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BOHR CLUSTER POINTS OF SIDON SETS

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It is a long standing open problem whether Sidon subsets of $\mathbb{Z}$ can be dense in the Bohr compactification of $\mathbb{Z}$ ([LR]). Yitzhak Katznelson came closest to resolving the issue with a random process in which almost all sets were Sidon and and almost all sets failed to be dense in the Bohr compactification [K]. This note, which does not resolve this open problem, supplies additional evidence that the problem is delicate: it is proved here that if one has a Sidon set which clusters at even one member of $\mathbb{Z}$, one can construct from it another Sidon set which is dense in the Bohr compactification of $\mathbb{Z}$. A weaker result holds for quasi-independent and dissociate subsets of $\mathbb{Z}$.

Cluster points. By the definition of the Bohr topology, a subset $E \subset \mathbb{Z}$ clusters at q if and only if, for all $\varepsilon \in \mathbb{R}^+$, for all $n \in \mathbb{Z}^+$, and for all $(t_1, \ldots, t_n) \in \mathbb{T}^n$, there is some $m \in E$ such that

$$\sup_{1 \le i \le n} |\langle m, t_i \rangle - \langle q, t_i \rangle| < \varepsilon. \tag{1}$$

Here $\mathbb{T}$ is the dual group of $\mathbb{Z}$ and $\langle m, t \rangle$ denotes the result of the character m acting on t. Thus, if $\mathbb{T}$ is represented as $[-\pi, \pi)$ with addition mod 2π,

$$\langle m, t \rangle = e^{imt}.$$

If, for all $(t_1, \ldots, t_n) \in \mathbb{T}^n$, there is at least one $m \in E$ such that inequality (1) holds, then E is said to *approximate* q *within* ε *on* $\mathbb{T}^n$.

Overview. Let E be a Sidon subset of the integers $\mathbb{Z}$ which clusters at the integer $q \in \mathbb{Z}$ in the topology of the Bohr compactification. The dense Sidon set will have the form

$$S = \bigcup_{j=1}^{\infty} S_j, \qquad \text{with } S_j = x_j + k_j(E_j - q),$$

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where $E_j \subset E$ approximates q within $1/m_j$ on $\mathbb{T}^{n_j}$ under an exhaustive enumeration (x_j, n_j, m_j) of $\mathbb{Z} \times \mathbb{Z}^+ \times \mathbb{Z}^+$. Lemma 1 below asserts that finite $E_j \subset E$ can always be found. Lemma 3 below says that S is dense, regardless of the dilation factors k_j. The final step of the argument is to choose k_j's so that S is Sidon. Lemma 4 does this in part for N-independent sets (N-independent generalizes quasi-independent and dissociate; it is defined below). It is then a short step to Sidon sets, using a criterion of Gilles Pisier's.

LEMMA 1 (Compactness). *Let $E \subset \mathbb{Z}$ cluster at $q \in \mathbb{Z}$ in the topology of the Bohr compactification of $\mathbb{Z}$. For every $\varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, there is a finite subset $E' \subset E$ which approximates q within ε on $\mathbb{T}^n$.*

Proof. Let $\varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ be given. For each $(t_1, \ldots, t_n) \in \mathbb{T}^n$ there is some $m \in E$ such that (1) holds with $\varepsilon/2$ in the role of ε. By the continuity of the characters m and q on $\mathbb{T}$ (both are in $\mathbb{Z}$), there is an open neighborhood U of $(t_1, \ldots, t_n) \in \mathbb{T}^n$ for which (1) is valid when $(s_1, \ldots, s_n) \in U$ are substituted for $(t_1, \ldots, t_n)$. By the compactness of $\mathbb{T}^n$, a finite number of such U's cover $\mathbb{T}^n$. The set of m's corresponding to the U's can be taken for the set E'. ∎

For integers k, y, and z, and for $S \subset \mathbb{Z}$, let $z + k(S - y)$ denote $\{z + k(x - y) \mid x \in S\}$.

LEMMA 2 (Dilation). *Let k, y, and z be integers. If S approximates y within ε on $\mathbb{T}^n$, then $z + k(S - y)$ approximates z within ε on $\mathbb{T}^n$.*

Proof. Let $(t_1, \ldots, t_n) \in \mathbb{T}^n$. There is some $m \in S$ such that

$$\sup_{1 \le i \le n} |\langle m, kt_i \rangle - \langle y, kt_i \rangle| < \varepsilon.$$

Because m and k are integers, $\langle m, kt \rangle = \langle mk, t \rangle$. Therefore,

$$\begin{aligned} |\langle z + k(m - y), t_i \rangle - \langle z, t_i \rangle| &= |\langle z - ky, t_i \rangle (\langle km, t_i \rangle - \langle ky, t_i \rangle)| \\ &= |\langle m, kt_i \rangle - \langle y, kt_i \rangle| < \varepsilon, \end{aligned}$$

for $1 \le i \le n$. ∎

LEMMA 3 (Denseness). *Let (x_j, n_j, m_j), $j \in \mathbb{Z}^+$, exhaustively enumerate $\{(x, n, m) \mid x \in \mathbb{Z},\ n \in \mathbb{Z}^+,\ m \in \mathbb{Z}^+\}$. Suppose there is a sequence $\{E_j\}_{j=1}^{\infty}$ of subsets of $\mathbb{Z}$ such that E_j approximates p_j within $1/m_j$ on $\mathbb{T}^{n_j}$. Then for any sequence of integers k_j, $S = \bigcup_{j=1}^{\infty} (x_j + k_j(E_j - p_j))$ is dense in the Bohr compactification of $\mathbb{Z}$.*

Proof. Since $\mathbb{Z}$ is dense in its Bohr compactification, it suffices to show that the closure of S includes every $x \in \mathbb{Z}$. Let $x \in \mathbb{Z}$. By the definition of the Bohr topology, we must show that S approximates x within ε on $\mathbb{T}^n$ for any $\varepsilon \in \mathbb{R}^+$ and any $n \in \mathbb{Z}^+$. Choose some $m \in \mathbb{Z}^+$ such that $1/m < \varepsilon$.

The triple (x,n,m) is (x_j,n_j,m_j) for some j. Since E_j approximates p_j within $1/m_j$ on $\mathbb{T}^{n_j}$, the Dilation Lemma implies that $x_j + k_j(E_j - p_j)$ approximates x_j within $1/m_j$ on $\mathbb{T}^{n_j}$ and hence x within ε on $\mathbb{T}^n$. ∎

Definition. Let N be a positive integer and G be an additive group. An N-*relation* is a linear combination

$$\sum_{x\in G} \alpha_x x = 0,$$

where α_x an integer in $[-N,N]$ for all x and with $\alpha_x \neq 0$ for at most finitely many x. A subset A of G is said to be N-*independent* if and only if the only N-relation among its elements is the trivial relation which has all coefficients equal to 0. The N-*relation hull* of A, written $[A]_N$, is

$$\Big\{ \sum_{x\in A} \alpha_x x \,\Big|\, \alpha_x \in \{-N,-N+1,\ldots,N\}\Big\}.$$

The hull of the empty set is understood to be $\{0\}$ (1).

Quasi-independent sets are the 1-independent sets, while dissociate sets are the 2-independent sets ([P], [LR]).

Lemma 4. *Let* $\{W_j\}_{j=1}^{\infty}$ *be a sequence of finite* N-*independent subsets of* $\mathbb{Z}$. *Let* x_j *be arbitrary integers*, $1\le j<\infty$. *Set* D_j *equal to the maximum absolute value of the elements of* $[\bigcup_{i<j}(x_i+k_iW_i)]_N$, *and let* M_j *denote the size of* W_j. *If* $k_j>D_j+NM_j|x_j|$ *for all* $j\ge 1$, *then* $\bigcup_{j=1}^{\infty}(x_j+k_jW_j)$ *is* N-*independent. Moreover*, *the sets* $x_j+k_jW_j$ *are disjoint for distinct values of* j.

Proof. Let $W_i' = x_i + k_iW_i$, and set

$$V_j = \bigcup_{i<j} W_i'.$$

Since $V_1=\emptyset$, it is certainly N-independent. Assume that V_j is N-independent for some $j\ge 1$, and that W_{i_1}' and W_{i_2}' are disjoint for $i_1\neq i_2$ with $i_1<j$ and $i_2<j$. Consider V_{j+1}. It will be proved first that W_j' is disjoint from V_j. Let $x\in W_j'$ and $y\in V_j$. Then $x = x_j + k_jx'$ for some $x'\in W_j$. Since W_j is N-independent, $0\notin W_j$ and thus $x'\neq 0$. Therefore, since $V_j\subset [V_j]_N$,

$$|x| = |x_j + k_jx'| \ge k_j - |x_j| > D_j + NM_j|x_j| - |x_j| \ge D_j \ge |y|.$$

Next, consider an N-relation on V_{j+1} with coefficients α_x for $x\in V_{j+1}$. Since W_j' is disjoint from V_j, one may write

$$\sum_{x\in W_j'} \alpha_x x = -\sum_{x\in V_j} \alpha_x x = \tau,$$

(1) This definition is distinct from that of J. Bourgain, who defined N-independence to be a weaker version of quasi-independence.

for some $\tau \in [V_j]_N$. Each $x \in W'_j$ has the form $x_j + k_j x'$ for some x' in W_j (x' is unique since $k_j > 0$). Thus,

$$k_j \sum_{x \in W'_j} \alpha_x x' = \tau - x_j \sum_{x \in W'_j} \alpha_x. \tag{2}$$

Suppose that $\sum_{x \in W'_j} \alpha_x x' \neq 0$. Then, by equation (2),

$$\begin{aligned} k_j &\le \Big| k_j \sum_{x \in W'_j} \alpha_x x' \Big| = \Big| \tau - x_j \sum_{x \in W'_j} \alpha_x \Big| \\ &\le |\tau| + |x_j| \cdot \Big| \sum_{x \in W'_j} \alpha_x \Big| \le D_j + |x_j| N M_j, \end{aligned}$$

which is contrary to $k_j > D_j + N M_j |x_j|$. Thus $\sum_{x \in W'_j} \alpha_x x' = 0$. This is an N-relation among the elements of W_j (since x' is unique for each x, and vice versa). Since W_j is N-independent, $\alpha_x = 0$ for $x \in W'_j$. It follows that equation (2) reduces to $\tau = 0$, which is an N-relation supported on V_j. Since V_j is N-independent, $\alpha_x = 0$ for all $x \in V_j$ and hence for all $x \in V_{j+1} = V_j \cup W'_j$. Thus only the trivial relation occurs among the N-relations on V_{j+1}.

Finally, since $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}^+$ and

$$S = \bigcup_{i=1}^{\infty} W'_i = \bigcup_{j=1}^{\infty} V_j,$$

the N-independence of the V_j's makes S be N-independent. [Any N-relation on S has at most finitely many non-zero coefficients (by definition); thus it must be supported on V_j for some j (since S is an increasing union of the V_j's) and hence is trivial because V_j is N-independent.] ■

PROPOSITION 5. *If there is a Sidon set E which clusters at some $n \in \mathbb{Z}$ in the topology of the Bohr compactification of $\mathbb{Z}$, then there is a Sidon set which is dense in the Bohr compactification of $\mathbb{Z}$.*

Proof. By Lemma 2, $E' = E - n$ clusters at 0 in the Bohr topology; it is well known that E' is Sidon, in fact with the same Sidon constant as E ([LR]). By the definition of cluster point, we may assume $0 \notin E'$. As provided by Lemma 1, for any positive integers n and m there are finite subsets $E(n,m) \subset E'$ such that $E(n,m)$ approximates 0 within $1/m$ on $\mathbb{T}^n$. As in Lemma 3, with $p_j = 0$, $E_j = E(n_j, m_j)$, and k_j yet to be determined, let

$$S = \bigcup_{j=1}^{\infty} (x_j + k_j E_j).$$

Then S is dense in the Bohr compactification of $\mathbb{Z}$.

It remains to be seen that S is Sidon, provided the k_j's are chosen well. Let the k_j's satisfy this criterion: $k_j > D_j + M_j|x_j|$ (as in Lemma 4), where M_j is the size of E_j (which is the same size as $x_j + k_jE_j$) and D_j is the maximum absolute value of the elements of $[\bigcup_{i<j}(x_i + k_iE_i)]_N$. This by itself guarantees that the sets $x_j + k_jE_j$ are disjoint for distinct values of j. To see this, consider $w \in x_j + k_jE_j$ and $\tau \in x_i + k_iE_i$ for $i < j$. Then $|\tau| \le D_j$ while, because $0 \notin E'$ and hence $0 \notin E_j \subset E'$, there is some $x \ne 0$ such that

$$|w| = |x_j + k_jx| \ge k_j - |x_j| > D_j \ge |\tau|.$$

Gilles Pisier discovered the following arithmetic condition for Sidonicity ([P]). Let $|H|$ denote the cardinality of H. A set Q is Sidon if and only if there is some $\lambda \in (0,1)$ such that, for every finite subset H of Q, there is a subset F of H such that F is quasi-independent and $|F| \ge \lambda|H|$. Let λ satisfy this criterion for the set E'.

It will be shown that λ also works for S. Let H be any finite subset of S. Then $H_j = H \cap (x_j + k_jE_j)$ is finite for each j; by the second paragraph of this proof, the H_j's are disjoint and thus

$$|H| = \sum_{j=1}^{\infty} |H_j|.$$

Since $k_j > 0$, $H_j = x_j + k_jH'_j$ and $|H'_j| = |H_j|$ for some $H'_j \subset E_j$. Recall that $E_j = E(n_j, m_j) \subset E'$. There is some $F'_j \subset H'_j$ such that F'_j is quasi-independent and $|F'_j| \ge \lambda|H'_j|$. Let

$$F = \bigcup_{j=1}^{\infty} (x_j + k_jF'_j).$$

Note that $M_j = |E_j| \ge |F'_j|$ and that D_j dominates the largest absolute value of

$$\Big[\bigcup_{i<j}(x_i + k_iF'_i)\Big]_N \subset \Big[\bigcup_{i<j}(x_i + k_iE_i)\Big]_N.$$

Thus the k_j's grow fast enough to allow Lemma 4 to apply to F with $N = 1$: F is quasi-independent and the sets $x_j + k_jF'_j$ are disjoint. Thus, $F \subset H$ and

$$\begin{aligned} |F| &= \sum_{j=1}^{\infty} |x_j + k_jF'_j| = \sum_{j=1}^{\infty} |F'_j| \\ &\ge \lambda \sum_{j=1}^{\infty} |H'_j| = \lambda \sum_{j=1}^{\infty} |H_j| = \lambda|H|. \end{aligned}$$

It follows that S is at least as Sidon as E' according to Gilles Pisier's criterion. ∎

The proof given above is easily modified for the N-independent sets. One of the early steps in the proof for Sidon sets does not work: when E is N-independent, $E - n$ need not be N-independent. For that reason, the theorem is weaker.

PROPOSITION 6. *Let $E \subset \mathbb{Z}$ be an N-independent set which clusters at* 0 *in the Bohr compactification of $\mathbb{Z}$. Then there is an N-independent subset $E' \subset \mathbb{Z}$ which is dense in the Bohr compactification of $\mathbb{Z}$.*

Proof. The N-independence of E excludes 0 from E. From this point, the proof for Sidon sets is easily adapted. One chooses $k_j > D_j + M_j N|x_j|$. Then S is dense in the Bohr group as before and the rest of the proof becomes easier. There is no need to consider a finite subset $H \subset S$. The choice of $k_j > D_j + M_j N|x_j|$ and Lemma 4 directly imply that S is N-independent. ∎

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