assumption (iii) we get $|f_j|(x) \sim |x|^{\deg f_j}$. Since $\omega|_{S_i}$ is of order m we see that $g_i|_{S_i}$ is of order $m - \sum \deg f_j + 1$. The latter means that g_i can be written as $g_i = f_i g_i' + g_i''$, $\deg g_i'' \le m - k + 1$.

Indeed, as the highest order homogeneous part $\widehat{f}_i$ of f_i is irreducible (for $p > 2$) or has only simple zeroes (on $S_\infty = P^1_{\mathbb{C}}$ for $p = 2$) the highest order homogeneous part $\widehat{g}_i$ of g_i vanishes on $\widehat{f}_i = 0$. So, $\widehat{f}_i$ divides $\widehat{g}_i$, $\widehat{g}_i = \kappa \widehat{f}_i$ and $g_i = \kappa f_i + g_{i1}$ with g_{i1} of smaller degree than $\deg g_i$. Then we apply the same to g_{i1} etc.

Hence,

$$\omega = \Big(\prod f_j\Big)\Big[\eta + \sum g_i' df_i\Big] + \omega', \quad \omega' = \sum g_i''\Big(\prod_{j\ne i} f_j\Big) df_i,$$

ω' is of degree $\le m$ and the form $\eta + \sum g_i' df_i$ has degree $m - k$.

If $m = k - 1$ then $\deg g_i = 0$, $g_i = \alpha_i$ =const. Because $\deg \prod f_j \ge m + 1$ we have $\eta = 0$. Therefore $\omega = M^{-1} dH$, where $H = \prod f_j^{\alpha_j}$ is a Darboux first integral and $M = \prod f_j^{\alpha_j - 1}$ is an integrating factor.

If $m < k - 1$ then we find $g_i = 0$ and $\eta = 0$. Thus $\omega = 0$. ∎

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Averages of unitary representations and weak mixing of random walks

by

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Abstract. Let $\mathcal{S}$ be a locally compact (σ-compact) group or semigroup, and let $T(t)$ be a continuous representation of $\mathcal{S}$ by contractions in a Banach space X. For a regular probability μ on $\mathcal{S}$, we study the convergence of the powers of the μ-average $Ux = \int T(t)x\,d\mu(t)$. Our main results for random walks on a group G are:

(i) The following are equivalent for an adapted regular probability on G: μ is strictly aperiodic; U^n converges weakly for every continuous unitary representation of G; U is weakly mixing for any ergodic group action in a probability space.

(ii) If μ is ergodic on G metrizable, and U^n converges strongly for every unitary representation, then the random walk is weakly mixing: $n^{-1}\sum_{k=1}^n |\langle \mu^k * f, g\rangle| \to 0$ for $g \in L_\infty(G)$ and $f \in L_1(G)$ with $\int f\,d\lambda = 0$.

(iii) Let G be metrizable, and assume that it is nilpotent, or that it has equivalent left and right uniform structures. Then μ is ergodic and strictly aperiodic if and only if the random walk is weakly mixing.

(iv) Weak mixing is characterized by the asymptotic behaviour of μ^n on $UCB_l(G)$.

1. Introduction. Let $\mathcal{S}$ be a locally compact (σ-compact) semigroup (always assumed Hausdorff). For a regular probability μ on $\mathcal{S}$, the convolution operator $\mu * f(t) = \int f(ts)\,d\mu(s)$ is a Markov operator on $C(\mathcal{S})$, which is the average of the translation operators $\delta_s * f(t) = f(ts)$. When $\mathcal{S} = G$ is a locally compact group with right Haar measure λ, the *regular* representation $s \to \delta_s$ is continuous in $L_p(G, \lambda)$, $1 \le p < \infty$, $C_0(G)$ and $UCB_l(G)$.

Let X be a Banach space, and let $T : \mathcal{S} \to B(X)$ be a bounded operator representation of $\mathcal{S}$ (i.e., $T(st) = T(s)T(t)$, and $\sup_s \|T(s)\| < \infty$). The representation is called *continuous* if $t \to T(t)x$ is continuous for every $x \in X$, and *weakly continuous* if $f(t) = \langle x^*, T(t)x\rangle$ is continuous for $x \in X^*$ and $x \in X$. For groups, this implies (strong) continuity [HRo, p. 340]. For a regular probability μ on $\mathcal{S}$, the μ-average $U_\mu x = \int T(t)x\,d\mu$ is defined in the

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strong operator topology for strongly continuous bounded representations. If X is reflexive and the representation is weakly continuous, then $U_\mu x$ is defined in the weak operator topology by $\langle x^*, U_\mu x\rangle = \int \langle x^*, T(t)x\rangle\, d\mu(t)$. It is easily checked that $U_{\mu*\nu} = U_\mu U_\nu$, with the convolution $\mu * \nu$ defined by $\mu * \nu(A) = \int\int 1_A(ts)\, d\mu(t)\, d\nu(s)$. Note that $U_\mu^* = \int T^*(t)\, d\mu(t)$ is always defined in the weak-$*$ topology of X^*.

An important property for the study of the asymptotic behaviour of U_μ^n is *ergodicity* of μ (i.e., the only $g \in C(\mathcal{S})$ satisfying $\int g(st)\, d\mu(s) = g(t)$ are the constant functions).

For a group G, ergodicity means that $\check{\mu} * g = g$ is satisfied only by constants ($\check{\mu}(A) = \mu(A^{-1})$ is the "reflection" of μ), and is equivalent to ergodicity of the random walk (i.e., $\|N^{-1}\sum_{k=1}^N \mu^k * f\|_1 \to 0$ for $f \in L_1(G)$ with $\int f\, d\lambda = 0$). There are two interesting problems in the study of the asymptotic behaviour of random walks on groups:

1. The "complete mixing problem": If μ is an ergodic strictly aperiodic probability (i.e., G is the smallest closed normal subgroup a coset of which contains the support of μ), is the random walk generated by μ completely mixing (i.e., $\|\mu^n * f\|_1 \to 0$ for $\int f\, d\lambda = 0$)?

2. The "concentration function problem": If G is non-compact, and μ is *adapted* (the closed group generated by its support is G) and strictly aperiodic, do we have $\|\mu^n * f\|_\infty \to 0$ for every $f \in C_0(G)$?

The approach to the second problem in [DL$_2$] was by studying the more general "unitary representation problem": If μ is adapted and strictly aperiodic, does U_μ^n converge strongly for any continuous unitary representation?

All these problems have a positive solution if G is Abelian (see e.g. [L]), or if μ is *spread-out* (i.e., μ^n is not singular for some n)—see [G] and [DL$_2$].

These problems were studied in [LW$_2$], via the study of the convergence properties of the iterates of the μ-average of a bounded continuous representation of G (or, more generally, of a semigroup $\mathcal{S}$). It was shown there that if $\mu \times \mu$ is ergodic on $\mathcal{S} \times \mathcal{S}$, then U_μ^n converges strongly for any continuous reprsentation of $\mathcal{S}$ by contractions in a Hilbert space. It follows from the general theory of L_∞-Markov operators that for a group G, ergodicity of $\mu \times \mu$ implies *weak mixing* of μ (i.e., $n^{-1}\sum_{k=1}^n |\langle \mu^k * f, g\rangle| \to 0$ for every $g \in L_\infty(G,\lambda)$ and $f \in L_1(G,\lambda)$ with $\int f\, d\lambda = 0$), which is equivalent to $\check{\mu} * g = \alpha g \in L_\infty$ with $|\alpha| = 1$ holding only for $\alpha = 1$ and g constant (see [AaLWe]). Weak mixing implies ergodicity and strict aperiodicity of μ [LW$_2$]. The "weak mixing problem" is whether the converse is true (though the general answer to the complete mixing problem may be negative).

The purpose of this paper is to study the relationship between weak mixing of μ, ergodicity and strict aperiodicity of μ, and convergence of U_μ^n for all continuous unitary representations. For μ adapted, we characterize in Section 2 weak convergence of all U_μ^n, and study strong convergence. In Section 3 we prove that a positive solution to the unitary representation problem implies a positive solution to the weak mixing problem (in metrizable groups), and apply this result to metrizable groups which are either nilpotent, or have equivalent left and right uniform structures.

2. Convergence of U^n for unitary representations. In this section we investigate the convergence of U_μ^n for μ adapted and strictly aperiodic on a locally compact σ-compact group G when U_μ is obtained from a continuous unitary representation in a Hilbert space (called, for short, just a *unitary* representation).

In the Abelian case, μ is necessarily completely mixing ([KeMa], [S]), and $U_\mu^n(U_\mu - I)$ converges strongly to 0 for any bounded continuous representation in a Banach space (see e.g. [L], where a semigroup extension is also given). On the other hand, in the non-amenable case μ is never ergodic [A], so the non-commutative (non-compact) case requires a more careful analysis (see e.g. [LW$_2$]). It was proved in [DL$_2$] that U_μ^n converges weakly, and strongly if μ is spread-out, for any isometric continuous representations of G in a uniformly convex Banach space (the weak convergence result was extended to semigroups in [LW$_2$]). Strong convergence for general μ, under the above hypothesis, was proved in [LW$_2$] for groups in the class [SIN] (i.e., with equivalent left and right uniform structures [HRo, p. 22]). The concentration function problem for μ irreducible (the closed semigroup generated by its support is G) is treated in [HoM], and solved in [Wi].

Recall [DL$_2$] that for μ adapted with support S, μ is strictly aperiodic if and only if the closed subgroup generated by $\bigcup_{n=1}^\infty (S^{-n}S^n \cup S^nS^{-n})$ is G.

Weak mixing properties of Hilbert space contractions are given by the Spectral Mixing Theorem [K, pp. 96–97]. An application to ergodic Markov operators with invariant probability, in particular to U_μ obtained from an ergodic action of G by probability preserving transformations, yields the equivalence of weak mixing to non-existence of unimodular eigenvalues different from 1.

THEOREM 2.1. *Let μ be an adapted probability on a locally compact σ-compact group G. Then the following are equivalent:*

(i) *μ is strictly aperiodic.*

(ii) *For every unitary representation of G, U_μ^n converges weakly.*

(iii) *For every unitary representation of G in a complex Hilbert space, U_μ has no unimodular eigenvalues $\neq 1$.*

(iv) *U_μ is weakly mixing for any ergodic group action in a probability space.*

Proof. (i)$\Rightarrow$(ii). We give a simpler proof than that of [DL$_2$], using ideas of [D].

Let $T(t)$ be a unitary representation. Denote U_μ by U. Let
$$H_1 = \{x \in \mathcal{H} \mid U^{*n}U^n x = x = U^n U^{*n} x \text{ for every } n\}.$$
By uniform convexity, if $x \in H_1$, then $T(t)x = x$ for every
$$t \in \bigcup_{n=1}^{\infty} [\operatorname{supp}(\check{\mu}^n * \mu^n) \cup \operatorname{supp}(\mu^n * \check{\mu}^n)] = \bigcup_{n=1}^{\infty} (\overline{S^{-n}S^n} \cup \overline{S^n S^{-n}}).$$
Since $\{t \mid T(t)x = x\}$ is a closed subgroup, the strict aperiodicity yields $x \in H_1 \Leftrightarrow T(t)x = x$, $\forall t \in G$. Hence $Ux = x$ if (and only if) $x \in H_1$. For $x \perp H_1$ we have $U^n x \to 0$ weakly [F, p. 85]. Hence U^n is weakly convergent.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) is obvious.

(iv)$\Rightarrow$(i). Let H be the smallest closed normal subgroup a coset of which contains S. By [DL$_2$, Prop. 1.6], G/H is a monothetic Abelian group—either compact or $\mathbb{Z}$. Assume $H \neq G$, and let φ be the canonical projection of G onto G/H. Then $\varphi(S) = \{p\}$, and p generates G/H [DL$_2$].

Assume first that G/H is compact. Then $\varphi(t)$ is an ergodic action in $(G/H, \lambda_{G/H})$, and $T(t)f(x) = f(\varphi(t)x)$ is a unitary representation, continuous since φ is. Since $\varphi(s) = p$ for any $s \in S$, $U_\mu f(x) = f(px)$. Since G/H is Abelian and $p \neq e$, there exists a character χ with $\chi(p) \neq 1$. Then $U_\mu \chi = \chi(p)\chi$, yielding a unimodular eigenvalue $\neq 1$, contradicting (iv).

If $G/H \cong \mathbb{Z}$, we denote by $\varphi(t)^* \in \mathbb{Z}$ the number corresponding to $\varphi(t)$. Then $\varphi(s)^* = p^* = 1$ for $s \in S$, and $t \to \varphi(t)^*$ is a continuous homomorphism of G onto $\mathbb{Z}$. Let Γ be the unit circle, and α an irrational rotation. For $f \in L_2(\Gamma)$ define $T(t)f(z) = f(z\alpha^{\varphi(t)^*})$. Then $T(t)$ is a continuous unitary representation of G, induced by an ergodic action of G, and $U_\mu f(z) = f(\alpha z)$. But $f(z) = z$ is then an eigenfunction with eigenvalue $\alpha \neq 1$, contradicting (iv).

Hence $H \neq G$ leads to a contradiction, and (i) holds.

Remarks. (1) If we know (ii) only for *real* Hilbert spaces, we deduce from (ii) the weak mixing condition (iv) and then obtain (i).

(2) It was shown in [DL$_1$] that $\mu \times \delta_1$ is adapted on $G \times \mathbb{Z}$ if and only if μ is adapted and strictly aperiodic.

(3) If G is amenable, the conditions of the theorem are equivalent to

(v) *For every bounded continuous representation of G in a Hillbert space, U_μ^n converges weakly.*

Clearly (v)$\Rightarrow$(ii), and, in the amenable case, (ii)$\Rightarrow$(v) using [P, p. 187], [Ly, p. 83].

It is not known if (i) (or any of the other equivalent conditions) implies *strong* convergence of U^n for every unitary representation (the "unitary representation problem"). It does if μ is spread-out [DL$_2$], or when G is in the class [SIN] (cf. [LW$_2$]).

THEOREM 2.2. *Let G be a σ-compact metric group and let μ be an adapted probability on G. Then, U_μ^n converges strongly for every unitary representation of G if and only if U_μ^n converges strongly for every irreducible unitary representation.*

Proof. Since G is separable, for a given representation $T(t)$ and $x \in \mathcal{H}$ we know that $\operatorname{clm}\{T(t)x : t \in G\}$ (where $\operatorname{clm} Y$ is the closed subspace generated by Y) is separable (and T-invariant). Hence we may assume $\mathcal{H}$ separable.

For an irreducible unitary representation $T_\gamma(t)$, let $U_{\mu,\gamma} = \int T_\gamma(t)\,d\mu(t)$. For the given representation $T(t)$, the direct integral representation [Di, §8.5] $T(t) = \int_\Lambda^\oplus T_\gamma(t)\,dF(\gamma)$ yields
$$U_\mu^n = \int_\Lambda^\oplus U_{\mu,\gamma}^n\,dF(\gamma).$$
We restrict ourselves to $\operatorname{clm} \bigcup_{t \in G} (I - T(t))\mathcal{H} = \overline{(I - U_\mu)\mathcal{H}}$ (the equality because $Ux = x \Leftrightarrow T(t)x = x$ for all t, since μ is adapted). Hence U_μ has no fixed points, so the irreducible trivial identity representation does not appear in the direct integral representation. For T_γ non-identity irreducible, $U_{\mu,\gamma}$ has no fixed points, so our assumption is $U_{\mu,\gamma}^n \to 0$ strongly. Let $x \in \mathcal{H}$, $\|x\| = 1$. Then $x = \int^\oplus x_\gamma\,dF(\gamma)$ with $x_\gamma \in \mathcal{H}_\gamma$, the Hilbert space in which T_γ represents G, and $\|x_\gamma\| = 1$. Hence, if x has $F\{\gamma \mid x_\gamma \neq 0\} < \infty$, then
$$\|U_\mu^n x\|^2 = \int \|U_{\mu,\gamma}^n x_\gamma\|^2\,dF(\gamma) \to 0$$
by our assumption and Lebesgue's theorem. But the vectors x of this form are dense in the Hilbert space.

Remark. We have adapted here the proof of Proposition 3.4 of [JRT] (which is now a trivial corollary).

THEOREM 2.3. *Let G be a σ-compact locally compact group, and let μ be an adapted strictly aperiodic probability. If $T(t)$ is a unitary representation of G with $\lim_n \|U_\mu^n\| \neq 0$, then there exist a dense subgroup H and a sequence $\{x_j\}$ with $\|x_j\| = 1$ such that $\|T(t)x_j - x_j\| \to 0$ for every $t \in H$.*

Proof. Denote U_μ by U. If $\|U^k\| < 1$, then $\lim_n \|U^n\| = \lim_n \|U^{kn}\| = 0$. Hence the given assumption is $\|U^n\| = 1$ for every n. Therefore for each n there is y_n with $\|y_n\| = 1$ and $\|U^n y_n\| > 1 - 1/n$.

Let $z_n = U^n y_{2n}$. Then
$$1 - 1/n \le \|U^{2n} y_{2n}\| = \|U^n z_n\| \le \|z_n\| \le \|y_{2n}\| = 1.$$

Hence $\|z_n\| \to 1$. We also have

$$1 \geq \|U^{*n} z_n\| = \|U^{*n} U^n y_{2n}\| \\ \geq \langle U^{*n} U^n y_{2n}, y_{2n} \rangle = \|U^n y_{2n}\|^2 \geq (1 - 1/n)^2$$

We now define $x_n = z_n / \|z_n\|$. Since $\|z_n\| \to 1$, we obtain $\|U^{*n} x_n\| \to 1$ and $\|U^n x_n\| \to 1$. For k fixed and $n \geq k$ we have $\|U^n x_n\| \leq \|U^k x_n\| \leq 1$ and $\|U^{*n} x_n\| \leq \|U^{*k} x_n\| \leq 1$, so $\lim_{n\to\infty} \|U^k x_n\| = \lim_{n\to\infty} \|U^{*k} x_n\| = 1$. This is equivalent to

$$(*) \qquad \lim_{n\to\infty} \langle U^{*k} U^k x_n, x_n \rangle = 1 = \lim_{n\to\infty} \langle U^k U^{*k} x_n, x_n \rangle,$$

which can be rewritten as

$$(**) \qquad \lim \langle U_{(\check{\mu}^k * \mu^k + \mu^k * \check{\mu}^k)/2} x_n, x_n \rangle = 1.$$

Now, for any probability ν, if $\langle U_\nu x_n, x_n \rangle \to 1$, then

$$\int [1 - \mathrm{Re}\langle T(t) x_n, x_n \rangle] \, d\nu = 1 - \mathrm{Re}\langle U_\nu x_n, x_n \rangle \to 0,$$

which yields $\mathrm{Re}\langle T(t) x_n, x_n \rangle \to 0$ in ν measure (the integrand is non-negative). Hence $\langle U_\nu x_n, x_n \rangle \to 1$ implies the existence of $\{n_j\}$ such that $\langle T(t) x_{n_j}, x_{n_j} \rangle \to 1$ ν-a.e.

Let $\eta = \sum_{k=1}^{\infty} 2^{-(k+1)} (\check{\mu}^k * \mu^k + \mu^k * \check{\mu}^k)$. By $(**)$ and the diagonal process, we have a subsequence $\{n_j\}$ with $\langle T(t) x_{n_j}, x_{n_j} \rangle \to 1$ η-a.e.

Let $H = \{t \mid \|T(t) x_{n_j} - x_{n_j}\| \to 0\}$. Then by the above, $\eta(H) = 1$, and clearly H is a subgroup. The support of η is $\overline{\bigcup_{k=1}^{\infty} (S^{-k} S^k \cup S^k S^{-k})}$ by its definition, and is contained in H. Hence $\overline{H}$ contains the closed subgroup generated by the support of η, which is G by strict aperiodicity. We take $\{x_{n_j}\}$ for the claimed $\{x_j\}$.

Remarks. (1) A special case of our theorem is Theorem 3.6 of [JRT] (where "strict aperiodicity" has the *stronger* meaning "S is not contained in a left coset of a closed subgroup of G", which is equivalent to "G is the closed subgroup generated by $S^{-1}S$").

(2) If G is metric, we can have a countable H in the theorem.

DEFINITION. An *approximate fixed point* for a representation T (in any Banach space) is a sequence $\{x_n\}$ with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|T(t) x_n - x_n\| = 0$ for every $t \in G$.

Remark. *If T is a unitary representation which has an approximate fixed point, then* $\lim \|U^n\| = 1$. This has to be proved only if the representation has no common fixed points, i.e., when $\mathcal{H} = \overline{(I - U)\mathcal{H}}$. But an approximate fixed point $\{x_n\}$ satisfies

$$\|(I - U^k) x_n\| = \left\| \int [x_n - T(t) x_n] \, d\mu^k(t) \right\| \\ \leq \int \|x_n - T(t) x_n\| \, d\mu^k(t) \underset{n\to\infty}{\longrightarrow} 0$$

by Lebesgue's theorem. Hence $I - U^k$ is not invertible, so $\|U^k\| = 1$ for every k.

The next two results, due to R. Howe (and used implicitly in [JRT]), were communicated to us by J. Rosenblatt.

LEMMA 2.4. *If $T(t)$ is a faithful irreducible unitary representation of G, and G has non-trivial center, then the representation has no approximate fixed points.*

Proof. Let $t_0 \neq e$ be in the center $Z(G)$. Since the representation is irreducible, $T(t_0) = \alpha I$ by Schur's lemma (see [HRo, 21.30]), and since the representation is faithful, $\alpha \neq 1$. If there is an approximate fixed point $\{x_n\}$, then

$$|1 - \alpha| = \|T(t)_0 x_n - x_n\| \to 0$$

yields a contradiction.

PROPOSITION 2.5. *Let G be nilpotent. If $T(t)$ is an irreducible unitary representation, not identically I, then $T(t)$ has no approximate fixed points.*

Proof. Let $G' = \{t \in G \mid T(t) = I\}$. Then G' is a closed normal subgroup, $G' \neq G$ and $G_1 = G/G'$ is also nilpotent. If $t_1 = t_2 \mod G'$, then $T(t_1) = T(t_2)$. Hence $T_1(\widehat{t}) = T(t)$ is well defined on G_1, and is clearly a faithful representation. Since T_1 and T have the same range, T_1 is also irreducible. Any approximate fixed point for T is such for T_1—but it cannot exist, as is seen by applying the lemma to G_1, which (being nilpotent) has non-trivial center.

COROLLARY 2.6. *Let μ be an adapted probability on a nilpotent σ-compact locally compact group. If $T(t)$ is a non-trivial irreducible unitary representation, then $\|n^{-1} \sum_{k=1}^{n} U_\mu^k\| \to 0$ as $n \to \infty$. If μ is also strictly aperiodic, then $\|U_\mu^n\| \to 0$.*

Proof. Assume first that μ is strictly aperiodic. We use Theorem 2.3, and then apply Proposition 2.5 to the dense subgroup H, which is also nilpotent.

If we drop the strict aperiodicity, we note that $\nu = (\delta_e + \mu)/2$ is adapted and strictly aperiodic since μ is adapted. Hence $(I - U_\nu)\mathcal{H} = (I - U_\mu)\mathcal{H}$ implies $\|U_\nu^n\| \to 0$ if and only if $\|n^{-1} \sum_{k=1}^{n} U_\mu^k\| \to 0$. The result is now obtained by applying to U_ν the result for the strictly aperiodic case.

COROLLARY 2.7. *Let G be a nilpotent σ-compact metric group. If μ is adapted and strictly aperiodic, then U_μ^n converges strongly, for any unitary representation.*

Proof. The previous corollary proves the result for non-identity irreducible representations. We now apply Theorem 2.2.

Remarks. (1) A nilpotent group need not be in [SIN], so the previous result does not follow from [LW$_2$].

(2) The corollary generalizes Corollary 3.8 of [JRT].

(3) For the regular representation in $L_2(G)$ we have $\|\mu^n\|_2 = 1$ for any μ, by amenability (using the existence of almost invariant vectors [Z]). Hence norm convergence need not hold without irreducibility.

In some cases, we can improve Theorem 2.3. As an application, we shall obtain uniform (operator-norm) convergence of U^n for a certain class of non-amenable groups—those with Kazhdan's property (T) (by [DGu], for μ adapted on any non-amenable group, $\|\mu^n\|_2 \to 0$).

The following lemma follows from the proof of [LW$_2$, Theorem 2.11].

LEMMA 2.8. *Let μ be a spread-out probability on G. Then for $\varepsilon > 0$ there exist a number N and a neighbourhood A of e such that for $t^{-1}s \in A$ and $n \geq N$ we have $\|T(t)U^n - T(s)U^n\| < \varepsilon$.*

THEOREM 2.9. *Let μ be an adapted strictly aperiodic spread-out probability on a locally compact σ-compact group. Let $T(t)$ be a unitary representation. If $\lim_{n\to\infty} \|U_\mu^n\| > 0$, then there exists $\{u_j\}$, $\|u_j\| = 1$, such that $\|T(t)u_j - u_j\| \to 0$ as $j \to \infty$ uniformly on compact sets.*

Proof. Let $\{y_n\}$ be as at the beginning of the proof of Theorem 2.3, i.e., $\|y_n\| = 1$ and $\|U^n y_n\| > 1 - 1/n$, and let $z_n = U^n y_{2n}$. It is shown in Theorem 2.3 that $\|z_n\| \to 1$, and $x_n = z_n/\|z_n\|$ has a subsequence $\{x_{n_j}\}$ such that $\|T(t)x_{n_j} - x_{n_j}\| \to 0$ for t in a dense subgroup H. Let $u_j = x_{n_j}$.

We will now show convergence for *every* $t \in G$, uniformly on compact sets. Let $C \subset G$ be a compact set. For $\varepsilon > 0$, let N and A be obtained by the previous lemma. Let A_1 be a neighbourhood of e such that $A_1^{-1}A_1 \subset A$, and let $\{t_i A_1\}_{i=1}^r$ be a finite covering of C, with $t_i \in C$. There exist $\{s_i\}_{i=1}^r$ with $s_i \in H$ and $s_i \in t_i A_1$, by density of H. Let $t \in C$. Then $t \in t_i A_1$ for some i, and $t^{-1}s_i \in A_1^{-1}A_1 \subset A$, so for $j \geq N$ we obtain

$$\begin{aligned}\|T(t)u_j - u_j\| &\leq \|T(t)u_j - T(s_i)u_j\| + \|T(s_i)u_j - u_j\| \\ &= \|T(t)U^{n_j}y_{2n_j} - T(s_i)U^{n_j}y_{2n_j}\|/\|z_{n_j}\| + \|T(s_i)u_j - u_j\| \\ &\leq 2\varepsilon + \|T(s_i)u_j - u_j\|,\end{aligned}$$

which shows uniform convergence to zero, since there are finitely many s_i.

THEOREM 2.11. *Let G be a locally compact σ-compact group in the class [SIN], let μ be an adapted and strictly aperiodic probability, and let $T(t)$ be a unitary representation. If $\lim_{n\to 0} \|U_\mu^n\| > 0$, then there exists $\{u_j\}$, $\|u_j\| = 1$, such that $\|T(t)u_j - u_j\| \to 0$ as $j \to \infty$ uniformly on compact sets.*

Proof. Since $\|U\| \leq 1$, the assumption is $\|U^n\| = 1$ for every n. By the uniform boundedness principle, there exists $x \in \mathcal{H}$ with $\sup \|nU^n x\| = \infty$. Let $a_n = \|nU^n x\|$. Then $\limsup_{n\to\infty}(a_{n+i}/a_n) \leq 1$ for each i. If $\limsup_{n\to\infty}(a_{n+i}/a_n) < 1$ for some i, then $\sum_{k=1}^\infty a_{ki+j} < \infty$ for each $0 \leq j < i$, so $a_n \to 0$, a contradiction. Hence

$$\limsup_{n\to\infty} (n+i)\|U^{n+i}x\|/(n\|U^n x\|) = 1 \qquad \text{for each } i \geq 1,$$

so there exists an increasing subsequence $\{k_i\}$ with $\|U^{k_i+i}x\|/\|U^{k_i}x\| > 1-1/i$. Let $y_i = U^{k_i}x/\|U^{k_i}x\|$. Then $\|y_i\| = 1$ and $\|U^i y_i\| > 1-1/i$. It follows from the proof of Theorem 2.3 that for $z_n = U^n y_{2n}$ we have $\|z_n\| \to 1$, and $x_n = z_n/\|z_n\|$ has a subsequence $u_j = x_{n_j}$ with $\|T(t)u_j - u_j\| \to 0$ for t in a dense sugroup H.

In the proof of [LW$_2$, Theorem 2.9], it is shown that if G is in [SIN], then for $x \in X$ and every $\varepsilon > 0$, there is a neighbourhood A of e such that for $t^{-1}s \in A$ and $n \geq 1$ we have $\|T(t)U^n x - T(s)U^n x\| < \varepsilon$.

The proof of the convergence $\|T(t)u_j - u_j\| \to 0$ uniformly on compact sets is now as in the previous theorem.

Remark. The results of Theorems 2.3, 2.9 and 2.10 are valid if we drop the strict aperiodicity, provided we replace $\|U_\mu^n\|$ by $\|n^{-1}\sum_{k=1}^n U_\mu^k\|$. See the proof of Corollary 2.6.

DEFINITION [Z]. A locally compact σ-compact group *has Kazhdan's property* (T) if any unitary representation T, with $\|T(t)y_j - y_j\| \to 0$ uniformly on compacta for $\|y_j\| = 1$, has a non-zero fixed point. A non-compact group with Kazhdan's property (T) is *non-amenable* [Z, p. 132].

COROLLARY 2.11. *Let G be a locally compact σ-compact group having Kazhdan's property* (T), *and let μ be adapted. Assume either that G is in* [SIN] *or that μ is spread-out. Then for every unitary representation of G, $\|N^{-1}\sum_{n=1}^N U_\mu^n - E\| \to 0$ (where E is the orthogonal projection on the common fixed points). If μ is also strictly aperiodic, then $\|U_\mu^n - E\| \to 0$.*

Proof. Assume first that μ is also strictly aperiodic. $(I-E)\mathcal{H}$ is invariant for the representation, so we look at the representation in $(I-E)\mathcal{H}$. Hence, it is enough to assume $E = 0$. Now, if $\lim \|U^n\| > 0$, we deduce, from either Theorem 2.10 or Theorem 2.9, and from property (T), that $\{T(t)\}$ has a non-zero common fixed point—a contradiction. Now $\|U_{(I-E)\mathcal{H}}^n\| \to 0 \Leftrightarrow$

$\|U^n - E\| \to 0$ yields the result. When μ is not strictly aperiodic, we obtain the result as before, using $\nu = \frac{1}{2}(\delta_e + \mu)$.

Remark. The convergence $\|U_\mu^n - E\| \to 0$ for G *discrete* with property (T), under the stronger assumption that $S^{-1}S$ generates G, is proved in [JRT].

3. Ergodic theorems and weak mixing of random walks. Weak mixing of a Markov operator on L_1 is a strictly weaker notion than complete mixing, and strictly between them we have ergodicity of the Cartesian square [AaLWe]. However, it is not clear if these notions are different for random walks on groups (they coincide if μ is spread-out [G], or if G is Abelian or compact). It was proved in [LW_2, Lemma 3.4] that if μ is weakly mixing, then it is ergodic and strictly aperiodic. The "weak mixing problem" is whether the converse is true. Recall [A] that ergodic random walks exist only in *amenable* groups (and every amenable group has ergodic random walks—cf. [R], [KaV]).

In this section we deal with weak mixing, in the sense of [JL_1] and [JL_2], of the average U_μ of a continuous representation of a semigroup $\mathcal{S}$. The subspace on which the limit behaviour should be studied is $\overline{(I-U_\mu)X}$, which is a subspace of $N = \mathrm{clm}\bigcup_{t\in\mathcal{S}}(I-T(t))X$.

Ergodicity of μ means that only constants satisfy $\int g(st)\,d\mu(s) = g(t)$ with g bounded continuous, so it is well defined also for *semigroups*. It was shown in [LW_2] that ergodicity of μ on $\mathcal{S}$ implies that for every bounded continuous representation $T(t)$ in a Banach space X, we have

$$\overline{(I-U_\mu)X} = N, \tag{3.1}$$

$$U_\mu^* y^* = y^* \Leftrightarrow T^*(t)y^* = y^* \text{ for every } t \in \mathcal{S}. \tag{3.2}$$

The proof in [LW_2] shows that (3.1) and (3.2) also hold if only *weak continuity* of $T(t)$ is assumed, provided that U_μ, defined now by $\langle x^*, U_\mu x\rangle = \int \langle x^*, T(t)x\rangle\,d\mu(t)$, maps X into X, and not just into X^{**}.

THEOREM 3.1. *Let $\mathcal{S}$ be a locally compact Hausdorff semigroup with countable base, and let μ be an ergodic probability on $\mathcal{S}$. Assume that for every continuous representation of $\mathcal{S}$ by isometries in a Hilbert space, the iterates V_μ^n of the μ-average converge strongly. Then for every bounded continuous representation of $\mathcal{S}$ in a Banach space X, we have*

$$\lim_{n\to\infty}\Big\{\sup_{\|x^*\|\le 1}\frac{1}{n}\sum_{k=1}^{n}|\langle x^*, U_\mu^k x\rangle|\Big\} = 0 \quad \forall x \in N. \tag{3.3}$$

Proof. By passing to an equivalent norm, we may assume $\|T(t)\| \le 1$ for $t \in \mathcal{S}$. We denote U_μ by U. Let $x_0 \in N$. Then $\|n^{-1}\sum_{k=1}^n U^k x_0\| \to 0$.

Assume there is $x^* \in X^*$ with $\limsup n^{-1}\sum_{k=1}^n |\langle x^*, U^k x_0\rangle| > 0$. Let $Y = \mathrm{clm}\{T(t)x_0 \mid t \in \mathcal{S}\}$. Since $\mathcal{S}$ is separable, Y is a separable closed subspace, invariant under each $T(t)$. Thus, we may assume X to be separable. Let $B = \{x^* \in X^* \mid \|x^*\| \le 1\}$. Then B is a compact metric space in the w^*-topology. By our assumption, $\limsup n^{-1}\sum_{k=1}^n |\langle x_0^*, U^k x_0\rangle| = \alpha > 0$ for some $x_0^* \in B$.

Define $R(t)$ on $C(B)$ by $(R(t)f)(x^*) = f(T^*(t)x^*)$. Since $T^*(t)$ is w^*-continuous, $R(t)f \in C(B)$, and $\|R(t)\| \le 1$. Clearly $R(t)$ is a representation of $\mathcal{S}$. We show that it is weakly continuous: Let $m \in C(B^*)$ be a probability. If $t_n \to t_0$, then $T^*(t_n)x^* \to T^*(t_0)x^*$ for every $x^* \in B$, by w^*-continuity of $t \to T^*(t)x^*$. Hence

$$\begin{aligned}\langle m, R(t_n)f\rangle &= \int f(T^*(t_n)x^*)\,dm(x^*)\\ &\to \int f(T^*(t_0)x^*)\,dm(x^*) = \langle m, R(t)f\rangle\end{aligned}$$

for $f \in C(B)$ by Lebesgue's theorem.

Let $f \in C(B)$. For $x^* \in B$ fixed, $f(T^*(t)x^*)$ is continuous on $\mathcal{S}$, by w^*-continuity of $t \to T^*(t)x^*$, so the integral

$$Vf(x^*) = \int f(T^*(t)x^*)\,d\mu(t) = \int R(t)f(x^*)\,d\mu(t)$$

is well defined. Since each $T^*(t)$ is continuous on B, if $x_n^* \to x^*$ we obtain

$$Vf(x_n^*) = \int f(T^*(t)x_n^*)\,d\mu(t) \to \int f(T^*(t)x^*)\,d\mu(t) = Vf(x^*)$$

by Lebesgue's theorem (for $(\mathcal{S},\mu)$), and since B is metrizable, $Vf \in C(B)$.

Now, every $x \in X$ defines $f_x \in C(B)$ by $f_x(x^*) = \langle x^*, x\rangle$. We then have

$$\begin{aligned}f_{Ux}(x^*) &= \langle x^*, Ux\rangle = \int \langle x^*, T(t)x\rangle\,d\mu(t)\\ &= \int \langle T^*(t)x^*, x\rangle\,d\mu(t) = Vf_x(x^*).\end{aligned}$$

Hence $f_{U^n x} = V^n f_x$ for any $x \in X$.

Let $m \in C(B)^*$ be an invariant probability for V, i.e., $V^*m = m$. By ergodicity of μ, we obtain $R^*(t)m = m$ for every $t \in \mathcal{S}$. Hence $R(t)$ is also (extendable to) an isometry of $L_2(B,m)$. For $f \in C(B)$ we have $\int |R(t_n)f - R(t)f|^2\,dm \to 0$ when $t_n \to t$, by Lebesgue's theorem (pointwise convergence of the integrand follows from weak continuity of $R(t)$). Hence $R(t)$ is a strongly continuous representation of $\mathcal{S}$ by isometries in $L_2(m)$, and the μ-average $\int R(t)\,d\mu(t)$ is V. By the property of μ in the theorem, V^n is strongly convergent in $L_2(m)$. For f_{x_0} we have $\|n^{-1}\sum_{k=1}^n V^k f_{x_0}\|_{C(B)} \to 0$, hence also in $L_2(m)$, so $\|V^n f_{x_0}\|_{L_2(m)} \to 0$. Hence $\int |V^n f_{x_0}|\,dm \to 0$ for every V-invariant probability m.

Let $\lim n_j^{-1}\sum_{k=1}^{n_j} |\langle x_0^*, U^k x_0\rangle| = \alpha$. Let m be a weak-$*$ limit point, in $C(B)^*$, of $n_j^{-1}\sum_{k=1}^{n_j} V^{*k}\delta_{x_0^*}$. Then m is invariant for V. Since B is metrizable, $C(B)$ is separable, so there is a subsequence (still denoted by $\{n_j\}$)

with $\langle m, f\rangle = \lim_j n_j^{-1}\sum_{k=1}^{n_j} V^k f(x_0^*)$ for every $f \in C(B)$. Moreover, for $i \geq 1$ we have

$$\langle m, |V^i f|\rangle = \lim_j \frac{1}{n_j}\sum_{k=1}^{n_j}(V^k|V^i f|)(x_0^*)$$
$$\geq \limsup_j \frac{1}{n_j}\sum_{k=1}^{n_j}|V^{k+i}f|(x_0^*) = \limsup_j \frac{1}{n_j}\sum_{k=1}^{n_j}|V^k f|(x_0^*).$$

Hence

$$\langle m, |V^i f_{x_0}|\rangle \geq \lim \frac{1}{n_j}\sum_{k=1}^{n_j}|V^k f_{x_0}|(x_0^*) = \lim_j \frac{1}{n_j}\sum_{k=1}^{n_j}|\langle x_0^*, U^k x_0\rangle| = \alpha,$$

contradicting $\int |V^n f_{x_0}|\,dm \to 0$ obtained before. It follows that $n^{-1}\sum_{k=1}^{n}|\langle x^*, U^k x_0\rangle| \to 0$ for every $x^* \in B$. The uniformity on B (in (3.3)) follows from [JL$_1$] (see also [K]).

Remarks. (1) The assumption of a countable base is used for characterizing continuity in terms of convergence over sequences. The theorem can be applied to a locally compact group (if and) only if the group is metrizable σ-compact, since the existence of a countable base at the identity is equivalent to metrizability [HRo, 8.3].

(2) The theorem shows that a *positive solution* to the "unitary representation problem" will imply that for metrizable groups, μ is weakly mixing if and only if it is ergodic and strictly aperiodic.

(3) The theorem will be of value (for semigroups) even if there is a positive solution to the complete mixing problem.

COROLLARY 3.2. *Let S be as in the theorem, and let μ be a probability such that $\mu\times\mu$ is ergodic. Then for every bounded continuous representation in a Banach space*, (3.3) *holds.*

Proof. Clearly μ is ergodic, and μ satisfies the assumption of the theorem by [LW$_2$].

Remark. For any Markov operator, ergodicity of its Cartesian square implies weak mixing [AaLWe]. Since this can happen with an invertible transformation preserving a σ-finite measure, in general there will be no L_2-norm convergence, unlike what we obtain for random walks on groups by [LW$_2$]. For an ergodic Markov operator with finite invariant measure, L_2-norm convergence is equivalent to complete mixing. Thus, ergodicity of $\mu\times\mu$ seems to be a strong condition for random walks on groups, and it is not known if it is implied by weak mixing (or even by the strong convergence of U_μ^n for every unitary representation).

THEOREM 3.3. *Let G be a metrizable locally compact σ-compact group. Assume that G is in* [SIN] *or that G is nilpotent. Then the following are equivalent for a probability μ:*

(i) *μ is ergodic and strictly aperiodic.*

(ii) *μ is ergodic, and for every unitary representation, U_μ^n is strongly convergent.*

(iii) *For every bounded continuous representation in a Banach space,* (3.3) *holds.*

(iv) *μ is weakly mixing.*

Proof. (ii)$\Rightarrow$(iii) by Theorem 3.1.

(iii)$\Rightarrow$(iv). We defined $N = \operatorname{clm}\bigcup_{t\in G}(I - T(t))X$ (without requiring ergodicity). When we take the canonical representation by right translations in $L_1(G)$, (3.3) yields exactly weak mixing, since $N = \{f \in L_1(G) : \int f\,d\lambda = 0\}$ (see [LW$_1$]).

(iv)$\Rightarrow$(i) is proved in [LW$_2$].

(i)$\Rightarrow$(ii) is proved in [LW$_2$] for G in [SIN], and in the previous section for G nilpotent.

Remarks. (1) A positive solution to the complete mixing problem will make the previous theorem obsolete.

(2) The previous proof shows that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i) in *any* metrizable group.

(3) If μ is *symmetric*, the conditions in the theorem are equivalent in any metrizable group, since (i)$\Rightarrow$(ii) is satisfied [DL$_2$].

(4) By [HRo, 8.18 and 4.14(g)], G metrizable is in [SIN] if and only if it has a two-sided invariant metric.

We now show that the last two conditions in the previous theorem are always equivalent, and characterize weak mixing by properties of the regular representation (by right translations) in $UCB_l(G)$.

Let ν be a signed measure on a locally compact group G. Then $\nu * f(t) = \int f(ts)\,d\nu(s)$ defines an operator on various function spaces (as in the introduction), with norm at most $\|\nu\|$ (the total variation of the signed measure ν).

LEMMA 3.4. *Let $\{\nu_n\}$ be a sequence of signed measures on a locally compact σ-compact group G with $\sup_n\|\nu_n\| < \infty$. If f is right uniformly continuous and bounded, then $\{\nu_n * f\}$ is equicontinuous.*

Proof. Since f is right uniformly continuous, for any $\varepsilon > 0$ there exists a neighbourhood V_ε of the unit e such that

$$ss'^{-1} \in V_\varepsilon \Rightarrow |f(s) - f(s')| < \varepsilon.$$

Hence if $ss'^{-1} \in V_\varepsilon$, then $(st)(s't)^{-1} = ss'^{-1} \in V_\varepsilon$ and $|f(st) - f(s't)| < \varepsilon$ for any $t \in G$. Integrating over t with the measure ν_n, we obtain

$$ss'^{-1} \in V_\varepsilon \Rightarrow |\nu_n * f(s) - \nu_n * f(s')| \le \varepsilon \|\nu_n\|$$

and the assertion follows. In fact, we have shown that the sequence $\{\nu_n * f\}$ is even right uniformly equicontinuous.

THEOREM 3.5. *Let G be a locally compact σ-compact group. Then the following are equivalent for a probability μ:*

(i) *For every bounded continuous representation in a Banach space,* (3.3) *holds.*

(ii) (3.3) *holds for the regular representation of G by right translations in $UCB_l(G)$.*

(iii) *For every $t \in G$ and $f \in UCB_l(G)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\mu^k * (f - \delta_t * f)(e)| = 0.$$

(iv) *μ is weakly mixing.*

Proof. We trivially have (i)⇒(ii)⇒(iii), and (i)⇒(iv). We first prove (iii)⇒(i).

Let T be a bounded continuous representation of G in a Banach space X. Fix $y \in X$ and $x^* \in X^*$, and define $f(s) = \langle x^*, T(s)y\rangle$. Then $f \in UCB_l(G)$ by continuity of the representation, and we have

$$(\mu^k * \delta_t * f)(e) = \int f(rt)\, d\mu^k(r) = \int \langle x^*, T(rt)y\rangle\, d\mu^k(r) = \langle x^*, U_\mu^k T(t)y\rangle.$$

Using the above also for $t = e$, we obtain

$$\frac{1}{n} \sum_{k=1}^{n} |\langle x^*, U_\mu^k (y - T(t)y)\rangle| = \frac{1}{n} \sum_{k=1}^{n} |\mu^k * (f - \delta_t * f)(e)| \xrightarrow[n\to\infty]{} 0.$$

By continuity, we obtain $\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |\langle x^*, U_\mu^k x\rangle| = 0$ for $x \in N$ and $x^* \in X^*$. The uniformity in (3.3) now follows from [JL$_1$].

(iv)⇒(iii) follows by applying the following proposition to $\nu_n = \mu^n - \mu^n * \delta_t$ for each $t \in G$.

PROPOSITION 3.6. *The following properties are equivalent for a bounded sequence of signed measures on a locally compact group G:*

(i) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\nu_k * h(e)| = 0 \quad (h \in UCB_l(G)),$$

(ii) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \int h(x) \nu_k * f(x) \lambda(dx) \right| = 0 \quad (h \in L_\infty,\ f \in L_1).$$

Proof. The proof of (i)⇒(ii) is similar to that of (iii)⇒(i) in the previous theorem.

To show the converse implication, we assume that (ii) holds. Let g be a bounded right uniformly continuous function. By Lemma 3.4, for $\varepsilon > 0$ there exists a neighbourhood V of the unit element e of G such that $s \in V$ implies $|\check\mu_n * g(s) - \check\mu_n * g(e)| < \varepsilon$ for any $n \in \mathbb{N}$. Thus, if $f_U \in L_1^+$ with $\int f_U \, d\lambda = 1$ and $\{f_U > 0\} \subset U$, then

$$\left|\check\mu_n * g(e) - \int f_U (\check\mu_n * g)\, d\lambda\right| \le \varepsilon \quad (n \in \mathbb{N}).$$

On the other hand, by (ii), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \int f_U (\check\mu_k * g)\, d\lambda \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \int g(\nu_k * f_U)\, d\lambda \right| = 0.$$

Hence we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\check\mu_n * g(e)| \le \varepsilon.$$

Making $\varepsilon > 0$ arbitrarily small, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\check\mu_n * g(e)| = 0 \quad (g \in UCB_r(G)).$$

Now, if $h \in UCB_l(G)$, then $g(s) = h(s^{-1})$ is right uniformly continuous and $\nu_n * h(e) = \check\mu_n * g(e)$, so (ii) follows from the above.

Remark. Condition (ii) of the theorem can be formulated for a Markov operator P on a locally compact σ-compact metric space by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\langle x^*, P^k(I - P)f\rangle| = 0 \quad \forall f \in UCB,\ \forall x^* \in UCB^*.$$

In a compact metric space K, this becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |P^k(I - P)f(t)| = 0 \quad \forall f \in C(K),\ \forall t \in K,$$

and implies complete mixing in L_1 of any ergodic invariant probability. Thus, we expect that the converse of Theorem 3.1 is true (at least for groups, even if there is a negative answer to the unitary representation problem).

THEOREM 3.7. *Let G be a locally compact σ-compact group. Then the following are equivalent for a probability μ:*

(i) *For every bounded continuous representation in a Banach space,*

(3.4) $$U_\mu^* y^* = \alpha y^* \text{ with } |\alpha| = 1 \Rightarrow T^*(t) y^* = y^* \text{ for every } t \in G.$$

(ii) *The only functions* $g \in UCB_r(G)$ *satisfying* $\check{\mu} * g = \alpha g$ *with* $|\alpha| = 1$ *are the constants.*

(iii) *The only functions* $g \in C(G)$ *satisfying* $\check{\mu} * g = \alpha g$ *with* $|\alpha| = 1$ *are the constants.*

(iv) μ *is weakly mixing.*

Proof. (iv)$\Rightarrow$(iii) because even in L_∞, only constants satisfy the equation when μ is weakly mixing (since for any contraction U in a Banach space, (3.3) implies that U^* has no unimodular eigenvalues different from 1).

(iii)$\Rightarrow$(ii) is clear.

(ii)$\Rightarrow$(i). Let $T(t)$ be a bounded continuous representation in the Banach space X, and let $y^* \in X^*$ satisfy $U_\mu^* y^* = \alpha y^*$ with $|\alpha| = 1$. For $x \in X$ define $g(t) = \langle T^*(t^{-1})y^*, x\rangle$. By continuity of the representation, $g \in UCB_r(G)$. Since $T^*(t)$ is an anti-representation in X^*, we have

$$\begin{aligned}\check{\mu} * g(s) &= \int \langle T^*(t^{-1}s^{-1})y^*, x\rangle\, d\check{\mu}(t) = \int \langle T^*(t)y^*, T(s^{-1})x\rangle\, d\mu(t)\\ &= \langle U_\mu^* y^*, T(s^{-1})x\rangle = \alpha\langle y^*, T(s^{-1})x\rangle = \alpha g(s).\end{aligned}$$

By (ii), g is constant. Since this is so for every $x \in X$, we have $T^*(t)y^* = y^*$ for every $t \in G$.

(i)$\Rightarrow$(iv). We apply (3.4) to the regular representation by right translations in $L_1(G)$, and obtain (iv) from the general weak mixing theorem for Markov operators proved in [AaLWe].

Remarks. (1) The regular representation by right translations is continuous in $UCB_r(G)$ if and only if G has equivalent left and right uniform structures (i.e., $G \in$ [SIN]).

(2) For a general contraction U on a Banach space, (3.3) may fail even if U^* has no unimodular eigenvalues $\neq 1$ [JL$_2$].

THEOREM 3.8. *Let $\mathcal{S}$ be a locally compact Hausdorff semigroup with countable base, and let μ be an ergodic probability. Then the following are equivalent*:

(i) *For every continuous representation of $\mathcal{S}$ by isometries in a Hilbert space*, (3.3) *holds.*

(ii) *For every bounded continuous representation in a complex Banach space, U_μ has no unimodular eigenvalues $\neq 1$.*

(iii) *For every bounded continuous representation in a reflexive Banach space*, (3.3) *holds.*

(iv) *For every bounded continuous representation in a Banach space X in which every bounded sequence has a weakly Cauchy subsequence*, (3.3) *holds.*

(v) *For every bounded continuous representation in a Banach space such that $\{U^n x\}$ is weakly conditionally compact for every $x \in X$*, (3.3) *holds.*

Proof. Clearly (iv)$\Rightarrow$(iii)$\Rightarrow$(i).

(i)$\Rightarrow$(iv). Let $x_0 \in N$, and proceed as in the proof of Theorem 3.1 to construct V. Since $\|N^{-1}\sum_{k=1}^N U^k x_0\| \to 0$, we have $\|N^{-1}\sum_{k=1}^N V^k f_{x_0}\| \to 0$ in $C(B)$, and hence in $L_2(m)$ for any V-invariant probability m. Applying (3.3) to V we obtain $N^{-1}\sum_{n=1}^N |\langle V^n f_{x_0}, h\rangle| \to 0$ for every $h \in L_2(m)$, so for some subsequence, $V^{k_i} f_{x_0} \to 0$ weakly in $L_2(m)$. Since $U^{k_i}x_0$ has a subsequence $\{n_i\}$ which is weakly Cauchy, we have $g(x^*) = \lim V^{n_i} f_{x_0}(x^*)$ well defined, and Borel measurable. Since $V^{n_i} f_{x_0} \to 0$ weakly in $L_2(m)$, $g = 0$ m-a.e. Hence $|V^{n_i} f_{x_0}| \to 0$ m-a.e., so $\|V^n f_{x_0}\|_2 \to 0$. Now we can use the remainder of the proof of Theorem 3.1.

(i)$\Rightarrow$(ii). Let $Ux_0 = \alpha x_0$ with $|\alpha| = 1$, $\alpha \neq 1$. Then $x_0 \in N$. For any subsequence $\{\alpha^{k_i}\}$ there is a convergent subsequence $\{\alpha^{n_i}\}$, so the proof of (i)$\Rightarrow$(iv) above applies.

(ii)$\Rightarrow$(i). Let $T(t)$ be a representation of $\mathcal{S}$ by contractions in a complex Hilbert space. Then its μ-average U has no unimodular eigenvalues different from 1, so by the well-known "weak mixing theorem" [K, p. 96], U satisfies (3.3). The complex case yields the real case.

(v)$\Rightarrow$(i) is obvious, and (i)$\Rightarrow$(v) is proved like (i)$\Rightarrow$(iv).

Remark. A (separable) Banach space satisfies the hypothesis in (iv) if and only if it does not contain an isomorphic copy of l_1 (see [Ros] for the real case, [Do] for the complex case). In particular, (iv) applies to spaces with separable duals.

COROLLARY 3.9. *Let μ be an ergodic probability on a locally compact σ-compact metrizable group G. Then the conditions of the previous theorem are all equivalent to strict aperiodicity of μ* (*and its equivalent conditions in Theorem* 2.1).

Proof. Since μ is ergodic, it is adapted. Theorem 3.8(ii) implies Theorem 2.1(iii), and Theorem 2.1(ii) implies Theorem 3.8(i).

Remark. We do not know if also for μ ergodic on a *semigroup*, the conditions of Theorem 3.8 imply the weak convergence of U_μ^n for any representation by contractions on a Hilbert space. A sufficient condition for this property is given in [LW$_2$].

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