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A SINGULAR INITIAL VALUE PROBLEM FOR SECOND AND THIRD ORDER DIFFERENTIAL EQUATIONS

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1. Introduction. In this paper we consider two nonlinear second and third order differential equations with homogeneous initial values. First we study the equation

$$u''(x) = g(x)u(x)^{\beta} \qquad (x > 0,\ -1 < \beta < 1), \tag{1.1}$$

with the initial condition

$$u(0) = u'(0) = 0. \tag{1.2}$$

Next we apply the existence and uniqueness results obtained for the problem (1.1), (1.2) to the study of the initial value problem

$$u''' = g(u(x)), \tag{1.3}$$

$$u(0) = u'(0) = u''(0) = 0. \tag{1.4}$$

Throughout the paper we assume that g satisfies the conditions

$$g \in C(0,\infty), \quad g(x) \ge 0 \ \text{ for } x > 0, \tag{1.5}$$

(1.6) there exists $m \ge 0$ such that $x^m g(x)$ is bounded as $x \to 0+$,

$$0 < \int_0^{\delta} g(s)s^{\beta}\, ds < \infty \qquad \text{for some } \delta > 0. \tag{1.7}$$

Recently the equation (1.1) with $g \le 0$ and $-1 < \beta < 0$ was considered in [3], [4] as a model for some problems of applied mathematics. Unfortunately, the technical arguments used therein involved the concavity properties of solutions. Therefore those methods are inapplicable in our case, where u is convex.

The results obtained in this paper generalize previous ones in [8], where the initial value problem (1.3), (1.4) was considered with g satisfying (1.6) with $m = 1/2$.

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We are interested in the existence of nonnegative solutions $u \in C[0,\infty) \cap C^2(0,\infty)$ to the problem (1.1), (1.2) and we study the maximal solution of this problem in the sense of [6].

Using the method of the initial values perturbation we see that the initial value problem

$$u_\varepsilon''(x) = g(x)u_\varepsilon(x)^\beta \qquad (x > \varepsilon),$$
$$u_\varepsilon(\varepsilon) = u_\varepsilon'(\varepsilon) = 0,$$

where $0 < \beta < 1$ and $\varepsilon > 0$ is chosen so that $g(\varepsilon) > 0$, has a solution u_ε positive for $x > \varepsilon$. Taking $u_\varepsilon(x) = 0$ for $0 \le x < \varepsilon$ the function u_ε becomes a solution of (1.1), (1.2). Hence it follows easily that in the case $0 < \beta < 1$ the maximal solution of (1.1), (1.2), if it exists, is positive for $x > 0$. If $-1 < \beta \le 0$ the same result is obtained immediately.

Before stating our results we introduce some auxiliary definitions and notations.

Let g satisfy (1.5), (1.6). We put

$$g^*(x) = x^{-m} \sup_{0<s<x} s^m g(s) \qquad \text{for } x > 0. \tag{1.8}$$

We easily see that $g(x) \le g^*(x)$ for $x > 0$ and $x^m g^*(x)$ is nondecreasing.

We will deal with two function classes $\mathcal{K}_0$ and $\mathcal{K}^*$ defined as follows:

$$\mathcal{K}_0 = \{g : g \text{ satisfies (1.5), (1.6) and } x^m g(x) \text{ is nondecreasing}\},$$
$$\mathcal{K}^* = \{g : g \text{ satisfies (1.5)–(1.7) and } \sup_{0<x} G^*(x)/G(x) < \infty\},$$

where

$$G(x) = \int_0^x g(s)s^\beta\,ds, \qquad G^*(x) = \int_0^x g^*(s)s^\beta\,ds.$$

Some a priori estimates of solutions to (1.1), (1.2) are established in the following theorem and remark.

Theorem 1.1. *Let $g \in \mathcal{K}_0$, $-1 < \beta < 1$ and u be a solution to* (1.1), (1.2) *positive for $x > 0$. Then there exist constants $c_1,\ c_2 > 0$ such that*

$$c_1 x\left(\frac{u(x)}{x}\right)^{1-\beta} \le \int_0^x (x-s)g(s)s^\beta\,ds \le c_2 x\left(\frac{u(x)}{x}\right)^{1-\beta}. \tag{1.9}$$

Remark 1.1. If $g \in \mathcal{K}^*$ *and* $-1 < \beta \le 0$, then the a priori estimates in (1.9) are still valid.

The existence result for (1.1), (1.2) is stated in the following theorem and its corollary.

Theorem 1.2. *Let $g \in \mathcal{K}_0$. Then the condition* (1.7) *is necessary and sufficient for the existence of a unique solution to the problem* (1.1), (1.2) *positive for $x > 0$.*

COROLLARY 1.2. *Let $g \in \mathcal{K}^*$. Then the problem* (1.1), (1.2) *has a maximal solution. If $-1 < \beta \le 0$, then it is the unique solution positive for $x > 0$.*

The above results applied to the study of the problem (1.3), (1.4) allow us to obtain

THEOREM 1.3. *Let $g \in \mathcal{K}^*$. Then the problem* (1.3), (1.4) *has a unique continuous solution u positive for $x > 0$ if and only if*

$$\int_0^\delta \Big\{ s^{1/2} \int_0^s (s-t) g(t) t^{-1/2}\, dt \Big\}^{-1/3} ds < \infty \tag{1.10}$$

for some $\delta > 0$.

We also give a condition for the blow-up of solutions which means that there exists $0 < L < \infty$ such that $\lim_{x \to L-} u(x) = \infty$.

THEOREM 1.4. *Let $g \in \mathcal{K}^*$. The continuous solution u to* (1.3), (1.4) *positive for $x > 0$ blows up if and only if*

$$\int_0^\infty \Big\{ s^{1/2} \int_0^s (s-t) g(t) t^{-1/2}\, dt \Big\}^{-1/3} ds < \infty.$$

The condition (1.10) is called the generalized Osgood condition for the problem (1.3), (1.4). Such conditions for ordinary differential equations $u^{(n)}(x) = g(u(x))$ with homogeneous initial values, and more generally for convolution type integral equations $u(x) = \int_0^x k(x-s) g(u(s))\, ds$, have been widely studied (see [5], [7], [2]). Unfortunately, only the case of nondecreasing functions g was considered there. Theorems 1.3 and 1.4 of the present paper are corresponding results obtained for functions g which can oscillate at 0. Some examples of the problem (1.3), (1.4) with g like $|\sin(1/x)|$ have been given in [8].

2. Proofs of theorems.

Technical arguments used in our considerations employ the fact that the considered solutions u are convex. Some properties of convex functions needed in the sequel are collected in the following lemma.

LEMMA 2.1. *Let $w''(x) \ge 0$ for $x > 0$ and $w(x) = \int_0^x (x-s) w''(s)\, ds$. Then*

(i) $\quad xw' - w$ *and* w/x *are nondecreasing for* $x > 0$;

if $x^m w''$ is nondecreasing for some $m \ge 0$, then

(ii) $\quad (xw' - w)^2 \le 2x^2 w'' w + m w (xw' - w) \quad (x > 0);$

if $w'' \in \mathcal{K}^*$, *then for each* $\gamma \in (-1/2, \infty)$ *there exist constants* $c_1(\gamma), c_2(\gamma) > 0$ *such that*

$$\text{(iii)}\qquad c_1(\gamma)x\left(\frac{w(x)}{x}\right)^{1+\gamma} \le \int_0^x (x-s)w''(s)\left(\frac{w(s)}{s}\right)^{\gamma} ds \le c_2(\gamma)x\left(\frac{w(x)}{x}\right)^{1+\gamma} \qquad (x>0).$$

P r o o f. The property (i) is well known for convex functions.

Since $xw' - w$ and $x^m w''$ are nondecreasing, (ii) can be obtained as follows:

$$x^m(xw'-w)^2 = 2\int_0^x s^{m+1}w''(sw'-w)\,ds + m\int_0^x s^{m-1}(sw'-w)^2\,ds \le 2x^{m+2}w''w + mx^m w(xw'-w) \qquad (x>0).$$

To prove (iii) we first consider an auxiliary function $\widetilde{w}$ defined by $\widetilde{w}(x) = \int_0^x (x-s)(w'')^*(s)\,ds$, where $(w'')^*$ is defined by (1.8). We will show that $\widetilde{w}$ satisfies (iii). Since $x^m\widetilde{w}''$ is nondecreasing and

$$\frac{1}{1+\gamma}\left(x\left(\frac{\widetilde{w}(x)}{x}\right)^{1+\gamma}\right)'' = \widetilde{w}''(x)\left(\frac{\widetilde{w}(x)}{x}\right)^{\gamma} + \gamma x^{-3}(x\widetilde{w}'(x) - \widetilde{w}(x))^2\left(\frac{\widetilde{w}(x)}{x}\right)^{\gamma-1} \qquad (\gamma \ne -1),$$

the required estimates will be obtained by an application of (ii).

In the case $-1/2 < \gamma \le 0$ we derive the inequalities

$$\frac{1}{1+\gamma}\left(x\left(\frac{\widetilde{w}(x)}{x}\right)^{1+\gamma}\right)'' \le \widetilde{w}''(x)\left(\frac{\widetilde{w}(x)}{x}\right)^{\gamma};$$

$$\text{(2.1)}\qquad (1+2\gamma)\widetilde{w}''(x)\left(\frac{\widetilde{w}(x)}{x}\right)^{\gamma} \le \frac{1}{1+\gamma}\left(x\left(\frac{\widetilde{w}(x)}{x}\right)^{1+\gamma}\right)'' - \frac{m\gamma}{1+\gamma}\left(\left(\frac{\widetilde{w}(x)}{x}\right)^{1+\gamma}\right)'$$

valid for $x>0$, which give the inequality (iii) for $\widetilde{w}$ with

$$\widetilde{c}_1(\gamma) = \frac{1}{1+\gamma} \quad \text{and} \quad \widetilde{c}_2(\gamma) = \frac{1-m\gamma}{(1+\gamma)(1+2\gamma)}.$$

In the case $\gamma > 0$ we can proceed as previously to derive two inequalities as (2.1) with reverse signs, from which it follows that the right inequality in (iii) is true for any $\gamma > 0$ with $\widetilde{c}_2(\gamma) = 1/(1+\gamma)$ and the left one for $0 < \gamma < 1/m$ with $\widetilde{c}_1(\gamma) = (1-m\gamma)/((1+\gamma)(1+2\gamma))$.

To complete the proof of (iii) for $\widetilde{w}$ we employ the Jensen inequality

$$\frac{1}{\widetilde{w}(x)}\int\limits_0^x (x-s)\widetilde{w}''(s)\left(\frac{\widetilde{w}(s)}{s}\right)^{n\gamma} ds \ge \left(\frac{1}{\widetilde{w}(x)}\int\limits_0^x (x-s)\widetilde{w}''(s)\left(\frac{\widetilde{w}(s)}{s}\right)^{\gamma} ds\right)^n$$

valid for $\gamma > 0$ and $n > 1$.

We also easily verify that

$$\lim_{\gamma\to 0+}\widetilde{c}_1(\gamma) = \lim_{\gamma\to 0+}\widetilde{c}_2(\gamma) = 1. \tag{2.2}$$

Now we are ready to consider w. By the definition of $\mathcal{K}^*$ we have

$$A\widetilde{w}'(x) \le w'(x) \le \widetilde{w}'(x) \qquad (x > 0), \tag{2.3}$$

for some constant $0 < A < 1$. Since $w''(x) \le \widetilde{w}''(x)$, from (2.3) we get

$$w''(x)\left(\frac{w(x)}{x}\right)^{\gamma} \le \max(1, A^{\gamma})\widetilde{w}''(x)\left(\frac{\widetilde{w}(x)}{x}\right)^{\gamma} \qquad (x > 0,\ \gamma > -1/2),$$

which gives the right inequality in (iii) with $c_2(\gamma) = \max(1, A^{\gamma})A^{-(1+\gamma)}\widetilde{c}_2(\gamma)$ for $\gamma > -1/2$.

We prove the left inequality in two steps.

When $\gamma \in (-1/2, 0]$, the proof is easy because $(w(s)/s)^{\gamma}$ is a nonincreasing function. In that case we can take $c_1(\gamma) = 1$.

In the case $\gamma > 0$ we first observe that

$$\int\limits_0^x (x-s)w''(s)\left(\frac{w(s)}{s}\right)^{\gamma} ds \ge A^{\gamma}\int\limits_0^x (x-s)w''(s)\left(\frac{\widetilde{w}(s)}{s}\right)^{\gamma} ds \tag{2.4}$$

for $x > 0$. An integration by parts applied to the integral on the right hand side and an application of (2.3) allow us to write

$$\begin{aligned}\int\limits_0^x (x-s)w''(s)\left(\frac{\widetilde{w}(s)}{s}\right)^{\gamma} ds &\ge \int\limits_0^x (x-s)\widetilde{w}''(s)\left(\frac{\widetilde{w}(s)}{s}\right)^{\gamma} ds \\ &\quad + (A-1)\int\limits_0^x \widetilde{w}'(s)\left(\frac{\widetilde{w}(s)}{s}\right)^{\gamma} ds \qquad (x > 0).\end{aligned} \tag{2.5}$$

The second integral on the right hand side can be estimated as follows:

$$\begin{aligned}\frac{1}{1+\gamma}x\left(\frac{\widetilde{w}(x)}{x}\right)^{1+\gamma} &\le \int\limits_0^x \widetilde{w}'(x)\left(\frac{\widetilde{w}(s)}{s}\right)^{\gamma} ds \\ &\le x\left(\frac{\widetilde{w}(x)}{x}\right)^{1+\gamma} \qquad (x > 0).\end{aligned} \tag{2.6}$$

Combining (2.4)–(2.6) we get

$$\int\limits_0^x (x-s)w''(s)\left(\frac{w(s)}{s}\right)^{\gamma} ds \ge c_1(\gamma)x\left(\frac{\widetilde{w}(x)}{x}\right)^{1+\gamma} \qquad (x > 0)$$

with $c_1(\gamma) = A^\gamma(\widetilde{c}_1(\gamma) + A - 1)$. Since in view of (2.2), $\lim_{\gamma\to 0+} c_1(\gamma) = A$, the left inequality in (iii) is valid for small $0 < \gamma$. For other values of $\gamma > 0$, we can use the same arguments as those based on the application of the Jensen inequality used in the case of $\widetilde{w}$.

The a priori estimates for solutions to the problem (1.1), (1.2) can be derived as follows.

Proof of Theorem 1.1. First we note that $u''(s)(u(s)/s)^{-\beta} = g(s)s^\beta$. We obtain, as in the proof of Lemma 1.1(ii), the inequality

$$\begin{aligned} x^m(xu'-u)^2 &= 2\int_0^x s^{m+3}u''(s)\left(\frac{u(s)}{s}\right)^{-\beta}\left(\frac{u(s)}{s}\right)'\left(\frac{u(s)}{s}\right)^{\beta} ds \\ &\quad + m\int_0^x s^{m-1}(su'-u)^2\, ds \\ &\le \frac{2}{1+\beta}x^{m+2}u''u + mx^m u(xu'-u) \end{aligned}$$

valid for $x > 0$, from which it follows that

$$\begin{aligned} (2.7)\qquad &(xu'(x)-u(x))^2 \\ &\le \frac{2}{1+\beta}x^2u''(x)u(x) + mu(x)(xu'(x)-u(x)) \qquad (x>0). \end{aligned}$$

Since

$$\begin{aligned} (2.8)\qquad &\frac{1}{1-\beta}\left(x\left(\frac{u(x)}{x}\right)^{1-\beta}\right)'' \\ &= u''(x)\left(\frac{u(x)}{x}\right)^{-\beta} - \beta x^{-3}(xu'(x)-u(x))^2\left(\frac{u(x)}{x}\right)^{-\beta-1} \qquad (x>0), \end{aligned}$$

in the case $0 < \beta$ we can apply (2.7) to obtain the following two inequalities:

$$\frac{1}{1-\beta}\left(x\left(\frac{u(x)}{x}\right)^{1-\beta}\right)'' \le u''(x)\left(\frac{u(x)}{x}\right)^{-\beta},$$

$$\frac{1-\beta}{1+\beta}u''(x)\left(\frac{u(x)}{x}\right)^{-\beta} \le \frac{1}{1-\beta}\left(x\left(\frac{u(x)}{x}\right)^{1-\beta}\right)'' + \frac{m\beta}{1-\beta}\left(\left(\frac{u(x)}{x}\right)^{1-\beta}\right)'$$

valid for $x > 0$, which give the required estimates with

$$c_1 = \frac{1}{1-\beta} \quad \text{and} \quad c_2 = \frac{(1+m\beta)(1+\beta)}{(1-\beta)^2}.$$

Now we can consider the case of $-1 < \beta \le 0$. From (2.8) we get

$$(2.9)\qquad 0 \le u''(x)\left(\frac{u(x)}{x}\right)^{-\beta} \le \frac{1}{1-\beta}\left(x\left(\frac{u(x)}{x}\right)^{1-\beta}\right)'' \qquad (x>0),$$

which gives the right inequality in (1.9) with $c_2 = 1/(1-\beta)$.

The left inequality can be proved as follows. In view of (2.9) we define an auxiliary function $w(x) = \int_0^x (x-s)g(s)s^\beta\,ds$ and obtain the inequality

$$w(x) \le \frac{1}{1-\beta} x \left(\frac{u(x)}{x}\right)^{1-\beta} \qquad (x>0),$$

from which it follows that

$$\begin{aligned} 0 \le u''(x) &= g(x)u^\beta(x) \\ &\le (1-\beta)^{\beta/(1-\beta)} w''(x) \left(\frac{w(x)}{x}\right)^{\beta/(1-\beta)} \qquad (x>0). \end{aligned}$$

Since for $-1 < \beta \le 0$ we have $-1/2 < \beta/(1-\beta) \le 0$, by an application of Lemma 1.1(iii) we obtain the inequality

$$\begin{aligned} u(x) &\le (1-\beta)^{\beta/(1-\beta)} \int_0^x (x-s) w''(s) \left(\frac{w(s)}{s}\right)^{\beta/(1-\beta)} ds \\ &\le c_2 x \left(\frac{w(x)}{x}\right)^{1/(1-\beta)} \end{aligned}$$

valid for $x > 0$, from which the required inequality follows immediately.

Proof of Remark 1.1. The proof is exactly the same as that of Theorem 1.1 in the case of $-1 < \beta \le 0$.

Now we are ready to consider the existence problem for (1.1), (1.2).

Proof of Theorem 1.2. In view of the proved a priori estimates the necessity part of the theorem is obvious.

Now assuming that the condition (1.7) is satisfied we can define auxiliary functions $w(x) = \int_0^x (x-s)g(s)s^\beta\,ds$ and $\varphi(x) = x(w(x)/x)^{1/(1-\beta)}$ $(x>0)$. We look for solutions to (1.1), (1.2) in the function cone

$$\mathcal{X}_\beta = \{v \in C[0,\infty) : \text{there exist constants } c_1, c_2 > 0 \text{ such that} \\ c_1\varphi(x) \le v(x) \le c_2\varphi(x),\ x>0\},$$

as fixed points of the integral operator

$$T_\beta v(x) = \int_0^x (x-s)g(s)v^\beta(s)\,ds$$

defined on $\mathcal{X}_\beta$. Since

$$T_\beta\varphi(x) = \int_0^x (x-s)w''(s)(w(s)/s)^{\beta/(1-\beta)}\,ds$$

and $\beta/(1-\beta) > -1/2$ for $-1 < \beta < 1$, from Lemma 1.1(iii) and the monotonicity properties of T_β it follows that T_β maps $\mathcal{X}_\beta$ into $\mathcal{X}_\beta$.

We introduce a pseudometric ϱ in $\mathcal{X}_\beta$ by

$$\varrho(v_1, v_2) = \ln \frac{M(v_1 \mid v_2)}{m(v_1 \mid v_2)} \quad (v_1, v_2 \in \mathcal{X}_\beta),$$

where

$$m(v_1 \mid v_2) = \inf_{s>0} \frac{v_1(s)}{v_2(s)}, \quad M(v_1 \mid v_2) = \sup_{s>0} \frac{v_1(s)}{v_2(s)},$$

which becomes a metric $\widetilde{\varrho}$ in the quotient space $\widetilde{\mathcal{X}}_\beta = \mathcal{X}_\beta/\sim$, where

$$v_1 \sim v_2 \quad \text{if and only if} \quad v_1 = \lambda v_2 \text{ for some } \lambda > 0.$$

Moreover, $(\widetilde{\mathcal{X}}_\beta, \widetilde{\varrho})$ is a complete metric space (see [1], [9]).

Since $T_\beta(\lambda v) = \lambda^\beta T_\beta(v)$ for any $v \in \mathcal{X}_\beta$ and $\lambda > 0$, we can consider T_β on $\widetilde{\mathcal{X}}_\beta$. From the monotonicity properties of T_β it follows that

$$\widetilde{\varrho}(T_\beta \widetilde{v}_1, T_\beta \widetilde{v}_2) \le |\beta| \widetilde{\varrho}(\widetilde{v}_1, \widetilde{v}_2) \quad \text{for any } \widetilde{v}_1, \widetilde{v}_2 \in \widetilde{\mathcal{X}}_\beta,$$

which allows us to find a unique solution $u \in \mathcal{X}_\beta$ to the problem (1.1), (1.2) by a contraction argument. In view of the a priori estimates (1.9) this must be the unique solution of that problem positive for $x > 0$.

Proof of Corollary 1.2. The same arguments as those used in the proof of Theorem 1.2 show that the problem (1.1), (1.2) has a unique solution u in $\mathcal{X}_\beta$. We will prove that it is maximal.

In the case $-1 < \beta \le 0$ the proof is easy because in view of Remark 1.1, u must be the unique continuous solution to (1.1), (1.2).

In the case $0 < \beta < 1$, for any solution v to (1.1), (1.2) we get

$$\begin{aligned} v(x) &= \int_0^x (x-s) g(s) s^\beta \left(\frac{v(s)}{s} \right)^\beta ds \\ &\le \int_0^x (x-s) g(s) s^\beta \, ds \left(\frac{v(x)}{x} \right)^\beta \quad (x > 0). \end{aligned}$$

Hence it follows that $v(x) \le \varphi(x)$ for $x > 0$. Therefore we can find a constant $c > 0$ such that $v(x) \le cu(x)$ for $x > 0$, and by using an iteration process we obtain the inequality

$$v(x) = T^n v(x) \le T^n(cu)(x) = c^{\beta^n} u(x) \quad (x > 0),$$

which gives the required result as $n \to \infty$.

Now we consider the initial value problem for the third order differential equation. Substituting $v(x) = 2^{-2/3} u'(u^{-1}(x))^2$ in the problem (1.3), (1.4), where u^{-1} is the inverse function to u, we see that v satisfies

$$\begin{aligned} v''(x) &= g(x) v^{-1/2}(x) \quad (x > 0) \\ v(0) &= v'(0) = 0. \end{aligned} \tag{2.10}$$

P r o o f o f T h e o r e m s 1.3 a n d 1.4. Since $v^{-1/2}(x) = 2^{1/3}(u^{-1})'(x)$, it suffices to apply the estimates

$$c_1 x\left(\frac{v(x)}{x}\right)^{3/2} \le \int_0^x (x-s)g(s)s^{-1/2}\,ds \le c_2 x\left(\frac{v(x)}{x}\right)^{3/2} \qquad (x>0)$$

for solutions of (2.10) obtained by Remark 1.1.

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