a measure.

We choose a linear ordering of $\mathcal{A}$ which will remain fixed. For K in $\mathbf{P}^+$ we define

$$\Lambda^K = \prod_{\alpha \in \mathcal{A}} \lambda_{\alpha, K_\alpha},$$

where $\prod$ denotes the convolution product of measures taken according to the ordering. For a subset E of $\mathcal{A}$ we define

$$\tau^E = \prod_{\alpha \in E} \sigma_{\alpha,0},$$

where the convolution product is taken according to the order of $\mathcal{A}$. We write

$$\sigma^r_{\alpha,k} = \delta_r \sigma_{\alpha,k}, \quad \lambda^r_{\alpha,k} = \delta_r \lambda_{\alpha,k}, \quad \Lambda^{K,r} = \delta_r \Lambda^K, \quad \tau^{E,r} = \delta_r \tau^E.$$

The Dirac measure concentrated at a point x is denoted by $\mathbf{e}_x$, and $\mathbf{e}$ is the measure concentrated at the identity of $\mathbf{G}$. We decompose $\mathbf{e}$ as

$$\mathbf{e} = \prod_{\alpha \in \mathcal{A}} [(\mathbf{e} - \sigma^r_{\alpha,0}) + \sigma^r_{\alpha,0}] = \prod_{\alpha \in \mathcal{A}} (\mathbf{e} - \sigma^r_{\alpha,0}) + \sum_{\emptyset \neq E \subseteq \mathcal{A}} c_E \tau^{E,r},$$

where the c_E are integers. Expanding $\mathbf{e} - \sigma^r_{\alpha,0} = \sum_{k>0} \lambda^r_{\alpha,k}$ for each α, we have

$$\mu = \sum_{K \in \mathbf{P}^+} \mu * \Lambda^{K,r} + \sum_{\emptyset \neq E \subseteq \mathcal{A}} c_E \mu * \tau^{E,r},$$

and dilating gives

$$\mu_J = \sum_{K \in \mathbf{P}^+} \mu_J * \Lambda^{K,r}_J + \sum_{\emptyset \neq E \subseteq \mathcal{A}} c_E \mu_J * \tau^{E,r}_J.$$

Thus to prove Theorem (6.1) we consider two operators

$$\mathcal{M}_1 : f \mapsto \sup_{J \in \mathbf{P}} \Big| f * \sum_{K \in \mathbf{P}^+} \mu_J * \Lambda^{K,r}_J \Big| \tag{6.2}$$

and

$$\mathcal{M}_2 : f \mapsto \sup_{J \in \mathbf{P}} \Big| f * \sum_{\emptyset \neq E \subseteq \mathcal{A}} c_E \mu_J * \tau^{E,r}_J \Big|, \tag{6.3}$$

and we prove that they satisfy the appropriate bounds on L^p. First we prove

(6.4) PROPOSITION. *For every $p > 1$ we have*

$$\|\mathcal{M}_1\|_{L^p \to L^p} \leq C_p (1 + \log^+\{\max |o_k|\})^q,$$

where $q = |\mathcal{A}| + \sum_{\alpha \in \mathcal{A}} d_\alpha$.

For a fixed K we define

$$M^{K,r} f(x) = \sup_{J \in \mathbf{P}} |f| * \mu_J * \Lambda^{K,r}_J(x). \tag{6.5}$$

Now we are going to prove the following propositions:

(6.6) PROPOSITION. *There are $\zeta, C, \varepsilon > 0$ such that for every r,*

$$\|M^{K,r}\|_{L^2\to L^2} \le Cr^{\zeta}e^{-\varepsilon|K|}.$$

(6.7) PROPOSITION. *For every $p > 1$ there exists a constant C_p such that for every r,*

$$\|M^{K,r}\|_{L^p\to L^p} \le C_p(1+|K|)^{\Sigma_{\alpha\in\mathcal{A}} d_\alpha}.$$

Proof of Proposition (6.6). For a fixed K we define

$$S_K f(x) = \Big(\sum_{J\in\mathbf{P}} |f * \mu_J * \varLambda_J^{K,r}(x)|^2\Big)^{1/2}.$$

Of course,

$$\sup_{J\in\mathbf{P}} |f * \mu_J * \varLambda_J^{K,r}| \le \Big(\sum_{J\in\mathbf{P}} |f * \mu_J * \varLambda_J^{K,r}(x)|^2\Big)^{1/2} = S_K f(x).$$

We write

$$T_J f = f * \mu_J * \varLambda_J^{K,r} \quad \text{and} \quad T = \sum_{J\in\mathbf{P}} \pm T_J$$

with an arbitrary choice of signs. We are going to prove that

$$\|T\|_{L^2\to L^2} \le Cr^{\zeta}e^{-\varepsilon|K|}, \tag{6.8}$$

where the constant is independent of the choice of signs. This will give the same bound on $\|S_K\|_{L^2\to L^2}$ and so the desired estimate on $\|M^{K,r}\|_{L^2\to L^2}$.

First notice that

$$\|T_J\|_{L^2\to L^2} \le C$$

uniformly in J in $\mathbf{P}$, K in $\mathbf{P}^+$ and $r > 0$, since the norms of the measures μ_J and $\varLambda_J^{K,r}$ are uniformly bounded.

We will prove that there are $\varepsilon, \zeta, c > 0$ such that

$$\|T_I^*T_J\|_{L^2\to L^2} + \|T_I T_J^*\|_{L^2\to L^2} \le cr^{\zeta}e^{-\varepsilon|I-J|-\varepsilon|K|} \tag{6.9}$$

for all $I, J \in \mathbf{P}$ and $K \in \mathbf{P}^+$. This, by the Cotlar–Stein lemma, implies

$$\|T\|_{L^2\to L^2} \le Cr^{\zeta}e^{-\varepsilon|K|}\sum_{I\in\mathbf{P}} e^{-\varepsilon|I|}$$

and so (6.8) follows. Thus it suffices to prove (6.9). To do this we write $\langle f, \mu_J^r\rangle = \langle f\circ\delta_{r^{-1}}, \mu_J\rangle$. Then

$$\langle f, \mu_J^r\rangle = \int_{\mathbb{R}^\chi} f\Big(\exp\Big\{\sum_{\alpha\in\mathcal{A}} e^{J_\alpha}u^\alpha X_\alpha\Big\}\Big)\psi_r(u)\,du,$$

where $\psi_r(u) = r^\chi\psi(ru_1,\ldots,ru_\chi)$. Moreover, $\mu_J * \varLambda_J^{K,r} = \delta_r(\mu_J^r * \varLambda_J^K)$.

We prove (6.9) where the operators T_J are replaced by the operators $f \mapsto f * \mu_J^r * \varLambda_J^K$. The support of the measure $\mu_J^r * \varLambda_J^K$ does not depend on

r any more and the estimate (6.9) is just Lemmas (4.3) and (4.4) of [Chr] except that the dependence on r is not explicit. To show that in fact it is as in (6.9) we examine carefully the proof of Lemma (4.5) in [Chr] and Lemma (3.4) in [Ch], which are the main tools in the proof of (4.3) and (4.4) in [Chr].

We reformulate Lemma (4.5) in [Chr] to emphasize the dependence of the estimate on $\|\psi_r\|_{C^1}$.

For two natural numbers $n \geq D$, with $D = \dim \mathbf{G}$, we consider a family $\mathcal{F}$ of functions $F : \mathbb{R}^n \to \mathbb{R}^D$ which satisfy the following conditions:

The coordinate functions F_i, $i = 1, \ldots, D$, are homogeneous polynomials (with respect to the usual dilations in $\mathbb{R}^n$) whose degrees are uniformly bounded by a number M, and for a compact set $\mathcal{K} \subset \mathbb{R}^n$,

$$\sup\{\|F\|_{C^\infty(\mathcal{K})} : F \in \mathcal{F}\} = C < \infty.$$

For a subset E of $\{1, \ldots, n\}$ with $|E| = D$, the Jacobian determinant

$$J_E = (\partial F/\partial x_\xi)_{\xi \in E}$$

is a homogeneous polynomial. We assume that for every F in $\mathcal{F}$ there is a set E_F and a multiindex γ_F such that $\partial^{\gamma_F} J_{E_F}/\partial x^{\gamma_F}$ is a constant and

$$\inf_{F \in \mathcal{F}} |\partial^{\gamma_F} J_{E_F}/\partial x^{\gamma_F}| > 0.$$

Finally, let $\phi \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp} \phi \subset \mathcal{K}$, and let $\xi_F = F_*(\phi dx)$ be the push-forward measure and $\mathcal{K}'$ a fixed compact set in $\mathbf{G} = \mathbb{R}^D$.

(6.10) (Reformulation of Lemma (4.5) of [Chr]). *Under the assumptions above there are constants $C, \varepsilon > 0$ such that for every measure σ supported in $\mathcal{K}'$, every $\varrho > 0$, every measure ν supported in a set of diameter ϱ such that $\int_{\mathbf{G}} d\nu = 0$ and $F \in \mathcal{F}$,*

$$\|\xi_F * \sigma * \nu\|_{L^1} \leq C\varrho^\varepsilon \|\phi\|_{C^1} \|\sigma\|_{L^1} \|\nu\|_{L^1}.$$

As in [Chr] we apply the above lemma to the function

$$\phi = \prod_{j=1}^{D} \prod_{i=1}^{4} \psi_r(x_j^i), \qquad x_j^i \in \mathbb{R}^\chi,$$

which satisfies

$$\|\phi\|_{C^1} \leq C r^{4D\chi+1},$$

and proceeding as in [Chr] we obtain (6.9). This completes the proof of Proposition (6.6).

To prove the L^p estimate, $p > 1$, we recall the following

(6.11) Lemma (cf. [St], [NS], [Sj]). *Let N be a nilpotent group and $E_1, \ldots$ $\ldots, E_n$ a basis of its Lie algebra. For every $p > 1$ there exists a constant c_p*

such that for the operator T defined by

$$Tf(x)=\sup_{m\in\mathbb{Z}^n}(r_1\dots r_n)^{-1}\int\limits_{|y_i-o_i|<r_i,\,i=1,\dots,n} f\Big(x\prod_{i=1}^n\exp(y_ie^{m_i}E_i)\Big)\,dy$$

we have

$$\|T\|_{L^p\to L^p}\le c_p\prod_{i=1}^n\bigg(1+\log_+\frac{|o_i|}{r_i}\bigg).$$

P r o o f o f P r o p o s i t i o n (6.7). Since $\mu=\int_{|u_i-o_i|<1}\delta_{F(u)}\psi(u)\,du$, it is enough to prove

$$\|\sup_J|f|*(\mathbf{e}_{F(u)}*\Lambda^{K,r})_J\|_{L^p}\le C_p(1+|K|)^{\Sigma_{\alpha\in\mathcal{A}}d_\alpha}.$$

For that consider the measure

$$\mathbf{e}_{\delta_{r^{-1}}F(u)}*\Lambda^K=\delta_{r^{-1}}(\mathbf{e}_{F(u)}*\Lambda^{K,r}).$$

Writing $\delta_{r^{-1}}F(u)=\prod_{\alpha\in\mathcal{A}}x_\alpha$, where $x_\alpha\in\exp\mathbf{g}^\alpha$, we have

$$\mathbf{e}_{\delta_{r^{-1}}F(u)}*\Lambda^K=\prod_{\alpha\in\mathcal{A}}\nu_\alpha,$$

where

$$\nu_\alpha=\mathbf{e}_{x_\alpha}*\mathbf{e}_{(\Pi_{\beta>\alpha}x_\beta)}*\lambda_{\alpha,K_\alpha}*\mathbf{e}_{(\Pi_{\beta>\alpha}x_\beta)^{-1}}$$

and all x_α belong to a compact set independent of r,u and K. Since $\exp\mathbf{g}^\alpha$ is a normal subgroup, ν_α is a smooth measure supported in $\exp\mathbf{g}^\alpha$. Moreover, there are $c,c_1,\dots,c_{d_\alpha}$ independent of r,u and K such that the density of ν_α is dominated by $|B_{K_\alpha}|^{-1}1_{B_{K_\alpha}}$, where $B_{K_\alpha}=\{x=\prod_{i=1}^{d_\alpha}\exp(x_iE_i):|x_i-c_i|<ce^{-K_\alpha}\}$. Therefore the operator

$$M_\alpha f(x)=\sup_J f*(\delta_r\nu_\alpha)_J(x)$$

both on $\exp\mathbf{g}^\alpha$ and G is dominated by the operator T of Lemma (6.11) with $|o_i|/r_i\le\ |c_i|ce^{K_\alpha}$, $i=1,\dots,d_\alpha$. Therefore

$$\|M_\alpha\|_{L^p\to L^p}\le C(1+K_\alpha)^{d_\alpha}.$$

Finally, the operator

$$\sup_J|f|*(\mathbf{e}_{F(u)}*\Lambda^{K,r})_J=\sup_J|f|*\Big(\prod_{\alpha\in\mathcal{A}}\delta_r\nu_\alpha\Big)_J$$

is dominated by the composition of the operators $M_\alpha,\alpha\in\mathcal{A}$, and Proposition (6.7) follows.

P r o o f o f P r o p o s i t i o n (6.4). To complete the proof of Proposition (6.4), we argue as in [Chr]. We take p' such that $1<p'<p<2$ and we interpolate between the L^2 estimate of Proposition (6.6) and the $L^{p'}$estimate of Proposition (6.7) to obtain

$$\| \sup_J |f| * \mu_J * \Lambda_J^{K,r} \|_{L^p} \le C_p r^{\zeta} e^{-\varepsilon|K|} \|f\|_{L^p}. \tag{6.12}$$

Consequently,

$$\begin{aligned}
\|\mathcal{M}_1\|_{L^p \to L^p} &\le \sum_{K \in \mathbf{P}} \|M^{K,r}\|_{L^p \to L^p} \\
&\le C_p \sum_{\{K : \max K < \varepsilon^{-1}(\zeta \log r + 1)\}} (1 + |K|)^{\Sigma_{\alpha \in \mathcal{A}} d_\alpha} \\
&\quad + C_p \sum_{\{K : \max K > \varepsilon^{-1}(\zeta \log r + 1)\}} r^{\zeta} e^{-\varepsilon |K|} \\
&\le C_p'(1 + \log r)^q,
\end{aligned}$$

which completes the proof of Proposition (6.4).

P r o o f o f T h e o r e m (6.1). To complete the proof of Theorem (6.1) it suffices to show that for $\mathcal{M}_2$ as defined in (6.3) we have

$$\|\mathcal{M}_2\|_{L^p \to L^p} \le C(1 + \log r)^q. \tag{6.13}$$

This will be proved as follows. First we prove our Theorem (6.1) for $|\mathcal{A}| = 1$, i.e. when $\mathbf{G} = \mathbb{R}$. Then we assume that the theorem is true for every $\mathcal{A}_0 \subset \mathcal{A}$ with $\mathcal{A}_0 \ne \mathcal{A}$. Under this assumption we prove (6.13) and thus complete the proof of Theorem (6.1).

If $|\mathcal{A}| = 1$, then $\mathbf{G} = \mathbb{R}$ and our maximal operator is of the form

$$\mathcal{M}f(x) = \sup_{\gamma \in \mathbb{Z}} \int_{\mathbb{R}^x} |f(x + e^{\gamma} u_1^{\alpha_1} \dots u_n^{\alpha_n})| \psi(u_1 \dots u_n)\, du_1 \dots du_n.$$

First we fix $u_2^{\alpha_2} \dots u_n^{\alpha_n} = b \ne 0$. Since γ runs over all integers, we may assume $1/2 \le b \le 1$ and so

$$\mathcal{M}f(x) \le 2^{1/\alpha_1} \int_{|u_k - o_k| < 2 k \ne 1} \Big(\sup_{\gamma \in \mathbb{Z}} \int_{|u_1 - o_1'| < 2} |f(x + e^{\gamma} u_1^{\alpha_1})|\, du_1 \Big) du_2 \dots du_n,$$

where $o_1' = b^{1/\alpha_1} o_1$. Now we consider the one-dimensional operator

$$\overline{\mathcal{M}} f = \sup_{\gamma \in \mathbb{Z}} \int_{|u - o| < 2} |f(x + e^{\gamma} u^{\alpha_1})|\, du,$$

and we prove that

$$\|\overline{\mathcal{M}}\|_{L^p \to L^p} \le C_p(1 + \log^+ |o|).$$

If $|o| > 3$ we change the variable $u^{\alpha_1} \to v$ and we see that

$$\begin{aligned}\overline{\mathcal{M}}f(x) &\le \sup_{\gamma\in\mathbb{Z}} \frac{1}{\alpha_1} \int\limits_{|v-o^{\alpha_1}|<c(|o|+2)^{\alpha_1-1}} |f(x+e^\gamma v)| v^{-(\alpha_1-1)/\alpha_1}\,dv \\ &\le \sup \frac{1}{\alpha_1}(|o|-2)^{-\alpha_1+1} \int\limits_{|v-o^{\alpha_1}|<c(|o|+2)^{\alpha_1-1}} |f(x+e^\gamma v)|\,dv,\end{aligned}$$

whence for $f \in L^p$ by Lemma (6.11),

$$\|\overline{\mathcal{M}}f\|_{L^p} \le C_p(1+\log^+|o|)\|f\|_{L^p}.$$

If $|o| < 3$, then

$$\overline{\mathcal{M}}f(x) \le \sup_{\gamma\in\mathbb{Z}} \int\limits_{|u|<4} |f(x+e^\gamma u^{\alpha_1})|\,du,$$

which is bounded by $C\|f\|_{L^p}$, as proved in [Chr].

Now we are going to prove the induction step. We show that for every $E \subset \mathcal{A}$ we have

$$\|\sup_J f * \mu_J * \tau_J^{E,r}\|_{L^p} \le C(1+\log r)^q \|f\|_{L^p}. \tag{6.14}$$

We fix E and we split $\mathbf{g}$ as $\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_\infty$, where

$$\mathbf{g}_0 = \operatorname{span}\{\mathbf{g}^\alpha : \alpha \in E\}, \qquad \mathbf{g}_\infty = \operatorname{span}\{\mathbf{g}_I : \forall_\alpha\ I_\alpha = 0\}.$$

Then $\mathbf{g}_0$ is an ideal in $\mathbf{g}$ and $\mathbf{g}_\infty$ is the free nilpotent Lie algebra of the same step as $\mathbf{g}$ on $|\mathcal{A}| - |E|$ generators.

Let $G_0 = \exp \mathbf{g}_0$ and $G_\infty = \exp \mathbf{g}_\infty$. Every element x in $\mathbf{G}$ admits a unique representation

$$x = wv, \qquad w \in G_\infty,\ v \in G_0, \tag{6.15}$$

$$x = v'w', \qquad w' \in G_\infty,\ v' \in G_0. \tag{6.16}$$

Since G_0 is a normal subgroup, $F(u) = F_\infty(u)F_0(u)$ with $F_\infty(u) = \exp(\sum_{\alpha\notin E} u^\alpha X_\alpha)$. Therefore

$$\mu * \tau^{E,r} = \int \mathbf{e}_{F_\infty(u)} * (\mathbf{e}_{F_0(u)} * \tau^{E,r})\psi(u)\,du.$$

Let $|\cdot|$ be a norm on $\mathbf{G}$ homogeneous with respect to the dilations δ_r and let $B_R = \{x \in G_0 : |x| < R\}$. Since $\delta_{r^{-1}}F(\operatorname{supp}\psi)$ and $\delta_{r^{-1}}F_\infty(\operatorname{supp}\psi)$ are contained in a bounded set, so does $\delta_{r^{-1}}F_0(\operatorname{supp}\psi)$, i.e. $F_0(\operatorname{supp}\psi) \subset B_{c_1 r}$. Also $\operatorname{supp}\tau^{E,r} \subset B_{c_2 r}$, whence

$$\operatorname{supp}(\mathbf{e}_{F_0(u)} * \tau^{E,r}) \subset B_{cr} \qquad \text{for } u \in \operatorname{supp}\psi.$$

Since $\tau^{E,r}$ has smooth density on G_0 and

$$\|\tau^{E,r}\|_{L^\infty} \le |B_r|^{-1}\|\tau^{E,1}\|_{L^\infty},$$

there exists a constant C independent of r such that

$$\mathbf{e}_{F_0(u)} * \tau^{E,r} \le C|B_{cr}|^{-1}\mathbf{1}_{B_{cr}}$$

and so

$$\mu * \tau^{E,r} \le C\Big(\int \mathbf{e}_{F_\infty(u)}\psi(u)\,du\Big) * |B_{cr}|^{-1}\mathbf{1}_{B_{cr}}.$$

The measure $\mu' = \int \mathbf{e}_{F_\infty(u)}\psi(u)\,du$ is supported in G_∞ and has the same properties as μ. Also both G_0 and G_∞ are invariant under the dilations $\mathbf{d}_J$, $J \in \mathbb{Z}^{\mathcal{A}}$. Therefore our maximal function is estimated by the composition of two operators,

$$M_1 f(x) = \sup\{f * \mu'_I(x) : I \in \mathbb{Z}^{\mathcal{A}\setminus E}\},$$

where μ'_I is defined in the same way as μ_J at the beginning of this section, and

$$\begin{aligned} N_1 f(x) &= \sup\{f * \mathbf{d}_J(\delta_r\nu)(x) : r \in \mathbb{R}^+,\ J \in \mathbb{Z}^{\mathcal{A}}\} \\ &\le \sup\{f * \mathbf{d}_J\nu(x) :\ J \in \mathbb{Z}^{\mathcal{A}}\} = N_2 f(x) \end{aligned}$$

where $\nu = |B_{\rm c}|^{-1}\mathbf{1}_{B_{\rm c}}$. In view of (6.15) and (6.16) it is sufficient to prove

$$\|M_1 f\|_{L^p(G_\infty)} \le C(1+\log r)^{|\mathcal{A}\setminus E| + \Sigma_{\alpha\in\mathcal{A}\setminus E} d_\alpha}\|f\|_{L^p(G_\infty)} \tag{6.17}$$

and

$$\|N_2 f\|_{L^p(G_0)} \le C\|f\|_{L^p(G_0)}. \tag{6.18}$$

But (6.17) is just our inductive hypothesis and (6.18) is proved by a simple iteration argument (Lemma (6.11)).

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