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Triebel–Lizorkin spaces for Hermite expansions

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Abstract. This paper develops some Littlewood–Paley theory for Hermite expansions. The main result is that certain analogues of Triebel–Lizorkin spaces are well-defined in the context of Hermite expansions.

1. Introduction and main results. Let $H_k(x)$ denote the kth Hermite polynomial (consult [4] for background), and let

$$h_k(x) = \pi^{-1/4}(2^k k!)^{-1/2} H_k(x) e^{-x^2/2}$$

denote the kth L^2-normalized Hermite function. Recall that the collection $\{h_k\}_{k=0}^{\infty}$ is a complete orthonormal basis for $L^2(\mathbb{R})$. The kth Hermite function h_k is an eigenfunction of the Hermite operator $H = -d^2/dx^2 + x^2$ with corresponding eigenvalue $2k+1$. If $m : \mathbb{R} \to \mathbb{C}$ is a bounded function, then we let $m(H)$ denote the bounded linear operator on L^2 defined by the property $m(H)h_k = m(2k+1)h_k$.

Suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is C^∞ and satisfies

(i) $\operatorname{supp}\varphi \subset [1/2, 2]$,
(ii) $|\varphi(x)| \ge c > 0$ if $x \in [3/4, 7/4]$.

Define operators $Q_\mu = \varphi(2^{-\mu}H)$ for each $\mu \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, and let L^2_{f} denote the space of all finite linear combinations of Hermite functions. For $f \in L^2_{\mathrm{f}}$ define the *Hermite–Triebel–Lizorkin norm*

$$\|f\|_{H_p^{\alpha q}} = \Big\| \Big(\sum_{\mu=0}^{\infty} (2^{\mu\alpha}|Q_\mu f|)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R})}.$$

See [5], [6] for a detailed description of the Triebel–Lizorkin spaces which occur in Fourier analysis. We assume throughout this paper that the parameters α, q, p satisfy $\alpha \in \mathbb{R}$, $1 < p < \infty$, and $1 < q \le \infty$. If $q = \infty$, the inner l^q norm is replaced by $\sup_\mu 2^{\mu\alpha}|Q_\mu f|$.

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It turns out that the completion of L^2_{f} with respect to $\|\cdot\|_{H^{\alpha q}_p}$, denoted by $H^{\alpha q}_p$, is essentially independent of the particular choice of φ chosen to satisfy conditions (i), (ii). To be precise, suppose $\varphi^{(1)}$ and $\varphi^{(2)}$ are two different C^∞ functions satisfying (i), (ii) and let $\|\cdot\|_{H^{\alpha q}_p(k)}$, $k=1,2$, denote the corresponding norms. We will show that there exist positive constants c_1, c_2 independent of $f \in L^2_{\mathrm{f}}$ such that

$$c_1\|f\|_{H^{\alpha q}_p(1)} \le \|f\|_{H^{\alpha q}_p(2)} \le c_2\|f\|_{H^{\alpha q}_p(1)}. \tag{1}$$

Let $H^{\alpha q}_p(k)$, $k=1,2$, denote the completion of L^2_{f} with respect to $\|\cdot\|_{H^{\alpha q}_p(k)}$. As a consequence of (1) we have

THEOREM 1.1. *Let $\alpha \in \mathbb{R}$, $1<p<\infty$, $1<q\le\infty$. Then the Banach spaces $H^{\alpha q}_p(1)$ and $H^{\alpha q}_p(2)$ are identical as sets, and have equivalent norms.*

Another consequence of (1) is that the operator H^β, initially defined on L^2_{f}, extends naturally to a bounded linear operator from $H^{\alpha q}_p$ to $H^{\alpha-\beta,q}_p$, for all $\beta \in \mathbb{R}$. The main idea is that if φ satisfies (i), (ii), then so does the function $\widetilde{\varphi}(x) = x^\beta\varphi(x)$.

As in the case of ordinary Triebel–Lizorkin spaces, the parameter setting $\alpha=0$, $q=2$ is recognizable.

THEOREM 1.2. *Let $1<p<\infty$. Then the spaces H^{02}_p and L^p are isomorphic, and have equivalent norms.*

The proofs of both theorems depend on estimates for the kernel of an operator $m(H)$, where $m:\mathbb{R}\to\mathbb{C}$ is C^∞ and compactly supported. Of course the kernel $K(x,y)$ of $m(H)$ is given by

$$K(x,y) = \sum_{k=0}^{\infty} m(2k+1)h_k(x)h_k(y),$$

but this expression is not very useful for getting decay in $x-y$. Section 3 contains a more useful formula for $K(x,y)$ based on Mehler's kernel. From this formula we derive two different types of estimates.

LEMMA 1.1. *Let $m:\mathbb{R}\to\mathbb{C}$ be C^∞ and compactly supported. For each $\mu\in\mathbb{N}_0$ let $K_\mu(x,y)$ denote the kernel of the operator $m(2^{-\mu}H)$. Then for every $\kappa>0$ there exists a constant $c<\infty$ independent of μ such that*

$$|K_\mu(x,y)| \le \frac{c2^{\mu/2}}{(1+2^{\mu/2}|x-y|)^4} + \frac{c2^{\mu/2}}{(1+2^{\mu\kappa}|x+y|)^4}$$

and also

$$|K_\mu(x,y)| \le \frac{c\,2^{3\mu/2}}{(1+2^{\mu/2}|x-y|)^2(x^2+y^2)} + \frac{c\,2^{3\mu/2}}{(1+2^{\mu\kappa}|x+y|)^2(x^2+y^2)}.$$

The standard techniques for proving Theorems 1.1 and 1.2 also require estimates on $(\partial/\partial x)K_\mu(x,y)$. Unfortunately, such estimates do not come as

easily as Lemma 1.1. On the other hand, it is trivial to handle the second derivative. We write

$$\begin{aligned}-\frac{\partial^2}{\partial x^2}K_\mu(x,y) &= \left(\left(-\frac{\partial^2}{\partial x^2}+x^2\right)-x^2\right)\sum_{k=0}^{\infty} m(2^{-\mu}(2k+1))h_k(x)h_k(y)\\ &= 2^\mu\sum_{k=0}^{\infty} m(2^{-\mu}(2k+1))\,(2^{-\mu}(2k+1))h_k(x)h_k(y)\\ &\quad - x^2\sum_{k=0}^{\infty} m(2^{-\mu}(2k+1))h_k(x)h_k(y).\end{aligned}$$

So, by Lemma 1.1 we have

LEMMA 1.2. *Let m and $K_\mu(x,y)$ be as in Lemma 1.1. Then for every $\kappa>0$ there exists a constant $c<\infty$ independent of $\mu\in\mathbb{N}_0$ such that*

$$\left|\frac{\partial^2}{\partial x^2}K_\mu(x,y)\right| \le \frac{c\,2^{3\mu/2}}{(1+2^{\mu/2}|x-y|)^2} + \frac{c\,2^{3\mu/2}}{(1+2^{\mu\kappa}|x+y|)^2}.$$

With a little work we can exploit this lemma in place of a first derivative estimate.

It is interesting to note that any improvement in the decay exponent 2 occurring in Lemma 1.2 would allow us to take the legal range of p, q in Theorem 1.1 below 1. On the other hand, a worse decay exponent would force us to take $\min\{p,q\} > c > 1$.

The remainder of this paper is organized as follows. The proof of Lemma 1.1 is postponed until Section 3. Assuming this lemma, Section 2 contains proofs of the main theorems. One of the auxiliary results there, used to prove Theorem 1.1, is that a Peetre type maximal inequality holds for Hermite expansions.

Previously Thangavelu [4] studied the more classical g-functions defined via the Hermite semigroup. He used these g-functions to prove a Marcinkiewicz type multiplier theorem for Hermite expansions. One advantage of working directly with the Hermite semigroup is that its kernel is explicitly known. Thangavelu established nice estimates on this kernel, from which the g-function characterization of L^p, $1<p<\infty$, follows almost immediately by standard Calderón–Zygmund theory.

It would be a natural outgrowth of this paper to study boundedness criteria for Hermite multiplier operators acting on the Hermite–Triebel–Lizorkin spaces $H^{\alpha q}_p$. We plan to discuss this elsewhere.

2. Proofs of theorems.

We need to define more operators. For $x>0$ let $\psi(x) = \overline{\varphi(x)}(\sum_{\mu\in\mathbb{Z}}|\varphi(2^{-\mu}x)|^2)^{-1}$. By (i), (ii) this function is C^∞ and supported on $[1/2,2]$. Also,

$$\sum_{\nu\in\mathbb{Z}}\varphi(2^{-\nu}x)\psi(2^{-\nu}x)=1 \quad \text{for all } x>0,$$

and so in particular,

$$\sum_{\nu=0}^{\infty}\varphi(2^{-\nu}x)\psi(2^{-\nu}x)=1 \quad \text{for all } x\geq 1.$$

For $\mu\in\mathbb{N}_0$ define $R_\mu=\psi(2^{-\mu}H)$ and $T_\mu=Q_{\mu-1}R_{\mu-1}+Q_\mu R_\mu+Q_{\mu+1}R_{\mu+1}$. Note that by (i), $Q_{-1}=0$ on L^2. Also note that $Q_\mu=T_\mu Q_\mu$.

Next, if $f\in L^2$, define

$$A_\mu f(x)=\sup_{y\in\mathbb{R}}\frac{|Q_\mu f(y)|}{\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}}$$

and

$$B_\mu f(x)=\sup_{y\in\mathbb{R}}\frac{|(Q_\mu f)''(y)|}{\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}}.$$

The operator A_μ is a Hermite analogue of Peetre's maximal function [3]. We write $\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}$ for $\min\{(1+2^{\mu/2}|x-y|)^\lambda,(1+2^{\mu/2}|x+y|)^\lambda\}$.

LEMMA 2.1. *For every $0<\lambda<1$ there exists a constant $c<\infty$ independent of $\mu\in\mathbb{N}_0$ and $f\in L^2$ such that $B_\mu f(x)\leq c2^\mu A_\mu f(x)$.*

Proof. Write

$$(Q_\mu f)''(y)=\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty}T_\mu(y,u)Q_\mu f(u)\,du$$

and then apply Lemma 1.2 to get

$$\begin{aligned}(2)\quad &\frac{|(Q_\mu f)''(y)|}{\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}}\\ &\leq\int_{-\infty}^{\infty}\Big(\sum_{\sigma=\pm1}\frac{c\,2^{3\mu/2}}{(1+2^{\mu/2}|y+\sigma u|)^2}\Big)\frac{|Q_\mu f(u)|}{\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}}\,du\\ &\leq A_\mu f(x)\int_{-\infty}^{\infty}\Big(\sum_{\sigma=\pm1}\frac{c\,2^{3\mu/2}}{(1+2^{\mu/2}|y+\sigma u|)^2}\Big)\frac{\min\{(1+2^{\mu/2}|x\pm u|)^\lambda\}}{\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}}\,du.\end{aligned}$$

The quantity $\min\{(1+2^{\mu/2}|x\pm u|)^\lambda\}$ is bounded above by

$$\begin{aligned}\min\{&(1+2^{\mu/2}|x+y|)^\lambda(1+2^{\mu/2}|-y+u|)^\lambda,\\ &(1+2^{\mu/2}|x-y|)^\lambda(1+2^{\mu/2}|y-u|)^\lambda\}\\ &=(1+2^{\mu/2}|y-u|)^\lambda\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}\end{aligned}$$

and also by

$$\begin{aligned}\min\{&(1+2^{\mu/2}|x-y|)^\lambda(1+2^{\mu/2}|y+u|)^\lambda,\\ &(1+2^{\mu/2}|x+y|)^\lambda(1+2^{\mu/2}|-y-u|)^\lambda\}\\ &=(1+2^{\mu/2}|y+u|)^\lambda\min\{(1+2^{\mu/2}|x\pm y|)^\lambda\}.\end{aligned}$$

Hence (2) is bounded by

$$\begin{aligned}A_\mu f(x)\int_{-\infty}^{\infty}\Big(\sum_{\sigma=\pm1}\frac{c2^{3\mu/2}}{(1+2^{\mu/2}|y+\sigma u|)^2}\Big)\min\{(1+2^{\mu/2}|y\pm u|)^\lambda\}\,du\\ \leq c2^\mu A_\mu f(x).\ \blacksquare\end{aligned}$$

In the next lemma M denotes the Hardy–Littlewood maximal operator.

LEMMA 2.2. *Let $r=1/\lambda$, where $0<\lambda<1$ is the parameter in the definition of A_μ. Then there exists a constant $c<\infty$ independent of $\mu\in\mathbb{N}_0$ and $f\in L^2$ such that*

$$A_\mu f(x)\leq c\sum_{\sigma=\pm1}(M(|Q_\mu f|^r)(\sigma x))^{1/r}.$$

Proof. Let $0<\delta<1$, to be chosen sufficiently small later. By Taylor's theorem,

$$Q_\mu f(\pm x-y)=Q_\mu f(\pm x-y-u)+u(Q_\mu f)'(\pm x-y)-\frac{u^2}{2}(Q_\mu f)''(\xi)$$

and

$$Q_\mu f(\pm x-y)=Q_\mu f(\pm x-y+u)-u(Q_\mu f)'(\pm x-y)-\frac{u^2}{2}(Q_\mu f)''(\eta),$$

where ξ is between $\pm x-y$ and $\pm x-y+u$ and η is between $\pm x-y$ and $\pm x-y-u$. We add these to get

$$\begin{aligned}2|Q_\mu f(\pm x-y)|\leq|Q_\mu f(\pm x-y-u)+Q_\mu f(\pm x-y+u)|\\ +\frac{u^2}{2}|(Q_\mu f)''(\xi)+(Q_\mu f)''(\eta)|.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{2|Q_\mu f(\pm x-y)|}{(1+2^{\mu/2}|y|)^\lambda}\\ &\leq\frac{1}{(1+2^{\mu/2}|y|)^\lambda}\Big(\frac{1}{2\cdot2^{-\mu/2}\delta}\int_{-2^{-\mu/2}\delta}^{2^{-\mu/2}\delta}|Q_\mu f(\pm x-y-u)\\ &\quad+Q_\mu f(\pm x-y+u)|^r\,du\Big)^{1/r}+2^{-\mu}\delta^2\sup_{|u|\leq2^{-\mu/2}\delta}\frac{|(Q_\mu f)''(\pm x-y-u)|}{(1+2^{\mu/2}|y|)^\lambda}\end{aligned}$$

$$\begin{aligned}
&\le \frac{c}{(1+2^{\mu/2}|y|)^{\lambda}}\Big(\frac{|y|+2^{-\mu/2}\delta}{2^{-\mu/2}\delta}M(|Q_\mu f|^r)(\pm x)\Big)^{1/r}\\
&\quad + c2^{-\mu}\delta^2 \sup_{|u|\le 2^{-\mu/2}\delta} \frac{(1+2^{\mu/2}|y+u|)^{\lambda}}{(1+2^{\mu/2}|y|)^{\lambda}}\,\frac{|(Q_\mu f)''(\pm x-y-u)|}{(1+2^{\mu/2}|y+u|)^{\lambda}}\\
&\le c\Big(\frac{|y|+2^{-\mu/2}\delta}{(1+2^{\mu/2}|y|)2^{-\mu/2}\delta}\Big)^{\lambda}(M(|Q_\mu f|^r)(\pm x))^{1/r}\\
&\quad + 2^{-\mu}\delta^2(1+\delta)^{\lambda}\sup_{u\in\mathbb{R}}\frac{\max\{|(Q_\mu f)''(\pm x-y-u)|\}}{(1+2^{\mu/2}|y+u|)^{\lambda}}\\
&\le c_1(\delta^{-1}+1)^{\lambda}(M(|Q_\mu f|^r)(\pm x))^{1/r}+c_2 2^{-\mu}\delta^2(1+\delta)^{\lambda}B_\mu f(x)
\end{aligned}$$

with c_1 and c_2 independent of δ. The result now follows from Lemma 2.1 by taking δ small enough. ∎

Proof of Theorem 1.1. It suffices to prove the first inequality in (1). So suppose $\varphi^{(k)}$, $k=1,2$, are two different C^∞ functions satisfying (i), (ii), and let $\psi^{(k)}$, $Q_\mu^{(k)}$, $R_\mu^{(k)}$, $A_\mu^{(k)}$ denote the corresponding objects as defined earlier. Then we can write

$$Q_\mu^{(1)} f = \sum_{\nu=\mu-1}^{\mu+1} Q_\mu^{(1)} R_\nu^{(2)} Q_\nu^{(2)} f.$$

Let $H_{\mu\nu}(x,y)$ denote the kernel of the operator $Q_\mu^{(1)}R_\nu^{(2)}$. According to Lemma 1.1,

$$|H_{\mu\nu}(x,y)|\le \sum_{\sigma=\pm1}\frac{c2^{\nu/2}}{(1+2^{\nu/2}|x+\sigma y|)^4}$$

with c independent of μ and $\nu=\mu-1,\mu,\mu+1$, so

$$\begin{aligned}
|Q_\mu^{(1)} f(x)| &\le \sum_{\nu=\mu-1}^{\mu+1}\sum_{\sigma=\pm1}\int_{-\infty}^{\infty}\frac{c2^{\nu/2}}{(1+2^{\nu/2}|x+\sigma y|)^4}|Q_\nu^{(2)} f(y)|\,dy\\
&\le \sum_{\nu=\mu-1}^{\mu+1} A_\nu^{(2)} f(x)\\
&\quad\times \sum_{\sigma=\pm1}\int_{-\infty}^{\infty}\frac{c2^{\nu/2}}{(1+2^{\nu/2}|x+\sigma y|)^4}\min\{(1+2^{\nu/2}|x\pm y|)^{\lambda}\}\,dy\\
&\le c\sum_{\nu=\mu-1}^{\mu+1} A_\nu^{(2)} f(x).
\end{aligned}$$

We finish the proof by taking the parameter λ such that $(\min\{p,q\})^{-1}\le\lambda=1/r<1$, applying Lemma 2.2, and using the Fefferman–Stein vector-valued

maximal inequality [1] to get

$$\begin{aligned}
\|f\|_{H_p^{\alpha q}(1)} &\le c\Big\|\Big(\sum_{\mu=0}^{\infty}(M(2^{\mu\alpha r}|Q_\mu^{(2)} f|^r))^{q/r}\Big)^{1/q}\Big\|_{L^p}\\
&= c\Big\|\Big(\sum_{\mu=0}^{\infty}(M(2^{\mu\alpha r}|Q_\mu^{(2)} f|^r))^{q/r}\Big)^{r/q}\Big\|_{L^{p/r}}^{1/r}\le c\|f\|_{H_p^{\alpha q}(2)}. \blacksquare
\end{aligned}$$

Now we prove Theorem 1.2. Let $L^2(l^2)_{\mathrm f}$ denote the subspace of $L^2(l^2)$ consisting of sequences $\{f_\mu\}_{\mu=0}^\infty$ such that only finitely many f_μ are nonvanishing. Define operators

$$Q: L^2_{\mathrm f}\to L^2(l^2)_{\mathrm f},\qquad R: L^2(l^2)_{\mathrm f}\to L^2_{\mathrm f}$$

by $Qg=\{Q_\mu g\}_{\mu=0}^\infty$ and $R(\{g_\mu\}_{\mu=0}^\infty)=\sum_{\mu=0}^\infty R_\mu g_\mu$.

LEMMA 2.3. *Q extends to a bounded linear operator from L^2 to $L^2(l^2)$ and R extends to a bounded linear operator from $L^2(l^2)$ to L^2.*

Proof. Trivial calculation. ∎

LEMMA 2.4. *Q is weak-type bounded from L^1 to $L^1(l^2)$.*

Proof. The method of proof is standard, except for a few minor detours due to the nature of the kernel estimates in Lemmas 1.1, 1.2. We need to show that there exists a constant $c<\infty$ such that

$$\Big|\Big\{x:\Big(\sum_{\mu=0}^\infty|Q_\mu f(x)|^2\Big)^{1/2}>\lambda\Big\}\Big|\le\frac{c}{\lambda}\|f\|_{L^1}$$

for all $f\in L^1$, $\lambda>0$. (Note that each of the operators Q_μ has a natural extension from $L^1\cap L^2$ to L^1.) So fix $f\in L^1$, $\lambda>0$, and apply the Calderón–Zygmund lemma to get a collection of disjoint dyadic open intervals $\{I_j\}$ such that

(a) $|f(x)|\le\lambda$ for a.e. $x\in\mathbb{R}\setminus\bigcup_j I_j$,
(b) $\sum_j|I_j|\le\lambda^{-1}\|f\|_{L^1}$,
(c) $\lambda\le|I_j|^{-1}\int_{I_j}|f(x)|\,dx\le2\lambda$ for all j.

Let z_j denote the centerpoint of I_j, and for $x\in I_j$ let

$$g(x)=\frac{1}{|I_j|}\int_{I_j}f(y)\,dy+\frac{12(x-z_j)}{|I_j|^3}\int_{I_j}f(y)(y-z_j)\,dy.$$

Also, if $x\in I_j$, let $b(x)=f(x)-g(x)$. For $x\notin\bigcup_j I_j$, let $g(x)=f(x)$ and $b(x)=0$. Thus $f=g+b$ everywhere. Note that if $x\in I_j$, then $|g(x)|\le8\lambda$.

Also, for a.e. $x \notin \bigcup_j I_j$, $|g(x)| \le \lambda$. Therefore, in the usual way,

$$\|g\|_{L^2}^2 = \int_{\mathbb{R}\setminus\bigcup I_j} |g(x)|^2\,dx + \int_{\bigcup I_j} |g(x)|^2\,dx$$
$$\le \lambda \int_{\mathbb{R}\setminus\bigcup I_j} |f(x)|\,dx + \sum_j (8\lambda)^2|I_j| \le \lambda\|f\|_{L^1} + 64\lambda^2\cdot\frac{1}{\lambda}\|f\|_{L^1}.$$

It follows by Chebyshev's inequality and Lemma 2.3 that

$$\Big|\Big\{x : \Big(\sum_{\mu=0}^{\infty}|Q_\mu g(x)|^2\Big)^{1/2} > \lambda/2\Big\}\Big| \le \frac{4}{\lambda^2}\|Qg\|^2_{L^2(l^2)} \le \frac{c}{\lambda}\|f\|_{L^1}.$$

Next we have to prove that the correct sort of estimate holds for

$$\Big|\Big\{x : \Big(\sum_{\mu=0}^{\infty}|Q_\mu b(x)|^2\Big)^{1/2} > \lambda/2\Big\}\Big|.$$

Define

$$I_j^* = (z_j - |I_j|, z_j + |I_j|) \cup (-z_j - |I_j|, -z_j + |I_j|).$$

Since $|\bigcup_j I_j^*| \le \frac{4}{\lambda}\|f\|_{L^1}$, it suffices to estimate

$$\Big|\Big\{x \in \mathbb{R}\setminus\bigcup_j I_j^* : \Big(\sum_{\mu=0}^{\infty}|Q_\mu b(x)|^2\Big)^{1/2} > \lambda/2\Big\}\Big|.$$

For each j let $b_j = b\chi_{I_j}$. Then we have $b = \sum_j b_j$ a.e., $\int b_j(x)\,dx = 0$, and $\int b_j(x)(x - z_j)\,dx = 0$. By Chebyshev's and Minkowski's inequalities,

$$(3)\qquad \Big|\Big\{x \in \mathbb{R}\setminus\bigcup_j I_j^* : \Big(\sum_{\mu=0}^{\infty}|Q_\mu b(x)|^2\Big)^{1/2} > \lambda/2\Big\}\Big|$$
$$\le \frac{2}{\lambda}\sum_j \int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\mu=0}^{\infty}|Q_\mu b_j(x)|^2\Big)^{1/2}dx.$$

For each j define the kernel

$$L_\mu^j(x,y) = \begin{cases} Q_\mu(x,y) & \text{if } 2^{\mu/2}|I_j| \ge 1,\\ Q_\mu(x,y) - (Q_\mu(x,z_j) + (y-z_j)(\partial Q_\mu/\partial x_2)(x,z_j)) & \\ & \text{if } 2^{\mu/2}|I_j| < 1,\end{cases}$$

where of course $Q_\mu(x,y)$ denotes the kernel of Q_μ. Because of the vanishing

moment conditions imposed on b_j, we have

$$\int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\mu=0}^{\infty}|Q_\mu b_j(x)|^2\Big)^{1/2}dx = \int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\mu=0}^{\infty}\Big|\int_{I_j} L_\mu^j(x,y)b_j(y)\,dy\Big|^2\Big)^{1/2}dx$$
$$\le \int_{I_j}|b_j(y)|\int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\mu=0}^{\infty}|L_\mu^j(x,y)|^2\Big)^{1/2}dx\,dy.$$

The last line follows from Minkowski's inequality for integrals. Now we need to estimate

$$(4)\qquad \int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\mu=0}^{\infty}|L_\mu^j(x,y)|^2\Big)^{1/2}dx \le \int_{\mathbb{R}\setminus I_j^*}\sum_{\mu=0}^{\infty}|L_\mu^j(x,y)|\,dx.$$

Suppose first that $2^{\mu/2}|I_j| \ge 1$. Then according to Lemma 1.1, if $y \in I_j$,

$$\int_{\mathbb{R}\setminus I_j^*}|L_\mu^j(x,y)|\,dx = \int_{\mathbb{R}\setminus I_j^*}|Q_\mu(x,y)|\,dx$$
$$\le c\int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\sigma=\pm1}\frac{c2^{\mu/2}}{(1+2^{\mu/2}|x+\sigma y|)^4}\Big)dx \le c(2^{\mu/2}|I_j|)^{-3}.$$

Suppose on the other hand that $2^{\mu/2}|I_j| < 1$. In this case we use the fact that

$$L_\mu^j(x,y) = \frac{1}{2}(y-z_j)^2\frac{\partial^2 Q_\mu}{\partial x_2^2}(x,\xi)$$

for some ξ between y and z_j. Then according to Lemma 1.2, if $y \in I_j$,

$$\int_{\mathbb{R}\setminus I_j^*}|L_\mu^j(x,y)|\,dx \le c\int_{\mathbb{R}\setminus I_j^*}(y-z_j)^2\Big(\sum_{\sigma=\pm1}\frac{c\,2^{3\mu/2}}{(1+2^{\mu/2}|x+\sigma y|)^2}\Big)dx$$
$$\le c(2^{\mu/2}|I_j|)^2\int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\sigma=\pm1}\frac{c2^{\mu/2}}{(1+2^{\mu/2}|x+\sigma y|)^2}\Big)dx$$
$$\le c2^{\mu/2}|I_j|.$$

Combining these estimates we have

$$(4) \le c\sum_{\mu=0}^{\infty}\min\{(2^{\mu/2}|I_j|)^{-3}, 2^{\mu/2}|I_j|\}$$
$$\le \text{const. independent of } j.$$

Since $\int_{I_j} |b_j(y)|\,dy \le 5\int_{I_j} |f(y)|\,dy$, the result is that

$$(3) \le \frac{c}{\lambda}\sum_j \int_{I_j} |f(y)|\,dy \le \frac{c}{\lambda}\|f\|_{L^1}. \ \blacksquare$$

For the next lemma define operators $R_n : L^1(l^2) \to L^1$, $n \in \mathbb{N}_0$, by $R_n(\{f_\mu\}) = \sum_{\mu=0}^n R_\mu f_\mu$.

LEMMA 2.5. *The operators R_n, $n \in \mathbb{N}_0$, are uniformly weak-$(L^1(l^2), L^1)$ bounded.*

Proof. We need to show that there exists a constant $c < \infty$ independent of $n \in \mathbb{N}_0$ such that

$$\left|\left\{x : \left|\sum_{\mu=0}^n R_\mu f_\mu(x)\right| > \lambda\right\}\right| \le \frac{c}{\lambda}\|\{f_\mu\}\|_{L^1(l^2)}$$

for all $\{f_\mu\} \in L^1(l^2)$, $\lambda > 0$. So fix $\{f_\mu\} \in L^1(l^2)$, $\lambda > 0$, let $h(x) = (\sum_{\mu=0}^\infty |f_\mu(x)|^2)^{1/2}$, and apply the Calderón–Zygmund lemma to get a collection of disjoint open intervals $\{I_j\}$ such that

(a) $|h(x)| \le \lambda$ for a.e. $x \in \mathbb{R} \setminus \bigcup_j I_j$,
(b) $\sum_j |I_j| \le \lambda^{-1}\|h\|_{L^1}$,
(c) $\lambda \le |I_j|^{-1}\int_{I_j} |h(x)|\,dx \le 2\lambda$ for all j.

Again let z_j denote the centerpoint of I_j, and for $x \in I_j$ let

$$g_\mu(x) = \frac{1}{|I_j|}\int_{I_j} f_\mu(y)\,dy + \frac{12(x-z_j)}{|I_j|^3}\int_{I_j} f_\mu(y)(y-z_j)\,dy.$$

For $x \in I_j$ let $b_\mu(x) = f_\mu(x) - g_\mu(x)$, and for $x \notin \bigcup_j I_j$ let $g_\mu(x) = f_\mu(x)$, $b_\mu(x) = 0$. If $x \in I_j$ we have

$$\begin{aligned}\Big(\sum_{\mu=0}^\infty |g_\mu(x)|^2\Big)^{1/2} &\le \Big(\sum_{\mu=0}^\infty \Big|\frac{1}{|I_j|}\int_{I_j} f_\mu(y)\,dy\Big|^2\Big)^{1/2} \\ &\quad + \Big(\sum_{\mu=0}^\infty \Big|\frac{12(x-z_j)}{|I_j|^3}\int_{I_j} f_\mu(y)(y-z_j)\,dy\Big|^2\Big)^{1/2} \\ &\le \frac{1}{|I_j|}\int_{I_j}\Big(\sum_{\mu=0}^\infty |f_\mu(y)|^2\Big)^{1/2} dy + \frac{3}{|I_j|}\int_{I_j}\Big(\sum_{\mu=0}^\infty |f_\mu(y)|^2\Big)^{1/2} dy \le 8\lambda.\end{aligned}$$

It follows that $\|\{g_\mu\}\|^2_{L^2(l^2)} \le c\lambda\|\{f_\mu\}\|_{L^1(l^2)}$, and therefore by Chebyshev's inequality and Lemma 2.3,

$$\left|\left\{x : \left|\sum_{\mu=0}^n R_\mu g_\mu(x)\right| > \lambda/2\right\}\right| \le \frac{c}{\lambda}\|\{f_\mu\}\|_{L^1(l^2)},$$

with c independent of n.

Next we have to estimate $|\{x : |\sum_{\mu=0}^n R_\mu b_\mu(x)| > \lambda/2\}|$. As in the previous lemma, it suffices to handle

$$(5)\quad \left|\left\{x \in \mathbb{R}\setminus\bigcup_j I_j^* : \left|\sum_{\mu=0}^n R_\mu b_\mu(x)\right| > \lambda/2\right\}\right| \le \frac{2}{\lambda}\sum_j \int_{\mathbb{R}\setminus I_j^*}\left|\sum_{\mu=0}^n R_\mu b_{\mu,j}(x)\right| dx.$$

Here of course $b_{\mu,j} = b_\mu\chi_{I_j}$. Now for each j define the kernel

$$\widetilde{L}^j_\mu(x,y) = \begin{cases} R_\mu(x,y) & \text{if } 2^{\mu/2}|I_j| \ge 1,\\ R_\mu(x,y) - (R_\mu(x,z_j) + (y-z_j)(\partial R_\mu/\partial x_2)(x,z_j)) & \\ & \text{if } 2^{\mu/2}|I_j| < 1.\end{cases}$$

Then

$$\begin{aligned}\int_{\mathbb{R}\setminus I_j^*}\left|\sum_{\mu=0}^n R_\mu b_{\mu,j}(x)\right| dx &= \int_{\mathbb{R}\setminus I_j^*}\left|\sum_{\mu=0}^n \int_{I_j}\widetilde{L}^j_\mu(x,y) b_{\mu,j}(y)\,dy\right| dx \\ &\le \int_{I_j}\Big(\sum_{\mu=0}^n |b_{\mu,j}(y)|^2\Big)^{1/2}\Big(\int_{\mathbb{R}\setminus I_j^*}\Big(\sum_{\mu=0}^n |\widetilde{L}^j_\mu(x,y)|^2\Big)^{1/2} dx\Big)\,dy \\ &\le c\int_{I_j}\Big(\sum_{\mu=0}^n |b_{\mu,j}(y)|^2\Big)^{1/2} dy \le c\int_{I_j}\Big(\sum_{\mu=0}^\infty |f_\mu(y)|^2\Big)^{1/2} dy\end{aligned}$$

with c independent of n. Substituting this in (5) finishes the proof. $\blacksquare$

In the next proof we use the fact that $L^2_{\rm f}$ is dense in L^p for $1 \le p < \infty$ (see for example [2], Lemma 2).

Proof of Theorem 1.2. Let $1 < p < \infty$. It suffices to show that there exist constants $c_1, c_2 > 0$ such that

$$c_1\|f\|_{H_p^{02}} \le \|f\|_{L^p} \le c_2\|f\|_{H_p^{02}}$$

for all $f \in L^2_{\rm f}$. First, by Lemmas 2.3, 2.4, Marcinkiewicz interpolation, and duality, the operator Q extends to a bounded linear operator from L^p to $L^p(l^2)$ for every $1 < p < \infty$. Hence

$$\|f\|_{H_p^{02}} = \|\{Qf\}\|_{L^p(l^2)} \le c\|f\|_{L^p}.$$

Next, by Lemmas 2.3, 2.5, Marcinkiewicz interpolation, and duality, each of the operators R_n extends to a bounded linear operator from $L^p(l^2)$ to L^p, for $1 < p < \infty$. Moreover, the operators $R_n : L^p(l^2) \to L^p$, $n \in \mathbb{N}_0$, are uniformly bounded. If $f \in L^2_{\mathrm{f}}$, then for large enough n,

$$f = \sum_{\mu=0}^{n} R_\mu Q_\mu f = R_n(\{Qf\}).$$

Hence, for such a function f,

$$\|f\|_{L^p} = \|R_n(\{Qf\})\|_{L^p} \le c\|\{Qf\}\|_{L^p(l^2)} = c\|f\|_{H^{02}_p}$$

with c independent of n. ∎

3. Kernel estimates. It remains to prove Lemma 1.1. Let m be as in the lemma, and let P_k denote the orthogonal projection in L^2 onto h_k. If we use the convention for the Fourier transform that $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx$, then, at least formally,

$$2\pi m(2^{-\mu}H) = \int_{-\infty}^{\infty} \widehat{m}(\xi)e^{i\xi 2^{-\mu}H}\,d\xi = \int_{-\infty}^{\infty} \widehat{m}(\xi)\Big(\sum_{k=0}^{\infty} e^{i\xi 2^{-\mu}(2k+1)}P_k\Big)\,d\xi.$$

The kernel of the operator inside the parentheses is given by Mehler's formula. We will use the following (slightly nonstandard) version.

LEMMA 3.1. *Suppose $z \in \mathbb{C}$, $|z| \le 1$, $z \ne \pm 1$, and let $f \in L^2_{\mathrm{f}}$. Then*

$$\sum_{k=0}^{\infty} z^k(P_kf)(x) = \pi^{-1/2}(1-z^2)^{-1/2} \times \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}\cdot\frac{1+z^2}{1-z^2}(x^2+y^2) + \frac{2z}{1-z^2}xy\Big)f(y)\,dy.$$

Note that $(1-z^2)^{-1/2}$ is defined by cutting $\mathbb{C}$ along the negative real axis. Also note that the sum over k is actually a finite sum.

Proof of Lemma 3.1. This is standard if $|z| < 1$ (see e.g. [4], p. 2). If $|z| = 1$, $z \ne \pm 1$, then

$$\sum_{k=0}^{\infty} z^k(P_kf)(x) = \lim_{r\to 1^-}\sum_{k=0}^{\infty}(rz)^k(P_kf)(x)$$
$$= \lim_{r\to 1^-}\pi^{-1/2}(1-(rz)^2)^{-1/2} \times \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}\cdot\frac{1+(rz)^2}{1-(rz)^2}(x^2+y^2) + \frac{2rz}{1-(rz)^2}xy\Big)f(y)\,dy$$
$$= \pi^{-1/2}(1-z^2)^{-1/2} \times \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}\cdot\frac{1+z^2}{1-z^2}(x^2+y^2) + \frac{2z}{1-z^2}xy\Big)f(y)\,dy.$$

The last line follows from the dominated convergence theorem. ∎

Now let $z = z(\xi) = e^{i2^{-\mu+1}\xi}$, and let

(6) $$K_\mu(x,y) = 2^{-1}\pi^{-3/2}\int_{-\infty}^{\infty} \widehat{m}(\xi)e^{i2^{-\mu}\xi}(1-z^2)^{-1/2} \times \exp\Big(-\frac{1}{2}\cdot\frac{1+z^2}{1-z^2}(x^2+y^2) + \frac{2z}{1-z^2}xy\Big)\,d\xi.$$

An application of Fubini's theorem and Lemma 3.1 shows that if $f \in L^2_{\mathrm{f}}$, then

$$(m(2^{-\mu}H)f)(x) = \int_{-\infty}^{\infty} K_\mu(x,y)f(y)\,dy.$$

It will be useful to put the integral in (6) in the more concise form

(7) $$2^{\mu-1}\int_{-\infty}^{\infty} \widehat{m}(2^{\mu-1}\xi)e^{i\xi/2}(1-e^{i2\xi})^{-1/2} \times \exp\Big(-\frac{i}{2}((x^2+y^2)\cot\xi - 2xy\csc\xi)\Big)\,d\xi.$$

Proof of Lemma 1.1. Since m is compactly supported, there exists some N such that for all $\mu = 0, 1, \ldots$,

$$|K_\mu(x,y)| = \Big|\sum_{k=0}^{2^\mu N} m(2^{-\mu}(2k+1))h_k(x)h_k(y)\Big|$$
$$\le \|m\|_{L^\infty}\Big(\sum_{k=0}^{2^\mu N} h_k^2(x)\Big)^{1/2}\Big(\sum_{k=0}^{2^\mu N} h_k^2(y)\Big)^{1/2}.$$

Now we recall the argument used to prove Lemma 3.2.1 in [4]. If $0 < r < 1$ and $L \ge 1$, then by Mehler's formula,

$$\sum_{k=0}^{L} h_k^2(x) \le r^{-L}\sum_{k=0}^{\infty} r^k h_k^2(x) = \pi^{-1/2}r^{-L}(1-r^2)^{-1/2}\exp\Big(-\frac{1-r}{1+r}x^2\Big).$$

Substituting $r = e^{-1/L}$ we get

$$\sum_{k=0}^{L} h_k^2(x) \le c_1L^{1/2}e^{-c_2L^{-1}x^2}.$$

Hence

$$|K_\mu(x,y)| \le c_1 2^{\mu/2} e^{-c_2 2^{-\mu}(x^2+y^2)} \le c\min\{2^{\mu/2}, 2^{3\mu/2}(x^2+y^2)^{-1}\}.$$

This proves the lemma if $2^{\mu/2}|x-y|<1$ or $2^{\mu\kappa}|x+y|<1$.

So suppose that both $2^{\mu/2}|x-y|\ge 1$ and $2^{\mu\kappa}|x+y|\ge 1$. To continue we integrate by parts in (7). Let

$$F(\xi) = \widehat{m}(2^{\mu-1}\xi)e^{i\xi/2}(1-e^{i2\xi})^{-1/2}$$

and

$$G(\xi) = (x^2+y^2)\cot\xi - 2xy\csc\xi.$$

Also let

$$\frac{1}{H(\xi)} = \frac{d}{d\xi}G(\xi) = -\frac{1}{\sin^2\xi}(x^2+y^2-2xy\cos\xi).$$

Then we can rewrite (7) as

$$c2^\mu \int_{-\infty}^{\infty} \frac{d}{d\xi}(F(\xi)H(\xi))e^{-iG(\xi)/2}\,d\xi.$$

Of course this integration by parts is justified by showing that when $|x-y|,|x+y|\ne 0$,

$$\int_{-\infty}^{\infty} \frac{d}{d\xi}(F(\xi)H(\xi)e^{-iG(\xi)/2})\,d\xi = 0.$$

The main ideas are that (1) the integrand $(d/d\xi)(\cdot)$ is integrable, (2) the function inside $(\cdot)$ is C^∞ away from the points $n\pi$, $n\in\mathbb{Z}$, and (3) as $\xi\to n\pi$, the function inside $(\cdot)$ tends to zero. Integrating by parts once more we get

$$K_\mu(x,y) = c2^\mu \int_{-\infty}^{\infty} \frac{d}{d\xi}\left(\frac{d}{d\xi}(F(\xi)H(\xi))H(\xi)\right)e^{-iG(\xi)/2}\,d\xi. \tag{8}$$

The rest of the proof consists of analyzing many separate integrals, depending on where the two differentiations in (8) fall. In each case we split up the integral over $\mathbb{R}$ into three integrals over subsets of $\mathbb{R}$. Let

$$A = \bigcup_{n\in\mathbb{Z}}[-\pi/3+2\pi n, \pi/3+2\pi n],$$

$$B = \bigcup_{n\in\mathbb{Z}}([\pi/3+2\pi n, 2\pi/3+2\pi n]\cup[4\pi/3+2\pi n, 5\pi/3+2\pi n]),$$

$$C = \bigcup_{n\in\mathbb{Z}}[2\pi/3+2\pi n, 4\pi/3+2\pi n].$$

Then $\mathbb{R}$ is the essentially disjoint union of the sets A,B,C. We will repeatedly use the trivial inequalities

$$x^2+y^2-2xy\cos\xi \ge \begin{cases} (x-y)^2/2+(1-\cos\xi)(x^2+y^2) & \text{if } \xi\in A,\\ (x^2+y^2)/2 & \text{if } \xi\in B,\\ (x+y)^2/2+(1+\cos\xi)(x^2+y^2) & \text{if } \xi\in C.\end{cases} \tag{9}$$

Also we will use the trivial inequality

$$(x^2+y^2)^{-1} \le \max\{(x-y)^{-2},(x+y)^{-2}\}. \tag{10}$$

Suppose now that both differentiations in (8) fall on $\widehat{m}(2^{\mu-1}\xi)$. Since $\widehat{m}$ has rapid decay, we end up bounding a quantity of the form

$$c2^{3\mu}\int_{-\infty}^{\infty}(1+2^\mu|\xi|)^{-K}(1-\cos 2\xi)^{-1/4}\sin^4\xi(x^2+y^2-2xy\cos\xi)^{-2}\,d\xi, \tag{11}$$

where K can be taken arbitrarily large.

The part of the integral over region A is bounded by

$$c2^{3\mu}\sum_{n\in\mathbb{Z}}\int_{-\pi/3}^{\pi/3}(1+2^\mu|\xi+2\pi n|)^{-K}|\xi|^{-1/2}|\xi|^4(x^2+y^2-2xy\cos\xi)^{-2}\,d\xi. \tag{12}$$

In proving the first inequality in Lemma 1.1 we apply (9) in the form

$$(x^2+y^2-2xy\cos\xi)^{-2} \le 4(x-y)^{-4}$$

and in proving the second inequality in Lemma 1.1 we apply (9) in the form

$$(x^2+y^2-2xy\cos\xi)^{-2} \le 2(x-y)^{-2}((1-\cos\xi)(x^2+y^2))^{-1}.$$

In the first case by taking K large enough we get

$$\begin{aligned}|(12)| &\le \frac{c2^{3\mu}}{(x-y)^4}\sum_{n\in\mathbb{Z}}\int_{-\pi/3}^{\pi/3}(1+2^\mu|\xi+2\pi n|)^{-K}|\xi|^{7/2}\,d\xi\\ &\le \frac{c2^{\mu/2}}{(2^{\mu/2}(x-y))^4}\sum_{n\in\mathbb{Z}}\int_{-\pi/3}^{\pi/3}(1+2^\mu|\xi+2\pi n|)^{-K}(2^\mu|\xi|)^{7/2}2^\mu\,d\xi\\ &\le \frac{c2^{\mu/2}}{(2^{\mu/2}(x-y))^4},\end{aligned}$$

and in the second case we get

$$\begin{aligned}|(12)| &\le \frac{c2^{3\mu}}{(x-y)^2(x^2+y^2)}\sum_{n\in\mathbb{Z}}\int_{-\pi/3}^{\pi/3}(1+2^\mu|\xi+2\pi n|)^{-K}|\xi|^{3/2}\,d\xi\\ &\le \frac{c2^{3\mu/2}}{(2^{\mu/2}(x-y))^2(x^2+y^2)}.\end{aligned}$$

Again using (9), the part of the integral (11) over region B is bounded by

$$(13)\qquad \frac{c2^{3\mu}}{(x^2+y^2)^2}\sum_{n\in\mathbb{Z}}\Big(\int\limits_{\pi/3}^{2\pi/3}(1+2^\mu|\xi+2\pi n|)^{-K}\,d\xi + \int\limits_{4\pi/3}^{5\pi/3}(1+2^\mu|\xi+2\pi n|)^{-K}\,d\xi\Big).$$

It follows that $|(13)|\le c2^{-\mu L}(x^2+y^2)^{-2}$, where L can be taken arbitrarily large. This is more than sufficient, when combined with (10).

Finally, the part of the integral (11) over region C is bounded by

$$(14)\qquad c2^{3\mu}\sum_{n\in\mathbb{Z}}\int\limits_{2\pi/3}^{4\pi/3}(1+2^\mu|\xi+2\pi n|)^{-K}|\xi-\pi|^{-1/2}|\xi-\pi|^4 \times(x^2+y^2-2xy\cos\xi)^{-2}\,d\xi.$$

As with (12) we use (9) to get

$$|(14)|\le c2^{-\mu L}(x+y)^{-4},\ c2^{-\mu L}(x+y)^{-2}(x^2+y^2)^{-1}$$

with L arbitrarily large. This completes the discussion of the case where both differentiations in (8) fall on $\widehat{m}$.

It is helpful to index the remaining cases. If we substitute the definitions of $F(\xi)$ and $H(\xi)$, then there are 7 factors in (8) where a $d/d\xi$ can fall: 3 coming from $F(\xi)$ and 2 coming from each $H(\xi)$. The first factor is $\widehat{m}$, the second factor is $e^{i\xi/2}$, and so on. Let (i,j) refer to the case where the inside $d/d\xi$ operates on the ith factor and the outside $d/d\xi$ operates on the jth factor. There are 17 *distinct* cases: (1,1)–(1,5), (2,2)–(2,5), (3,3)–(3,5), (4,4)–(4,6), (5,5), (5,7). Cases (1,2)–(1,4), (2,2)–(2,4), (3,3), (3,4), (4,4), and (4,6) can be handled like the model case (1,1) discussed above. The remaining 6 cases involve differentiations on $(x^2+y^2-2xy\cos\xi)^{-1}$. For example we bound the (1,5) case by a quantity of the form

$$(15)\qquad c2^{2\mu}\int\limits_{-\infty}^{\infty}(1+2^\mu|\xi|)^{-K}(1-\cos 2\xi)^{-1/4}\sin^4\xi\,\frac{|xy\sin\xi|}{(x^2+y^2-2xy\cos\xi)^3}\,d\xi.$$

It follows from (9) that

$$\frac{|xy\sin\xi|}{x^2+y^2-2xy\cos\xi}\le\begin{cases} c|\sin\xi|(1-\cos\xi)^{-1} & \text{if } \xi\in A,\\ c & \text{if } \xi\in B,\\ c|\sin\xi|(1+\cos\xi)^{-1} & \text{if } \xi\in C.\end{cases}$$

With this observation (15) can be handled just like (11). An interested reader can now easily check the remaining 5 cases.

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