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Double exponential integrability, Bessel potentials and embedding theorems

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Abstract. This paper is a continuation of [5] and provides necessary and sufficient conditions for double exponential integrability of the Bessel potential of functions from suitable (generalized) Lorentz–Zygmund spaces. These results are used to establish embedding theorems for Bessel potential spaces which extend Trudinger's result.

1. Introduction. In a celebrated paper [4], Brézis and Wainger showed that estimates for the convolution of functions in Lorentz spaces $L^{p,q}$ could be used to establish the exponential integrability of the Riesz potential of suitable functions. This is related to the well-known result of Trudinger [18] that Sobolev spaces can, in certain limiting cases, be continuously embedded in spaces of Orlicz type, with a Young function of exponential character. Very recently, the authors [5] showed that use of a *double* limiting form of Hardy's inequalities coupled with replacement of $L^{p,q}$ by (generalized) Lorentz–Zygmund spaces enabled *double* exponential integrability of the Riesz potential of functions in appropriate spaces of this type to be proved.

The present paper continues this line of enquiry, its principal aim being to establish embedding theorems for Bessel potential spaces which generalize Trudinger's result. We provide necessary and sufficient conditions for double exponential integrability of Bessel potential of functions in suitable (generalized) Lorentz–Zygmund spaces. To explain this in a little more detail, suppose that $p \in [1,\infty]$, $\sigma \in (0,n)$, let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and let $Q \subset \mathbb{R}^n$ have finite Lebesgue n-measure and non-empty interior. Then we show in

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particular that given $A \in (0,\infty)$, there exist $a, M \in (0,\infty)$ such that

$$\int_Q \exp(A\exp[a|(g_\sigma * f)(x)|^{-1/\alpha}])\,dx \le M \tag{1.1}$$

for all f in the unit ball of the Lorentz–Zygmund space $L^{n/\sigma,p}(\log L)^{1/p'}(\mathbb{R}^n)$ if, and only if,

$$\alpha < 0, \quad \alpha + 1/p' \le 0. \tag{1.2}$$

Here g_σ is the Bessel kernel.

Analogous conclusions are obtained for Riesz potentials when, however, the functions f are also required to belong to $L^1(\mathbb{R}^n)$ because the Riesz kernel does not decay as quickly at infinity as the Bessel kernel. These results for Riesz potentials improve those given in [5] in two respects: they give necessary and sufficient conditions for (1.1) to hold, while [5] gives only sufficient conditions; and they extend the range of values of p from $(1,\infty)$ to $[1,\infty]$. The Bessel potential results are used to show that certain Bessel potential spaces, modelled upon (generalized) Lorentz–Zygmund spaces, may be embedded in appropriate Orlicz spaces. For example, it is proved that if $p \in (1,\infty)$ and $Q \subset \mathbb{R}^n$ has finite Lebesgue n-measure, then given any $A \in (0,\infty)$, the Bessel potential space

$$H^{n/p}L^p(\log L)^{1/p'}(\mathbb{R}^n) = \{g_{n/p} * f : f \in L^p(\log L)^{1/p'}(\mathbb{R}^n)\}$$

is continuously embedded in the Orlicz space $L_{\Phi_{A,1/p'}}(Q)$, where $\Phi_{A,1/p'}$ is a Young function such that $\Phi_{A,1/p'}(t) = \exp(A\exp(t^{p'}))$ for all large enough t. Results of a similar nature concerning compact embeddings are also proved, as are analogous theorems relating to the continuous or compact embeddings of Bessel potential spaces on domains with finite n-measure, rather than on the whole of $\mathbb{R}^n$.

We also indicate rather briefly the results obtainable by the use of a *single* limiting form of Hardy's inequality. Then our techniques establish theorems about *single* exponential integrability which resemble those derived by Fusco, Lions and Sbordone [8] and, in the higher-order case, in [6], and also extend results from [5, Remark 3.11(iv)].

The paper is organized as follows. Section 2 contains the notation and basic definitions, while Section 3 is devoted to preliminary results. The next section contains necessary and sufficient conditions for double exponential integrability of Bessel and Riesz potentials, and Section 5 provides embedding theorems. Finally, Section 6 gives results about single exponential integrability.

2. Notation. Let Q be a measurable subset of $\mathbb{R}^n$ (with respect to n-dimensional Lebesgue measure); by $|Q|_n$ we mean its n-volume; χ_Q will represent the characteristic function of Q.

For a measurable (real or complex) function f on Q which is finite a.e., the *distributional function* μ_f is given by

$$\mu_f(\lambda) = |\{x \in Q : |f(x)| > \lambda\}|_n, \quad \lambda > 0, \tag{2.1}$$

and the *non-increasing rearrangement* of f is the function f^* defined on $(0,\infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\}, \quad t > 0. \tag{2.2}$$

If $p, q \in (0,\infty]$ and $\lambda, \varepsilon \in \mathbb{R}$ the *generalized Lorentz–Zygmund space*

$$L^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon = L^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon(Q) \tag{2.3}$$

consists of all functions f on Q for which the quantity

$$\|f\|_{p,q,\lambda,\varepsilon} = \begin{cases} \left(\int_0^\infty [t^{1/p}(e+|\log t|)^\lambda \log^\varepsilon(e+|\log t|)f^*(t)]^q \frac{dt}{t}\right)^{1/q} & \text{for } q<\infty, \\ \sup_{t\in(0,\infty)} t^{1/p}(e+|\log t|)^\lambda \log^\varepsilon(e+|\log t|)f^*(t), & \text{for } q=\infty, \end{cases} \tag{2.4}$$

is finite.

As in the case of Lorentz spaces, the expression (2.4) is, in general, only a quasi-norm. However, it can be shown, by use of a convenient form of the Hardy inequality, that for $p \in (1,\infty]$, $q \in [1,\infty]$ and $\lambda, \varepsilon \in \mathbb{R}$, the quantity $\|\cdot\|_{p,q,\lambda,\varepsilon}$ defined as in (2.4) with $f^*(t)$ replaced by $f^{**}(t) := t^{-1}\int_0^t f^*(s)\,ds$ is a norm on $L^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon$ which is equivalent to the quasi-norm (2.4). In particular, for $p \in (1,\infty)$, the space $L^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon$ is (equivalent to) a Banach function space.

Let us mention that for $\lambda = \varepsilon = 0$ this space coincides with the classical *Lorentz space* $L^{p,q}$, with the quasi-norm $\|\cdot\|_{p,q}$, while for $\varepsilon = 0$ it is (equivalent to) the *Lorentz–Zygmund space* $L^{p,q}(\log L)^\lambda$ endowed with the quasi-norm $\|\cdot\|_{p,q,\lambda}$ and introduced in [2]. Consequently, for $\varepsilon = 0$ and $p = q$ we obtain the *Zygmund space* $L^p(\log L)^\lambda$ (cf. [2]), and if in addition $\lambda = 0$ we have the classical *Lebesgue space* $L^p = L^p(Q)$ with the norm $\|\cdot\|_p = \|\cdot\|_{p,Q}$, which is a Banach function space also for $p = 1$ and $p = \infty$. For more details we refer to [10], [2] and [3].

Throughout the paper the symbol I_σ, $\sigma \in (0,n)$, is used to denote the *kernel of the Riesz potential*, i.e. $I_\sigma(x) = |x|^{\sigma-n}$, $x \in \mathbb{R}^n$. The *Bessel kernel* g_σ, $\sigma > 0$, is defined as that function whose Fourier transform is

$$\widehat{g}_\sigma(x) = (2\pi)^{-n/2}(1+|x|^2)^{-\sigma/2}, \tag{2.5}$$

where the Fourier transform of the function f is given by

$$\widehat{f}(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot y} f(y)\,dy.$$

Let $0 < K \leq \infty$. By $\mathfrak{M}^+(0,K)$ we denote the set of all non-negative measurable functions on $(0,K)$. If $f \in \mathfrak{M}^+(0,K)$, the symbol $f\downarrow$ means that the function f is non-increasing on $(0,K)$. We put

$$\mathfrak{M}^+(0,K;\downarrow) := \{f \in \mathfrak{M}^+(0,K) : f\downarrow\},$$
$$\mathcal{W}(0,K) := \{w \in \mathfrak{M}^+(0,K) : 0 < w < \infty \text{ a.e. on } (0,K)\};$$

the elements of $\mathcal{W}(0,K)$ are called *weights.*

For $p \in [1,\infty]$ we put

$$Y(p) = L^p\left(0,1;\frac{dt}{t}\right), \quad Z(p) = L^p\left(0,1;\frac{dt}{t(e-\log t)}\right)$$

and endow these spaces with the norms

$$\|f\|_{Y(p)} = \left(\int_0^1 |f(t)|^p \frac{dt}{t}\right)^{1/p},$$
$$\|f\|_{Z(p)} = \left(\int_0^1 |f(t)|^p \frac{dt}{t(e-\log t)}\right)^{1/p} \quad \text{if } p < \infty,$$
$$\|f\|_{Y(\infty)} = \|f\|_{Z(\infty)} = \|f\|_{\infty,(0,1)} = \operatorname*{ess\,sup}_{t\in(0,1)} |f(t)|.$$

For $\varrho \in (0,\infty)$ and $x \in \mathbb{R}^n$ let $B_n(x,\varrho)$ denote the open ball in $\mathbb{R}^n$ of radius ϱ and centre x; the symbol ω_n stands for the volume of the unit ball in $\mathbb{R}^n$, i.e. $\omega_n = |B_n(0,1)|_n$.

Let Q be a measurable subset in $\mathbb{R}^n$ and let $L_\Phi(Q)$ be the *Orlicz space* with Young function Φ (by a *Young function* Φ we mean a continuous, non-negative, strictly increasing and convex function on $[0,\infty)$ satisfying $\lim_{t\to 0+}\Phi(t)/t = 0 = \lim_{t\to\infty} t/\Phi(t)$). The symbols $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi$ are used to denote the corresponding Orlicz and Luxemburg norms, respectively; these two norms are equivalent (cf. [9, Theorem 3.8.5]).

Let $p \in (1,\infty)$, $q \in [1,\infty]$, $\lambda, \varepsilon \in \mathbb{R}$, and $s > 0$. The *space of Bessel potentials*

$$H^sL^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon(\mathbb{R}^n) \tag{2.6}$$

is defined by

$$H^sL^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon(\mathbb{R}^n) := \{u = g_s * f : f \in L^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon(\mathbb{R}^n)\}$$

and it is equipped with the quasi-norm

$$\|u\|_{s;p,q,\lambda,\varepsilon} := \|f\|_{p,q,\lambda,\varepsilon}. \tag{2.7}$$

If $\Omega \subset \mathbb{R}^n$ is a domain, then

$$H^sL^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon(\Omega) := \{\widetilde{u}|_\Omega : \widetilde{u} \in H^sL^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon(\mathbb{R}^n)\}$$

and it is endowed with the quasi-norm

$$\|u\|_{s;p,q,\lambda,\varepsilon,\Omega} := \inf\{\|\widetilde{u}\|_{s;p,q,\lambda,\varepsilon} : u = \widetilde{u}|_\Omega\}.$$

Given two Banach spaces X and Y, we write $X \hookrightarrow Y$ or $X \hookrightarrow\hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous or compact, respectively.

As usual, the symbols $C, C_1, C_2, \ldots, c, c_1, c_2, \ldots$ signify positive constants independent of appropriate quantities. For non-negative expressions (i.e. functions or functionals) F_1, F_2 we use the symbols $F_1 \lesssim F_2$ and $F_1 \gtrsim F_2$, respectively, provided that $F_1 \leq CF_2$ and $CF_1 \geq F_2$, with some constant $C \in (0,\infty)$ independent of the variables in the expressions F_1, F_2. If $F_1 \lesssim F_2$ and simultaneously $F_2 \lesssim F_1$, we write $F_1 \approx F_2$.

We shall use the following convention:

$$\frac{1}{\infty} = 0, \quad \frac{1}{0} = \infty, \quad \frac{0}{0} = 0 \quad \text{and} \quad \frac{\infty}{\infty} = 0. \tag{2.8}$$

If $p \in [1,\infty]$, then the conjugate number p' is defined by $1/p + 1/p' = 1$.

3. Preliminaries. In this section we present some auxiliary assertions which we need in subsequent sections. We start with the following two lemmas which are particular cases of more general results on Hardy-type inequalities from [7].

3.1. Lemma (a non-limiting case). *Suppose $p \in [1,\infty]$ and $\alpha,\beta,\gamma,\delta \in \mathbb{R}$.*

(i) *Let $\nu < 0$. Then there exists a constant $c \in (0,\infty)$ such that the Hardy inequality*

$$\Big\|t^\nu(e-\log t)^\gamma \log^\alpha(e-\log t)\int_0^t g(\tau)\,d\tau\Big\|_{Y(\infty)} \leq c\|t^\nu(e-\log t)^\delta \log^\beta(e-\log t)tg(t)\|_{Y(p)} \tag{3.1}$$

holds for every $g \in \mathfrak{M}^+(0,1)$ if, and only if, one of the following conditions is satisfied:

$$\gamma - \delta < 0; \tag{3.2}$$
$$\gamma - \delta = 0, \quad \alpha - \beta \leq 0. \tag{3.3}$$

(ii) *Let $\nu > 0$. Then there exists a constant $c \in (0,\infty)$ such that the Hardy inequality*

$$\left\| t^{\nu}(e-\log t)^{\gamma}\log^{\alpha}(e-\log t)\int_t^1 g(\tau)\,d\tau\right\|_{Y(\infty)} \le c\|t^{\nu}(e-\log t)^{\delta}\log^{\beta}(e-\log t)tg(t)\|_{Y(p)} \tag{3.4}$$

holds for every $g\in\mathfrak{M}^+(0,1)$ if, and only if, one of conditions (3.2), (3.3) *is satisfied.*

3.2. Lemma (a double limiting case). *Suppose $p\in[1,\infty]$ and $\alpha,\beta\in\mathbb{R}$, $\alpha\ne 0$, $\beta\ne 1/p'$.*

(i) *There exists a constant $c\in(0,\infty)$ such that the Hardy inequality*

$$\left\|\log^{\alpha}(e-\log t)\int_0^t g(\tau)\,d\tau\right\|_{Z(\infty)} \le c\|\log^{\beta}(e-\log t)t(e-\log t)g(t)\|_{Z(p)} \tag{3.5}$$

holds for every $g\in\mathfrak{M}^+(0,1)$ if, and only if,

$$\beta>1/p' \tag{3.6}$$

and

$$\alpha-\beta+1/p'\le 0. \tag{3.7}$$

(ii) *There exists a constant $c\in(0,\infty)$ such that the Hardy inequality*

$$\left\|\log^{\alpha}(e-\log t)\int_t^1 g(\tau)\,d\tau\right\|_{Z(\infty)} \le c\|\log^{\beta}(e-\log t)t(e-\log t)g(t)\|_{Z(p)} \tag{3.8}$$

holds for every $g\in\mathfrak{M}^+(0,1)$ if, and only if,

$$\alpha<0 \tag{3.9}$$

and (3.7) *is satisfied.*

3.3. Remark. If we omit the assumptions $\alpha\ne 0$ and $\beta\ne 1/p'$ in Lemma 3.2, then the lemma continues to hold provided that we assume either (3.6) and (3.7), or

$$1=p,\quad \beta=0,\quad \alpha\le 0 \tag{3.10}$$

in part (i); and either (3.9) and (3.7), or

$$1<p\le\infty,\quad \alpha=0,\quad -\beta+1/p'<0, \tag{3.11}$$

or

$$1=p,\quad \alpha=0,\quad -\beta\le 0 \tag{3.12}$$

in part (ii).

The simple observation formulated in the next remark will be useful later.

3.4. Remark. Let $\gamma=0$ and $\delta=1/p'$. Then condition (3.7) implies one of (3.2), (3.3). Indeed, we have $\gamma-\delta=-1/p'\le 0$, which implies (3.2) provided that $1/p'\ne 0$. If $1/p'=0$, then $\gamma-\delta=0$. Moreover, condition (3.7) yields $\alpha-\beta\le 0$. Hence, (3.3) is satisfied.

While the kernel I_σ of the Riesz potential was given explicitly, the Bessel kernel g_σ was introduced by (2.5). However, it is well known (cf. [14] or [19]) that

(3.13) g_σ is a positive, integrable function which is analytic except at the origin;

$$g_\sigma * g_\tau = g_{\sigma+\tau}\quad \text{if } \sigma,\tau>0; \tag{3.14}$$

$$g_\sigma(x)\le C_1|x|^{\sigma-n}e^{-C_2|x|}\quad \text{for } \sigma\in(0,n) \text{ and all } x\in\mathbb{R}^n; \tag{3.15}$$

$$g_\sigma(x)\approx |x|^{\sigma-n}\quad \text{as } |x|\to 0. \tag{3.16}$$

The estimate (3.15) shows that the behaviour of the Bessel kernel as $|x|\to\infty$ is much better than that of the Riesz potential. The following lemma provides us with the important estimate (3.17) for the non-increasing rearrangement of the Bessel kernel.

3.5. Lemma. *Let $0<\sigma<n$. Then there exist constants $A,B\in(0,\infty)$ such that for all $t>0$,*

$$g_\sigma^*(t)\le At^{\sigma/n-1}\exp(-Bt^{1/n}) \tag{3.17}$$

and

$$g_\sigma^{**}(t)\le \frac{n}{\sigma}At^{\sigma/n-1}. \tag{3.18}$$

Proof. By (3.13) and (3.15),

$$0\le g_\sigma(x)\le H(x),\quad x\in\mathbb{R}^n,$$

where $H(x)=h(|x|)$ with $h(t)=C_1t^{\sigma-n}\exp(-C_2t)$, $t\ge 0$. Thus it is sufficient to show that estimates (3.17) and (3.18) hold with H instead of g_σ.

Since h is decreasing, $\mu_h(\lambda)=h^{-1}(\lambda)$ for all $\lambda>0$. Hence,

$$\mu_H(\lambda)=|B_n(0,h^{-1}(\lambda))|_n=\omega_n[h^{-1}(\lambda)]^n\quad \text{for all } \lambda>0,$$

and

$$\begin{aligned}H^*(t)&=\inf\{\lambda>0:\omega_n[h^{-1}(\lambda)]^n\le t\}=h((t/\omega_n)^{1/n})\\ &=At^{\sigma/n-1}\exp(-Bt^{1/n}),\quad t>0,\end{aligned}$$

where $A = C_1\omega_n^{1-\sigma/n}$ and $B = C_2\omega_n^{-1/n}$. Consequently,

$$H^{**}(t) = \frac{1}{t}\int_0^t As^{\sigma/n-1}\exp(-Bs^{1/n})\,ds \le \frac{n}{\sigma}At^{\sigma/n-1}, \qquad t>0,$$

and the proof is complete. ■

An estimate for the non-increasing rearrangement of a function from a generalized Lorentz–Zygmund space is given by

3.6. LEMMA. *Let $\lambda,\varepsilon \in \mathbb{R}$ and let $p\in(0,\infty)$ and $q\in(0,\infty]$, or $p=\infty=q$. Then there exists a constant c such that*

$$f^*(t) \le ct^{-1/p}(e+|\log t|)^{-\lambda}\log^{-\varepsilon}(e+|\log t|)\|f\|_{p,q,\lambda,\varepsilon}$$

for every $f\in L^{p,q}(\log L)^\lambda(\log\log L)^\varepsilon$ and all $t\in(0,\infty)$.

The proof is similar to that of Lemma 3.3 of [5], where the case $p\in(1,\infty)$, $q\in[1,\infty]$ was investigated. ■

3.7. Remark. Let $0<K\le\infty$, $1\le p\le\infty$ and $g,v\in\mathfrak{M}^+(0,K)$. Then since the spaces L^p and $L^{p'}$ are mutually associated (cf. [3, Chapter 1, Theorem 2.5]), we have

$$\sup_{f\in\mathfrak{M}^+(0,K)}\frac{\int_0^K g(t)f(t)\,dt}{\|fv\|_p} = \left\|\frac{1}{v}g\right\|_{p'} \tag{3.19}$$

(note that we use the convention (2.8)).

In the following lemma it is shown that the analogue of (3.19) with the supremum restricted to $\mathfrak{M}^+(0,K;\downarrow)$ enables us to replace a weighted inequality on $\mathfrak{M}^+(0,K;\downarrow)$ by an equivalent inequality on $\mathfrak{M}^+(0,K)$. When $p\in(1,\infty)$ such an analogue was obtained by Sawyer [13, Theorem 1]. His result is formulated as part (i) of Lemma 3.9 below. In parts (ii) and (iii) of that lemma, we give estimates of the mentioned quantity from below provided that $p=1$ and $p=\infty$. Such one-sided estimates appear to be sufficient for our purposes. Note that here we use some ideas from [16], where an alternative proof of Sawyer's result is given.

3.8. LEMMA. *Let $0<K\le\infty$ and let $w,v\in\mathcal{W}(0,K)$. Suppose that*

$$T,T' : \mathfrak{M}^+(0,K)\to\mathfrak{M}^+(0,K) \tag{3.20}$$

are two operators satisfying

$$\int_0^K Tf(t)\cdot g(t)\,dt = \int_0^K f(t)\cdot T'g(t)\,dt \tag{3.21}$$

for all functions $f,g\in\mathfrak{M}^+(0,K)$, $f\downarrow$. Let $p,q\in[1,\infty]$. Then the inequality

$$\|(Tf)w\|_q \le C\,\|fv\|_p \tag{3.22}$$

holds for all $f\in\mathfrak{M}^+(0,K;\downarrow)$ if, and only if, the inequality

$$\left\|\frac{1}{v}(T'g)\right\|_{p',v,\downarrow} \le C\left\|\frac{1}{w}g\right\|_{q'} \tag{3.23}$$

holds for all $g\in\mathfrak{M}^+(0,K)$, where for $h\in\mathfrak{M}^+(0,K)$,

$$\|h\|_{p',v,\downarrow} := \sup_{f\in\mathfrak{M}^+(0,K;\downarrow)}\frac{\int_0^K fvh\,dx}{\|fv\|_p}. \tag{3.24}$$

(The constant C is the same in inequalities (3.22), (3.23) and does not depend on the functions f, g.)

Proof. (i) Assume that (3.22) holds for all $f\in\mathfrak{M}^+(0,K;\downarrow)$. Taking $g\in\mathfrak{M}^+(0,K)$ and using successively (3.24), (3.21), (3.19) and (3.22), we obtain

$$\begin{aligned}\left\|\frac{1}{v}(T'g)\right\|_{p',v,\downarrow} &= \sup_{f\in\mathfrak{M}^+(0,K;\downarrow)}\frac{\int_0^K f\cdot(T'g)\,dx}{\|fv\|_p}\\ &= \sup_{f\in\mathfrak{M}^+(0,K;\downarrow)}\frac{\int_0^K (Tf)\cdot g\,dx}{\|fv\|_p}\\ &\le \sup_{f\in\mathfrak{M}^+(0,K;\downarrow)}\frac{\|(Tf)w\|_q\left\|\frac{1}{w}g\right\|_{q'}}{\|fv\|_p} \le C\left\|\frac{1}{w}g\right\|_{q'}.\end{aligned}$$

(ii) Suppose that (3.23) holds on $\mathfrak{M}^+(0,K)$. Let $f\in\mathfrak{M}^+(0,K;\downarrow)$. Then, by (3.19), (3.21), (3.24) and (3.23),

$$\begin{aligned}\frac{\|(Tf)w\|_q}{\|fv\|_p} &= \sup_{g\in\mathfrak{M}^+(0,K)}\frac{\int_0^K (Tf)\cdot g\,dx}{\left\|\frac{1}{w}g\right\|_{q'}}\cdot\frac{1}{\|fv\|_p}\\ &= \sup_{g\in\mathfrak{M}^+(0,K)}\frac{\int_0^K f\cdot(T'g)\,dx}{\left\|\frac{1}{w}g\right\|_{q'}}\cdot\frac{1}{\|fv\|_p}\\ &\le \sup_{g\in\mathfrak{M}^+(0,K)}\frac{1}{\left\|\frac{1}{w}g\right\|_{q'}}\sup_{f\in\mathfrak{M}^+(0,K;\downarrow)}\frac{\int_0^K fv\cdot\frac{1}{v}(T'g)\,dx}{\|fv\|_p}\\ &= \sup_{g\in\mathfrak{M}^+(0,K)}\frac{\left\|\frac{1}{v}(T'g)\right\|_{p',v,\downarrow}}{\left\|\frac{1}{w}g\right\|_{q'}} \le C. \ ■\end{aligned}$$

3.9. LEMMA. *Let $0<K\le\infty$ and $v\in\mathcal{W}(0,K)$.*

(i) *If $p\in(1,\infty)$ and v satisfies*

$$(3.25)\qquad \int_0^t v^p(\tau)\,d\tau < \infty, \qquad t \in (0,K),$$

then for any $h \in \mathfrak{M}^+(0,K)$,

$$(3.26)\qquad \|h\|_{p',v,\downarrow} \approx \Big\| v(x)^{p/p'} \int_x^K \frac{v(y)h(y)}{\int_0^y v^p}\,dy \Big\|_{p'}$$

$$\approx \Big\| v(x)^{p/p'} \Big(\int_0^x v^p\,dy\Big)^{-1} \int_0^x vh\,dy \Big\|_{p'} + \frac{\int_0^K vh\,dx}{\|v\|_p}.$$

(ii) *If* $p = \infty$ *and* $\widetilde{v} \in \mathcal{W}(0,K)$ *is such that*

$$(3.27)\qquad \Big\| \Big(\int_x^K g(y)\,dy\Big) v(x) \Big\|_\infty \le \widetilde{C} \|g\widetilde{v}\|_\infty$$

for all $g \in \mathfrak{M}^+(0,K)$ *(with a constant* $\widetilde{C}$ *independent of* g*), then for any* $h \in \mathfrak{M}^+(0,K)$,

$$(3.28)\qquad \|h\|_{p',v,\downarrow} = \|h\|_{1,v,\downarrow} \gtrsim \Big\| v(x)h(x) \int_x^K \frac{1}{\widetilde{v}(y)}\,dy \Big\|_1 + \frac{\int_0^K vh\,dx}{\|v\|_\infty}$$

$$= \Big\| \Big(\int_0^y v(x)h(x)\,dx\Big) \frac{1}{\widetilde{v}(y)} \Big\|_1 + \frac{\int_0^K vh\,dx}{\|v\|_\infty}.$$

(iii) *If* $p = 1$ *and* v *satisfies* (3.25), *then for any* $h \in \mathfrak{M}^+(0,K)$,

$$(3.29)\qquad \|h\|_{p',v,\downarrow} = \|h\|_{\infty,v,\downarrow} \gtrsim \Big\| \Big(\int_0^x v\,dy\Big)^{-1} \int_0^x vh\,dy \Big\|_\infty + \frac{\int_0^K vh\,dx}{\|v\|_1}.$$

Proof. (i) For the proof of part (i) see [13, proof of Theorem 1].

(ii) Let $h \in \mathfrak{M}^+(0,K)$. Then, by (3.24), (3.27), Fubini's theorem and (3.19),

$$\|h\|_{1,v,\downarrow} \ge \sup_{g\in\mathfrak{M}^+(0,K)} \frac{\int_0^K (\int_x^K g(y)\,dy) v(x)h(x)\,dx}{\|(\int_x^K g(y)\,dy)v(x)\|_\infty}$$

$$\ge (\widetilde{C})^{-1} \sup_{g\in\mathfrak{M}^+(0,K)} \frac{\int_0^K g(y)(\int_0^y v(x)h(x)\,dx)\,dy}{\|g\widetilde{v}\|_\infty}$$

$$= (\widetilde{C})^{-1} \Big\| \Big(\int_0^y v(x)h(x)\,dx\Big)\frac{1}{\widetilde{v}(y)} \Big\|_1$$

$$= (\widetilde{C})^{-1} \Big\| v(x)h(x) \int_x^K \frac{1}{\widetilde{v}(y)}\,dy \Big\|_1.$$

Moreover, upon taking $f \equiv c$, where c is a positive constant, (3.24) implies

$$(3.30)\qquad \|h\|_{1,v,\downarrow} \ge \frac{\int_0^K vh\,dx}{\|v\|_\infty}$$

and the result follows.

(iii) Let $h \in \mathfrak{M}^+(0,K)$. Then, using (3.24), Fubini's theorem and (3.19), we have

$$\|h\|_{\infty,v,\downarrow} \ge \sup_{g\in\mathfrak{M}^+(0,K)} \frac{\int_0^K (\int_x^K g(y)\,dy) v(x)h(x)\,dx}{\|(\int_x^K g(y)\,dy)v(x)\|_1}$$

$$= \sup_{g\in\mathfrak{M}^+(0,K)} \frac{\int_0^K g(y)(\int_0^y v(x)h(x)\,dx)\,dy}{\|g(y)\int_0^y v(x)\,dx\|_1}$$

$$= \Big\| \Big(\int_0^y v\,dx\Big)^{-1} \Big(\int_0^y vh\,dx\Big) \Big\|_\infty.$$

This and the choice $f \equiv c$, $c > 0$, in (3.24) yield the result. (It is easy to see that the last term in (3.29) can be omitted since it is bounded by the previous one.) ∎

In [11], [15] weighted inequalities

$$\|(Tf)w\|_q \le C\,\|fv\|_p$$

were investigated for certain integral operators T provided that $p, q \in (1,\infty)$. Using techniques from the theory of Banach function spaces, one can prove Lemma 3.10, which deals with the limiting values $p = 1$ and $q = \infty$.

3.10. Lemma. *Suppose* $\sigma \in (0,n)$ *and* $\widetilde{w}, \widetilde{v} \in \mathcal{W}(0,1)$; *put*

$$(3.31)\qquad k(y,t) = y^{\sigma/n} - t^{\sigma/n}, \qquad 0 < t \le y < 1.$$

Let there exist $C \in (0,\infty)$ *such that*

$$(3.32)\qquad \Big\| \widetilde{w}(y) \int_0^y k(y,t)g(t)\,dt \Big\|_\infty \le C \|\widetilde{v}(y)g(y)\|_1$$

for all $g \in \mathfrak{M}^+(0,1)$. *Then*

$$(3.33)\qquad \sup_{\xi\in(0,1)} \|\widetilde{w}\chi_{(\xi,1)}\|_\infty \Big\| \frac{1}{\widetilde{v}} k(\xi,\cdot)\chi_{(0,\xi)} \Big\|_\infty \le C,$$

$$(3.34)\qquad \sup_{\xi\in(0,1)} \|\widetilde{w}\,k(\cdot,\xi)\chi_{(\xi,1)}\|_\infty \Big\| \frac{1}{\widetilde{v}} \chi_{(0,\xi)} \Big\|_\infty \le C.$$

Proof. (i) To prove (3.33), we take $g \in \mathfrak{M}^+(0,1)$, with $\|g\widetilde{v}\|_1 \le 1$, and $\xi \in (0,1)$. On using (3.32) and the monotonicity of $k(\cdot,t)$, we obtain

$$C \ge C\|g\widetilde{v}\|_1 \ge \left\|\widetilde{w}(y)\int_0^y k(y,t)g(t)\,dt\right\|_\infty$$
$$\ge \left\|\widetilde{w}(y)\chi_{(\xi,1)}(y)\int_0^y k(y,t)g(t)\,dt\right\|_\infty$$
$$\ge \|\widetilde{w}\chi_{(\xi,1)}\|_\infty \int_0^\xi k(\xi,t)g(t)\,dt$$
$$= \|\widetilde{w}\chi_{(\xi,1)}\|_\infty \int_0^1 \frac{1}{\widetilde{v}(t)}k(\xi,t)\chi_{(0,\xi)}(t)\widetilde{v}(t)g(t)\,dt.$$

Now, taking the supremum over all g and ξ, we arrive at (3.33).

To prove (3.34), take $f \in \mathfrak{M}^+(0,1)$ satisfying $\|\frac{1}{\widetilde{w}}f\|_1 \le 1$. If $g \in \mathfrak{M}^+(0,1)$ and $\|g\widetilde{v}\|_1 \le 1$, we obtain by (3.32), the Hölder inequality and Fubini's theorem,

$$C\left\|\frac{1}{\widetilde{w}}f\right\|_1 \ge C\|g\widetilde{v}\|_1\left\|\frac{1}{\widetilde{w}}f\right\|_1 \ge \left\|\widetilde{w}(y)\int_0^y k(y,t)g(t)\,dt\right\|_\infty\left\|\frac{1}{\widetilde{w}}f\right\|_1$$
$$\ge \int_0^1\Big(\int_0^y k(y,t)g(t)\,dt\Big)f(y)\,dy$$
$$= \int_0^1 \frac{1}{\widetilde{v}(t)}\Big(\int_t^1 k(y,t)f(y)\,dy\Big)g(t)\widetilde{v}(t)\,dt.$$

Taking the supremum over all $g \in \mathfrak{M}^+(0,1)$ with $\|g\widetilde{v}\|_1 \le 1$, we get

$$\left\|\frac{1}{\widetilde{v}(t)}\int_t^1 k(y,t)f(y)\,dy\right\|_\infty \le C\left\|\frac{1}{\widetilde{w}}f\right\|_1 \tag{3.35}$$

(which is the "associated inequality" to (3.32)).

Now we proceed as in the previous case. Let $\xi \in (0,1)$. Then, taking $f \in \mathfrak{M}^+(0,1)$ with $\|\frac{1}{\widetilde{w}}f\|_1 \le 1$ and using the monotonicity of $k(y,\cdot)$, we have from (3.35),

$$C \ge \left\|\frac{1}{\widetilde{v}(t)}\chi_{(0,\xi)}(t)\int_t^1 k(y,t)f(y)\,dy\right\|_\infty \ge \int_\xi^1 k(y,\xi)f(y)\,dy\cdot\left\|\frac{1}{\widetilde{v}}\chi_{(0,\xi)}\right\|_\infty$$
$$= \int_0^1 \widetilde{w}(y)k(y,\xi)\chi_{(\xi,1)}(y)\frac{1}{\widetilde{w}(y)}f(y)\,dy\cdot\left\|\frac{1}{\widetilde{v}}\chi_{(0,\xi)}\right\|_\infty,$$

and the proof is finished by taking the supremum over all $f \in \mathfrak{M}^+(0,1)$ satisfying $\|\frac{1}{\widetilde{w}}f\|_1 \le 1$ and over all $\xi \in (0,1)$. ∎

4. Double exponential integrability. In this section we shall look for necessary and sufficient conditions for double exponential integrability of the Bessel potential of a function from an appropriate generalized Lorentz–Zygmund space. We start with

4.1. Lemma. *Let $p \in [1,\infty]$ and $\alpha,\beta \in \mathbb{R}$, $\alpha \ne 0$, $\beta \ne 1/p'$, $\sigma \in (0,n)$. Suppose that*

$$f \in L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n), \qquad u = g_\sigma * f.$$

Then the inequality

$$\sup_{t\in(0,1)}[\log^\alpha(e-\log t)u^*(t)] \le C\|f\|_{n/\sigma,p,1/p',\beta} \tag{4.1}$$

holds with a constant C which depends only on p, α, β, σ and n if, and only if,

$$\alpha < 0 \quad \text{and} \quad \alpha - \beta + 1/p' \le 0. \tag{4.2}$$

Proof. (4.2)⇒(4.1). Since $u = g_\sigma * f$, O'Neil's lemma [12, Lemma 1.5] implies that

$$u^*(t) \le u^{**}(t) \le tg_\sigma^{**}(t)f^{**}(t) + \int_t^\infty g_\sigma^*(\tau)f^*(\tau)\,d\tau. \tag{4.3}$$

The estimates (3.18) and (4.3) yield for every $t \in (0,1)$,

$$u^*(t) \le \frac{n}{\sigma}At^{\sigma/n-1}\int_0^t f^*(\tau)\,d\tau + \int_t^1 g_\sigma^*(\tau)f^*(\tau)\,d\tau + \int_1^\infty g_\sigma^*(\tau)f^*(\tau)\,d\tau. \tag{4.4}$$

Putting

$$X = L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n),$$

by Lemma 3.6 we have for all $t > 0$,

$$f^*(t) \le ct^{-\sigma/n}(e+|\log t|)^{-1/p'}\log^{-\beta}(e+|\log t|)\|f\|_X. \tag{4.5}$$

Using (3.17) and (4.5) we obtain

$$\int_1^\infty g_\sigma^*(\tau)f^*(\tau)\,d\tau \tag{4.6}$$
$$\le Ac\|f\|_X\int_1^\infty t^{\sigma/n-1}\exp(-Bt^{1/n})t^{-\sigma/n}(e+|\log t|)^{-1/p'}\log^{-\beta}(e+|\log t|)\,dt$$
$$= C_1\|f\|_X.$$

The estimates (4.4) and (4.6) imply

$$\|\log^\alpha(e-\log t)u^*(t)\|_{Z(\infty)} \le \frac{n}{\sigma}AN_1 + N_2 + C_1\|f\|_X N_3, \tag{4.7}$$

where

$$N_1 = \Big\|t^{\sigma/n-1}\log^\alpha(e-\log t)\int_0^t f^*(\tau)\,d\tau\Big\|_{Z(\infty)},$$

$$N_2 = \Big\|\log^\alpha(e-\log t)\int_t^1 g_\sigma^*(\tau)f^*(\tau)\,d\tau\Big\|_{Z(\infty)},$$

$$N_3 = \|\log^\alpha(e-\log t)\|_{Z(\infty)}.$$

As $\alpha<0$,

$$N_3<\infty. \tag{4.8}$$

Applying Lemma 3.1(i) (cf. Remark 3.4), we have

$$\begin{aligned} N_1 &= \Big\|t^{\sigma/n-1}\log^\alpha(e-\log t)\int_0^t f^*(\tau)\,d\tau\Big\|_{Y(\infty)} \\ &\le c\|t^{\sigma/n-1}(e-\log t)^{1/p'}\log^\beta(e-\log t)tf^*(t)\|_{Y(p)} \le c\|f\|_X. \end{aligned} \tag{4.9}$$

Finally, Lemma 3.2(ii) and the estimate $g_\sigma^*(t)\le At^{\sigma/n-1}$ (cf. (3.17)) yield

$$\begin{aligned} N_2 &\le c\|\log^\beta(e-\log t)t(e-\log t)g_\sigma^*(t)f^*(t)\|_{Z(p)} \\ &\le cA\|t^{\sigma/n}(e-\log t)\log^\beta(e-\log t)f^*(t)\|_{Z(p)} \le cA\|f\|_X. \end{aligned} \tag{4.10}$$

The result follows from inequalities (4.7)–(4.10).

(4.1)⇒(4.2). Take $f:\mathbb{R}^n\to[0,\infty]$ with compact support in the ball $B_n(0,R)$, $R=\omega_n^{-1/n}$, such that $\|f\|_{n/\sigma,p,1/p',\beta}<\infty$. Putting $\widetilde{f}(y)=f^*(\omega_n|y|^n)$, $y\in\mathbb{R}^n$, we have $\widetilde{f}(y)=0$ for $|y|>R$. Moreover, $(\widetilde{f})^*=f^*$, and consequently,

$$\|\widetilde{f}\|_{n/\sigma,p,1/p',\beta} = \|f\|_{n/\sigma,p,1/p',\beta}. \tag{4.11}$$

For $x\in\mathbb{R}^n$ put

$$\mathcal{B}_\sigma(x) = (g_\sigma*\widetilde{f})(x), \qquad \mathcal{R}_\sigma(x)=(I_\sigma*\widetilde{f})(x)$$

(recall that $I_\sigma(x)=|x|^{\sigma-n}$). By (3.16) and (3.13), there exists a constant $c_1\in(0,\infty)$ such that $g_\sigma(x)\ge c_1 I_\sigma(x)$ for all $x\in B_n(0,2R)\setminus\{0\}$. This easily yields that $\mathcal{B}_\sigma(x)\ge c_1\mathcal{R}_\sigma(x)$ for all $x\in B_n(0,R)$. Hence

$$\mathcal{B}_\sigma^*\ge[\mathcal{B}_\sigma\chi_{B_n(0,R)}]^*\ge c_1[\mathcal{R}_\sigma\chi_{B_n(0,R)}]^*. \tag{4.12}$$

By (3.15) of [5],

$$\mathcal{R}_\sigma(x)\ge c_2F(\omega_n|x|^n), \qquad x\in\mathbb{R}^n,$$

$(c_2=2^{\sigma-n}\omega_n^{1-\sigma/n})$ where

$$F(t)=t^{\sigma/n-1}\int_0^t f^*(s)\,ds+\int_t^\infty s^{\sigma/n-1}f^*(s)\,ds, \qquad t>0. \tag{4.13}$$

Since the functions $y\mapsto F(\omega_n|y|^n)$ and $t\mapsto F(t)$ are equimeasurable, they have the same non-increasing rearrangement. Moreover, F is non-increasing on $(0,\infty)$. Thus

$$\begin{aligned} [\mathcal{R}_\sigma\chi_{B_n(0,R)}]^*(t) &\ge c_2F(t)\chi_{(0,1)}(t) \\ &\ge c_2\int_t^1 s^{\sigma/n-1}f^*(s)\,ds, \qquad t\in(0,1). \end{aligned}$$

Using this estimate, (4.12), (4.1) and (4.11), we obtain

$$\Big\|\log^\alpha(e-\log t)\int_t^1 s^{\sigma/n-1}f^*(s)\,ds\Big\|_{Z(\infty)}\le c_3\|f\|_{n/\sigma,p,1/p',\beta}. \tag{4.14}$$

This easily implies (cf. [5, Remark 3.7]) that the inequality

$$\|(Tf)w\|_\infty\le c\|fv\|_p \tag{4.15}$$

(with $c=c_3$) holds for all $f\in\mathfrak{M}^+(0,1;\downarrow)$, where

$$\begin{cases} (Tf)(t)=\int_t^1 s^{\sigma/n-1}f(s)\,ds, \\ w(t)=\log^\alpha(e-\log t), \\ v(t)=t^{\sigma/n-1/p}(e-\log t)^{1/p'}\log^\beta(e-\log t) \end{cases} \tag{4.16}$$

for $t\in(0,1)$.

Defining the operator T' by

$$(T'g)(t):=t^{\sigma/n-1}\int_0^t g(\tau)\,d\tau, \qquad t\in(0,1),$$

we deduce from Lemma 3.8 that the inequality

$$\Big\|\frac{1}{v}(T'g)\Big\|_{p',v,\downarrow}\le C\Big\|\frac{1}{w}g\Big\|_1 \tag{4.17}$$

holds for all $g\in\mathfrak{M}^+(0,1)$. Put $K=1$. If $p\in[1,\infty)$, then the function v obviously satisfies (3.25).

(i) Let $p\in(1,\infty)$. Then Lemma 3.9(i) and the inequality (4.17) yield

$$\Big\|v(x)^{p/p'}\int_x^1\frac{(T'g)(t)}{\int_0^t v^p}\,dt\Big\|_{p'}\le C\Big\|\frac{1}{w}g\Big\|_1 \tag{4.18}$$

for all $g \in \mathfrak{M}^+(0,1)$. The monotonicity of

$$t \mapsto (G'g)(t) := \int_0^t g(\tau)\,d\tau$$

implies

$$\int_x^1 \frac{(T'g)(t)}{\int_0^t v^p}\,dt \geq (G'g)(x) \int_x^1 t^{\sigma/n-1}\Big(\int_0^t v^p(\tau)\,d\tau\Big)^{-1} dt.$$

Thus, putting

$$\overline{v}(x) = v(x)^{-p/p'}\Big[\int_x^1 t^{\sigma/n-1}\Big(\int_0^t v(\tau)^p\,d\tau\Big)^{-1} dt\Big]^{-1},$$

we have from (4.18),

$$\Big\|\frac{1}{\overline{v}}(G'g)\Big\|_{p'} \leq C_1 \Big\|\frac{1}{w}g\Big\|_1$$

for all $g \in \mathfrak{M}^+(0,1)$. Passing to associated spaces, we obtain

$$\|(Gg)w\|_\infty \leq C_1 \|g\overline{v}\|_p \tag{4.19}$$

for every $g \in \mathfrak{M}^+(0,1)$, where

$$(Gg)(t) = \int_t^1 g(\tau)d\tau, \qquad t \in (0,1).$$

Since

$$\overline{v}(x) \approx x^{1-1/p}(e-\log x)^{1-1/p}\log^\beta(x-\log x), \qquad x \in (0,1),$$

the inequalities (4.19) and (3.8) are equivalent and (4.2) follows from Lemma 3.2(ii).

(ii) Let $p = \infty$. Put

$$\widetilde{v}(t) = t^{1+\sigma/n}(e-\log t)\log^\beta(e-\log t), \qquad t \in (0,1).$$

Then, by Lemma 3.1(ii), inequality (3.27) (with v from (4.16)) holds on $\mathfrak{M}^+(0,1)$. Thus, Lemma 3.9(ii) and inequality (4.17) yield

$$\Big\|(T'g)(x)\int_x^1 \frac{1}{\widetilde{v}(y)}\,dy\Big\|_1 \leq C_1 \Big\|\frac{1}{w}g\Big\|_1$$

for all $g \in \mathfrak{M}^+(0,1)$. Passing to associated spaces, we obtain

$$\|(Th)w\|_\infty \leq C_1 \Big\|h(x)\Big(\int_x^1 \frac{1}{\widetilde{v}(y)}\,dy\Big)^{-1}\Big\|_\infty$$

for all $h \in \mathfrak{M}^+(0,1)$, i.e.,

$$\Big\|\Big(\int_x^1 s^{\sigma/n-1}h(s)\,ds\Big)w(x)\Big\|_\infty \leq C_1 \Big\|h(x)\Big(\int_x^1 \frac{1}{\widetilde{v}(y)}\,dy\Big)^{-1}\Big\|_\infty$$

for all $h \in \mathfrak{M}^+(0,1)$, which is equivalent to

$$\Big\|\Big(\int_x^1 g(s)\,ds\Big)w(x)\Big\|_\infty \leq C_1 \Big\|g(x)x^{1-\sigma/n}\Big(\int_x^1 \frac{1}{\widetilde{v}(y)}\,dy\Big)^{-1}\Big\|_\infty \tag{4.20}$$

for all $g \in \mathfrak{M}^+(0,1)$. Observing that

$$x^{1-\sigma/n}\Big(\int_x^1 \frac{1}{\widetilde{v}(y)}\,dy\Big)^{-1} \approx x(e-\log x)\log^\beta(e-\log x)$$

for all $x \in (0,1)$, we get the equivalence of (4.20) and (3.8) and the result follows from Lemma 3.2(ii).

(iii) Finally, suppose that $p = 1$. Then Lemma 3.9(iii) and the inequality (4.17) yield for all $g \in \mathfrak{M}^+(0,1)$,

$$\Big\|\Big(\int_0^x v\,dy\Big)^{-1}\int_0^x (T'g)(y)\,dy\Big\|_\infty \leq C_1 \Big\|\frac{1}{w}g\Big\|_1.$$

Moreover, Fubini's theorem implies

$$\begin{aligned}\int_0^x (T'g)(y)\,dy &= \int_0^x y^{\sigma/n-1}\Big(\int_0^y g(t)\,dt\Big)dy \\ &= \int_0^x g(t)\Big(\int_t^x y^{\sigma/n-1}\,dy\Big)dt \\ &\approx \int_0^x k(x,t)g(t)\,dt, \qquad x \in (0,1),\end{aligned}$$

where k is given by (3.31). Since

$$\Big(\int_0^x v(y)\,dy\Big)^{-1} = \Big(\int_0^x y^{\sigma/n-1}\log^\beta(e-\log y)\,dy\Big)^{-1} \approx \widetilde{w}(x),$$

where

$$\widetilde{w}(x) = x^{-\sigma/n}\log^{-\beta}(e-\log x), \qquad x \in (0,1),$$

we see that the inequality (3.32) holds on $\mathfrak{M}^+(0,1)$ with a *finite* constant $C = C_1$ and with the weights $\widetilde{w}$ and $\widetilde{v} := 1/w$. Consequently, Lemma 3.10 yields (3.33) and (3.34). Hence for arbitrary $\xi \in (0,1)$,

$$\infty > \Big\|\frac{1}{\widetilde{v}}\chi_{(0,\xi)}\Big\|_\infty = \|w\chi_{(0,\xi)}\|_\infty \geq \lim_{x\to 0+} w(x) = \lim_{x\to 0+} \log^\alpha(e-\log x),$$

which, together with the assumption $\alpha \neq 0$ in Lemma 4.1, implies the first condition in (4.2). Moreover, for all sufficiently small ξ in (0,1),

$$\|\widetilde{w}\chi_{(\xi,1)}\|_\infty = \xi^{-\sigma/n}\log^{-\beta}(e-\log\xi). \tag{4.21}$$

Further,

$$\left\|\frac{1}{\widetilde{v}}k(\xi,\cdot)\chi_{(0,\xi)}\right\|_\infty \geq k(\xi,\xi/2)/\widetilde{v}(\xi/2) \approx \xi^{\sigma/n}\log^{\alpha}(e-\log\xi) \tag{4.22}$$

for all $\xi \in (0,1)$, and the second inequality in (4.2) follows from (3.33), (4.21) and (4.22). ∎

To obtain double exponential integrability of Bessel potentials we need the following two assertions.

4.2. LEMMA. *Let $K \in (0,\infty)$, $\alpha < 0$ and let $\mathfrak{S} \subset \mathfrak{M}^+(0,K)$, $\mathfrak{S} \neq \emptyset$. Then the following statements are equivalent.*

(i) *There exist $a_1, a_2, L \in (0,\infty)$ such that*

$$\int_0^K \exp\left(a_2\exp[a_1 f(t)^{-1/\alpha}]\right) dt \leq L \quad \textit{for all } f \in \mathfrak{S}.$$

(ii) *There exists $B \in (0,\infty)$ such that*

$$\sup_{0<t<K} f^*(t)\log^{\alpha}(e+|\log t|) \leq B \quad \textit{for all } f \in \mathfrak{S}. \tag{4.23}$$

(iii) *Given any $A \in (0,\infty)$, there exist $a, M \in (0,\infty)$ such that*

$$\int_0^K \exp\left(A\exp[af(t)^{-1/\alpha}]\right) dt \leq M \quad \textit{for all } f \in \mathfrak{S}. \tag{4.24}$$

Proof. The equivalence (i)⇔(ii) is proved in [5, Lemma 3.10]. The proof of the implication (ii)⇒(iii) is the same as that of (ii)⇒(i). The implication (iii)⇒(i) is obvious. ∎

4.3. Remark. Let $\Phi : [0,\infty] \to [0,\infty]$ be a strictly monotone function. Let $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$ and let $H : Q \to [0,\infty]$ be a measurable function. Then the functions $x \mapsto \Phi(H(x))$ and $t \mapsto \Phi(H^*(t))$ are equimeasurable, and consequently they have the same non-increasing rearrangement, say F. Hence

$$\int_Q \Phi(H(x))\,dx = \int_0^\infty F(t)\,dt = \int_0^\infty \Phi(H^*(t))\,dt.$$

The main result of this section reads:

4.4. THEOREM. *Let $p \in [1,\infty]$, $\alpha,\beta \in \mathbb{R}$, $\alpha \neq 0$, $\beta \neq 1/p'$ and $\sigma \in (0,n)$. Suppose that $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$.*

(i) *Let*

$$\alpha < 0 \quad \textit{and} \quad \alpha - \beta + 1/p' \leq 0. \tag{4.2}$$

Then, given $A \in (0,\infty)$, there exist a, $M \in (0,\infty)$ such that

$$\int_Q \exp\left(A\exp[a|(g_\sigma * f)(x)|^{-1/\alpha}]\right) dx \leq M \tag{4.25}$$

for all f satisfying

$$f \in L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^{\beta}(\mathbb{R}^n), \qquad \|f\|_{n/\sigma,p,1/p',\beta} \leq 1. \tag{4.26}$$

(ii) *Let Q have non-empty interior and let the estimate (4.25) hold with some $A, a, M \in (0,\infty)$ for all f satisfying (4.26). Then (4.2) is satisfied.*

Proof. Statement (i) follows from Lemmas 4.1 and 4.2, and Remark 4.3. To prove statement (ii), suppose that (4.25) holds for all f satisfying (4.26). Set

$$\Phi(t) = \exp[A\exp(at^{-1/\alpha})], \qquad t > 0.$$

Since $\operatorname{int} Q \neq \emptyset$, there are $x_0 \in Q$ and $R > 0$ such that $B_n(x_0,R) \subset Q$. Thus, (4.25) implies

$$\int_{B_n(x_0,R)} \Phi(|(g_\sigma * f)(x)|)\,dx \leq M \tag{4.27}$$

for any f which satisfies (4.26).

Suppose that $x_0 = 0$ and $R = \omega_n^{-1/n}$. This implies that $|B_n(0,R)|_n = 1$. Take $f : \mathbb{R}^n \to [0,\infty]$ with

$$\operatorname{supp} f \subset B_n(0,R), \qquad 0 < \|f\|_{n/\sigma,p,1/p',\beta} < \infty. \tag{4.28}$$

Then

$$\operatorname{supp} f^* \subset [0,1]. \tag{4.29}$$

Putting

$$\widetilde{f}_\varepsilon(y) = \varepsilon f^*(\omega_n|y|^n), \qquad y \in \mathbb{R}^n,\ \varepsilon \in (0,\infty), \tag{4.30}$$

we obtain $\operatorname{supp}\widetilde{f}_\varepsilon \subset B_n(0,R)$ and, moreover, $(\widetilde{f}_\varepsilon)^*(t) = \varepsilon f^*(t)$, $t > 0$. Consequently,

$$\|\widetilde{f}_\varepsilon\|_{n/\sigma,p,1/p',\beta} = \varepsilon\|f\|_{n/\sigma,p,1/p',\beta}.$$

By (3.13), (3.15) and (3.16), there is $C_0 \in (1,\infty)$ such that

$$C_0 I_\sigma(x) \geq g_\sigma(x) \geq C_0^{-1} I_\sigma(x) \quad \text{for all } x \in B_n(0,2R)\setminus\{0\}.$$

Hence,

$$C_0(I_\sigma * \widetilde{f}_\varepsilon)(x) \geq (g_\sigma * \widetilde{f}_\varepsilon)(x) \geq C_0^{-1}(I_\sigma * \widetilde{f}_\varepsilon)(x) \tag{4.31}$$

for all $x \in B_n(0,R)$.

Assume first that $\alpha > 0$. Then

$$\Phi \text{ is decreasing on } (0,\infty). \tag{4.32}$$

Together with (4.31) and (4.27) this gives

$$\int_{B_n(0,R)} \Phi(C_0|(I_\sigma * \widetilde{f}_\varepsilon)(x)|)\,dx \le M \tag{4.33}$$

provided that

$$0 < \varepsilon \le \|f\|^{-1}_{n/\sigma,p,1/p',\beta}. \tag{4.34}$$

Since the function

$$F(t) = \frac{n}{\sigma}\omega_n^{1-\sigma/n}\Big[t^{\sigma/n-1}\int_0^t f^*(s)\,ds + \int_t^1 s^{\sigma/n-1}f^*(s)\,ds\Big]$$

is non-increasing on (0,1), Lemma 3.4 of [5] (applied to $\widetilde{f}_\varepsilon$ in place of $\widetilde{g}$) and (4.29) yield for any $t \in (1/2,1)$,

$$(I_\sigma * \widetilde{f}_\varepsilon)^*(t) \le \varepsilon F(t) \le \varepsilon F(1/2).$$

Taking

$$H(x) := C_0(I_\sigma * \widetilde{f}_\varepsilon)(x)\chi_{B_n(0,R)}(x), \quad x \in \mathbb{R}^n,$$

we have

$$H^*(t) \le C_0(I_\sigma * \widetilde{f}_\varepsilon)^*(t), \quad t > 0.$$

Together with Remark 4.3, the last two estimates and (4.32) imply

$$\begin{aligned}\int_{B_n(0,R)} &\Phi(C_0|(I_\sigma * \widetilde{f}_\varepsilon)(x)|)\,dx \\ &= \int_{B_n(0,R)} \Phi(H(x))\,dx = \int_0^1 \Phi(H^*(t))\,dt \ge \int_{1/2}^1 \Phi(\varepsilon C_0 F(1/2))\,dt \\ &= \frac{1}{2}\exp[A\exp(a\varepsilon^{-1/\alpha}C_0^{-1/\alpha}F(1/2)^{-1/\alpha})] \to \infty \quad \text{as } \varepsilon \to 0,\end{aligned}$$

which contradicts (4.33), (4.34), and consequently $\alpha < 0$ (note that $\alpha \ne 0$—cf. Theorem 4.4).

The inequality $\alpha < 0$ implies that

$$\Phi \text{ is increasing on } (0,\infty). \tag{4.35}$$

This, (4.31) and (4.27) give

$$\int_{B_n(0,R)} \Phi(C_0^{-1}|(I_\sigma * \widetilde{f}_\varepsilon)(x)|)\,dx \le M \tag{4.36}$$

provided that (4.34) is satisfied. By (3.15) of [5] and (4.30),

$$(I_\sigma * \widetilde{f}_\varepsilon)(x) \ge C_1\varepsilon F(\omega_n|x|^n) \ge C_1\varepsilon G(x), \quad x \in \mathbb{R}^n, \tag{4.37}$$

where F is from (4.13) and $G(x) := F(\omega_n|x|^n)\chi_{B_n(0,R)}(x)$, $C_1 = C_1(\sigma,n)$. Applying (4.35)–(4.37), we obtain

$$M \ge \int_{B_n(0,R)} \Phi(C_2\varepsilon G(x))\,dx = \int_0^1 \Phi(C_2\varepsilon G^*(t))\,dt,$$

where $C_2 = C_0^{-1}C_1$. With $g(t) := F(t)\chi_{(0,1)}(t)$ for $t > 0$ and making use of the fact that F is non-increasing on $(0,\infty)$, we have

$$G^*(t) = g^*(t) = F(t), \quad t \in (0,1).$$

Since $\operatorname{supp} f^* \subset [0,1]$,

$$F(t) \ge \int_t^1 s^{\sigma/n-1}f^*(s)\,ds, \quad t \in (0,1).$$

Thus,

$$\begin{aligned}M &\ge \int_0^1 \Phi\Big(C_2\varepsilon\int_t^1 s^{\sigma/n-1}f^*(s)\,ds\Big)\,dt \\ &= \int_0^1 \exp\Big[A\exp\Big(a\Big(C_2\varepsilon\int_t^1 s^{\sigma/n-1}f^*(s)\,ds\Big)^{-1/\alpha}\Big)\Big]\,dt\end{aligned}$$

for all functions f satisfying (4.28) and any ε from (4.34). Consequently, by Lemma 4.2 (with $K = 1$), there is $B \in (0,\infty)$ such that

$$\sup_{0<t<1}\Big(\int_t^1 s^{\sigma/n-1}f^*(s)\,ds\Big)\log^\alpha(e-\log t) \le B(C_2\varepsilon)^{-1}$$

for the same family of functions f and the same set of ε. Hence, choosing

$$\varepsilon = \|f\|^{-1}_{n/\sigma,p,1/p',\beta},$$

we have

$$\sup_{0<t<1}\Big(\int_t^1 s^{\sigma/n-1}f^*(s)\,ds\Big)\log^\alpha(e-\log t) \le BC_2^{-1}\|f\|_{n/\sigma,p,1/p',\beta}.$$

This implies that the inequality (4.14) holds for all $f \in \mathfrak{M}^+(0,1)$, and the same argument as that used in the proof of Lemma 4.1 yields the condition $\alpha - \beta + 1/p' \le 0$.

The general case $x_0 \in \mathbb{R}^n$ and $R \in (0,\infty)$ can be reduced to the case $x_0 = 0$ and $R = \omega_n^{-1/n}$ by using the transformation

$$y = (z - x_0)\frac{\omega_n^{-1/n}}{R}, \qquad z \in \mathbb{R}^n,$$

which maps $B_n(x_0,R)$ onto $B_n(0,\omega_n^{-1/n})$, and the property of the Riesz potential that

$$(I_\sigma * \bar{f})(x) = \left(\frac{R}{\omega_n^{-1/n}}\right)^\sigma (I_\sigma * f)\left((x-x_0)\frac{\omega_n^{-1/n}}{R}\right), \qquad x \in B_n(x_0,R),$$

where

$$f(y) := \bar{f}\left(y\frac{R}{\omega_n^{-1/n}} + x_0\right), \qquad y \in B_n(0,\omega_n^{-1/n}).$$

Note that then $\operatorname{supp} f \subset B_n(0,\omega_n^{-1/n})$ provided that $\operatorname{supp}\bar{f} \subset B_n(x_0,R)$ and that $\|f\|_{n/\sigma,p,1/p',\beta} = C\,\|\bar{f}\|_{n/\sigma,p,1/p',\beta}$, where C is a constant independent of f. ∎

As was said in Section 3, the decay of the Bessel kernel g_σ as $|x| \to \infty$ is much faster than that of the Riesz kernel I_σ. The same is true for the non-increasing rearrangements g_σ^* and I_σ^* (cf. the estimate (3.17) and the formula $I_\sigma^*(t) = (t/\omega_n)^{\sigma/n-1}$, $t > 0$). Because of this, the analogue of the estimate (4.6) with I_σ^* instead of g_σ^* does not hold for all functions f from $L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n)$. In order to get such an analogue, it suffices, for example, to assume that

$$f \in L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n) \cap L^1(\mathbb{R}^n).$$

Moreover, if we realize that for $f \in L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n)$ with bounded support we have the estimate

$$\|f\|_1 \le C\,\|f\|_{n/\sigma,p,1/p',\beta}$$

(with a constant C independent of f), we can see that the following analogues of Lemma 4.1 and Theorem 4.4 hold (cf. the corresponding results in [5]).

4.5. LEMMA. *Let $p \in [1,\infty]$ and $\alpha,\beta \in \mathbb{R}$, $\alpha \ne 0$, $\beta \ne 1/p'$, $\sigma \in (0,n)$. Suppose that*

$$f \in L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \qquad u = I_\sigma * f.$$

Then the inequality

$$\sup_{t\in(0,1)} [\log^\alpha(e - \log t)u^*(t)] \le c(\|f\|_{n/\sigma,p,1/p',\beta} + \|f\|_1)$$

holds with a constant c which depends only on p, α, β, σ and n if, and only if,

$$\alpha < 0 \quad \text{and} \quad \alpha - \beta + 1/p' \le 0.$$

4.6. THEOREM. *Let $p \in [1,\infty]$ and $\alpha,\beta \in \mathbb{R}$, $\alpha \ne 0$, $\beta \ne 1/p'$, $\sigma \in (0,n)$. Suppose that $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$.*

(i) *Let*

(4.2) $$\alpha < 0 \quad \text{and} \quad \alpha - \beta + 1/p' \le 0.$$

Then, given $A \in (0,\infty)$, there exist a, $M \in (0,\infty)$ such that

(4.38) $$\int_Q \exp(A\exp[a|(I_\sigma * f)(x)|^{-1/\alpha}])\,dx \le M$$

for all $f \in X := L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with

$$\|f\|_X := \|f\|_{n/\sigma,p,1/p',\beta} + \|f\|_1 \le 1.$$

(ii) *Let Q have a non-empty interior and let the estimate* (4.38) *hold with some $A, a, M \in (0,\infty)$ for all $f \in X$, $\|f\|_X \le 1$. Then* (4.2) *is satisfied.*

5. Embedding theorems. In this section we make use of Theorem 4.4 to obtain embeddings of Bessel potential spaces in appropriate Orlicz ones.

5.1. NOTATION. Let $A, B \in (0,\infty)$ be fixed numbers and let $\Phi_{A,B}$ be a Young function satisfying

(5.1) $$\Phi_{A,B}(t) = \exp(A\exp t^{1/B}) \quad \text{for all } t \ge t_0$$

with some $t_0 \in (0,\infty)$.

For a function u from the Orlicz space $L_{\Phi_{A,B}}(Q)$ and for a measurable subset $G \subset Q$, we put

(5.2) $$\varrho_{A,B}(u,G) = \int_G \Phi_{A,B}(|u(x)|)\,dx.$$

5.2. THEOREM. *Let $p \in [1,\infty]$ and $\alpha,\beta \in \mathbb{R}$, $\alpha \ne 0$, $\beta \ne 1/p'$, $\sigma \in (0,n)$. Suppose that $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$.*

(i) *Let*

(5.3) $$\alpha < 0 \quad \text{and} \quad \alpha - \beta + 1/p' \le 0.$$

Then, for any $A \in (0,\infty)$,

(5.4) $$H^\sigma L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow L_{\Phi_{A,-\alpha}}(Q).$$

(ii) *Let Q have non-empty interior and suppose that* (5.4) *holds with some $A \in (0,\infty)$. Then the numbers α, β and p satisfy* (5.3).

Proof. Let

$$u \in HL := H^\sigma L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n), \qquad \|u\|_{HL} \le 1. \tag{5.5}$$

Then $u = g_\sigma * f$, where $\|f\|_{n/\sigma,p,1/p',\beta} = \|u\|_{HL}$. Let $A \in (0,\infty)$ and let a, M be numbers from Theorem 4.4, and t_0 from (5.1). Putting

$$Q_1 = \{x \in Q : |u(x)|/a^\alpha \le t_0\}, \qquad Q_2 = Q \setminus Q_1 \tag{5.6}$$

and using (5.1), (5.2), (4.25) and the fact that $\Phi_{A,-\alpha}$ is increasing, we have

$$\begin{aligned}\varrho_{A,-\alpha}(u/a^\alpha, Q) &= \varrho_{A,-\alpha}(u/a^\alpha, Q_2) + \varrho_{A,-\alpha}(u/a^\alpha, Q_1)\\ &\le \int_Q \exp\left(A\exp[a|u(x)|^{-1/\alpha}]\right)dx + |Q|_n \Phi_{A,-\alpha}(t_0)\\ &\le M + |Q|_n\Phi_{A,-\alpha}(t_0) =: C.\end{aligned} \tag{5.7}$$

Moreover, Young's inequality implies

$$\|u/a^\alpha\|_{\Phi_{A,-\alpha}} \le \varrho_{A,-\alpha}(u/a^\alpha, Q) + 1,$$

which, together with (5.7), yields $\|u/a^\alpha\|_{\Phi_{A,-\alpha}} \le C+1$. Consequently, $\|u\|_{\Phi_{A,-\alpha}} \le a^\alpha(C+1)$ for any $u \in HL$, $\|u\|_{HL} \le 1$, and the proof of statement (i) is complete.

Assume now that $\operatorname{int} Q \ne 0$ and that (5.4) holds with some $A \in (0,\infty)$. Then there exists $C \in (0,\infty)$ such that

$$\|u\|_{\Phi_{A,-\alpha}} \le C \quad \text{ for all } u \in HL,\ \|u\|_{HL} \le 1$$

(we use the notation from (5.5)). Thus, $\|u/C\|_{\Phi_{A,-\alpha}} \le 1$, which implies (cf. [9, Lemma 3.8.4])

$$\varrho_{A,-\alpha}(u/C, Q) \le 1 \quad \text{ for all } u \in HL,\ \|u\|_{HL} \le 1, \tag{5.8}$$

and, on putting

$$Q_1 = \{x \in Q : |u(x)/C| \le t_0\}, \qquad Q_2 = Q \setminus Q_1,$$

we have from (5.8) and (5.1),

$$1 \ge \varrho_{A,-\alpha}(u/C, Q_2) = \int_{Q_2} \exp(A\exp|u(x)/C|^{-1/\alpha})\,dx.$$

Moreover,

$$\int_{Q_1} \exp(A\exp|u(x)/C|^{-1/\alpha})\,dx \le |Q|_n \exp(A\exp t_0^{-1/\alpha}) =: K.$$

Consequently,

$$\int_Q \exp(A\exp|u(x)/C|^{-1/\alpha})\,dx \le K+1 \quad \text{ for all } u \in HL,\ \|u\|_{HL} \le 1,$$

which means that statement (ii) of Theorem 4.4 is true with $a = C^{1/\alpha}$, $M = K+1$. Thus, condition (5.3) follows from Theorem 4.4. ∎

Taking $\sigma = n/p$ with $p \in (1,\infty)$ in Theorem 5.2, we obtain

5.3. EXAMPLE. Suppose $p \in (1,\infty)$ and $\alpha < 0$, $\beta \in \mathbb{R}$, $\beta \ne 1/p'$ and $\alpha - \beta + 1/p' \le 0$. Let $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$. Then, given any $A \in (0,\infty)$,

$$H^{n/p}L^{p,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow L_{\Phi_{A,-\alpha}}(Q). \tag{5.9}$$

In particular,

$$H^{n/p}L^{p,p}(\log L)^{1/p'}(\mathbb{R}^n) \hookrightarrow L_{\Phi_{A,1/p'}}(Q). \tag{5.10}$$

On putting $p = n > 1$, we have from (5.9) and (5.10),

$$H^1L^{n,n}(\log L)^{1/n'}(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow L_{\Phi_{A,-\alpha}}(Q)$$

and

$$H^1L^{n,n}(\log L)^{1/n'}(\mathbb{R}^n) \hookrightarrow L_{\Phi_{A,1/n'}}(Q),$$

respectively.

The following theorem deals with compact embeddings of Bessel potential spaces.

5.4. THEOREM. *Suppose that $p \in (1,\infty)$, $\gamma < 0$, $\beta \in \mathbb{R}$, $\beta \ne 1/p'$ and $\gamma - \beta + 1/p' < 0$. Let Ω be a measurable, bounded set in $\mathbb{R}^n$. Then, for any $A \in (0,\infty)$,*

$$H^{n/p}L^{p,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{\Phi_{A,-\gamma}}(\Omega). \tag{5.11}$$

Proof. Let $A \in (0,\infty)$ and let $\alpha \in (\gamma, 0)$ satisfy $\alpha - \beta + 1/p' \le 0$. Then, by Example 5.3, the embedding (5.9) (with $Q = \Omega$) holds. Consequently, if S is a bounded subset in

$$X := H^{n/p}L^{p,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n),$$

then S is bounded in $L_{\Phi_{A,-\alpha}}(\Omega)$. Since the Young function $\Phi_{A,-\gamma}$ increases more slowly than $\Phi_{A,-\alpha}$ near infinity, Theorem 8.23 of [1] shows that it is sufficient to prove that S is precompact in $L^1(\Omega)$.

Obviously, $L^{p,p}(\log L)^{1/p'}(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$, and hence

$$X \hookrightarrow H^{n/p}L^p(\mathbb{R}^n). \tag{5.12}$$

Moreover, we have with some $B := B_n(0,R)$ containing Ω (cf. [17]),

$$H^{n/p}L^p(\mathbb{R}^n) \hookrightarrow H^{n/p}L^p(B) \hookrightarrow\hookrightarrow L^p(B) \hookrightarrow L^p(\Omega),$$

which, together with (5.12), implies that S is precompact in $L^p(\Omega)$, and the result follows. ∎

5.5. COROLLARY. *Suppose $p \in (1,\infty)$ and $a > 1/p'$. Let Ω be a measurable, bounded set in $\mathbb{R}^n$. Then for any $A \in (0,\infty)$,*

$$H^{n/p}L^{p,p}(\log L)^{1/p'}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{\Phi_{A,a}}(\Omega).$$

In particular,

$$H^1 L^{n,n}(\log L)^{1/n'}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{\Phi_{A,b}}(\Omega)$$

provided that $n > 1$ and $b > 1/n'$.

Now, we are going to apply our results to obtain embedding theorems for Bessel potential spaces defined on domains in $\mathbb{R}^n$.

5.6. THEOREM. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega|_n < \infty$. Suppose $p \in [1,\infty]$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, $\beta \neq 1/p'$, $\sigma \in (0,n)$.*

(i) *Let*

$$\alpha < 0 \quad \text{and} \quad \alpha - \beta + 1/p' \le 0. \tag{5.3}$$

Then, given any $A \in (0,\infty)$,

$$H^\sigma L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\Omega) \hookrightarrow L_{\Phi_{A,-\alpha}}(\Omega). \tag{5.13}$$

(ii) *Suppose that the embedding* (5.13) *holds with some $A \in (0,\infty)$. Then the numbers α, β and p satisfy* (5.3).

Proof. Put $\mathcal{Y}(\Omega) = L_{\Phi_{A,-\alpha}}(\Omega)$.

(i) Assume that (5.3) is satisfied. Let

$$u \in HL(\Omega) := H^\sigma L^{n/\sigma,p}(\log L)^{1/p'}(\log\log L)^\beta(\Omega).$$

Then there exists $\widetilde{u} \in HL(\mathbb{R}^n)$ (we use the notation from (5.5)) such that $u = \widetilde{u}|_\Omega$. By Theorem 5.2,

$$\|\widetilde{u}\|_{\mathcal{Y}(\Omega)} \le c\|\widetilde{u}\|_{HL(\mathbb{R}^n)} \quad \text{for all } \widetilde{u} \in HL(\mathbb{R}^n)$$

with a constant c independent of $\widetilde{u}$. Since $\|u\|_{\mathcal{Y}(\Omega)} = \|\widetilde{u}\|_{\mathcal{Y}(\Omega)}$, we have

$$\|u\|_{\mathcal{Y}(\Omega)} \le C\|\widetilde{u}\|_{HL(\mathbb{R}^n)},$$

and taking the infimum over all $\widetilde{u} \in HL(\mathbb{R}^n)$ satisfying $u = \widetilde{u}|_\Omega$, we obtain (5.13).

(ii) Let (5.13) hold with some $A \in (0,\infty)$. Then there is $c \in (0,\infty)$ such that

$$\|u\|_{\mathcal{Y}(\Omega)} \le c\|u\|_{HL(\Omega)} \quad \text{for all } u \in HL(\Omega). \tag{5.14}$$

However, (5.14) yields $\|\widetilde{u}\|_{\mathcal{Y}(\Omega)} \le c\|\widetilde{u}\|_{HL(\mathbb{R}^n)}$ for any $\widetilde{u} \in HL(\mathbb{R}^n)$, and the result follows from Theorem 5.2. ∎

Using Theorem 5.4, one easily obtains

5.7. THEOREM. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega|_n < \infty$. Suppose $p \in (1,\infty)$, $\gamma < 0$, $\beta \in \mathbb{R}$, $\beta \neq 1/p'$ and $\gamma - \beta + 1/p' < 0$. Then, for any $A \in (0,\infty)$,*

$$H^{n/p}L^{p,p}(\log L)^{1/p'}(\log\log L)^\beta(\Omega) \hookrightarrow\hookrightarrow L_{\Phi_{A,-\gamma}}(\Omega).$$

6. Single exponential type results. In the previous sections we have obtained double exponential integrability of the Bessel potentials of functions from generalized Lorentz–Zygmund spaces. An essential role was played by a double limiting case of Hardy's inequality (Lemma 3.2). If we replace this double limiting form of Hardy's inequality by a single limiting one and use Lorentz–Zygmund spaces instead of generalized Lorentz–Zygmund ones, analogous arguments lead to single exponential integrability results for functions from Lorentz–Zygmund spaces $L^{p,q}(\log L)^\lambda$ as well as to corresponding embedding theorems for Bessel potential spaces. Starting with a formulation of a single limiting case of Hardy's inequality, we give here only a survey of results (cf. results in [6] and Remark 3.11(iv) of [5]).

6.1. LEMMA (a single limiting case). *Suppose $p \in [1,\infty]$ and $\gamma, \delta \in \mathbb{R}$, $\gamma \neq 0$, $\delta \neq 1/p'$.*

(i) *There exists a constant $c \in (0,\infty)$ such that the Hardy inequality*

$$\Big\|(e - \log t)^\gamma \int_0^t g(\tau)\,d\tau\Big\|_{Y(\infty)} \le c\,\|(e - \log t)^\delta t g(t)\|_{Y(p)}$$

holds for every $g \in \mathfrak{M}^+(0,1)$ if, and only if, $\delta > 1/p'$ and

$$\gamma - \delta + 1/p' \le 0. \tag{6.1}$$

(ii) *There exists a constant $c \in (0,\infty)$ such that the Hardy inequality*

$$\Big\|(e - \log t)^\gamma \int_t^1 g(\tau)\,d\tau\Big\|_{Y(\infty)} \le c\,\|(e - \log t)^\delta t g(t)\|_{Y(p)}$$

holds for all $g \in \mathfrak{M}^+(0,1)$ if, and only if, $\gamma < 0$ and (6.1) *is satisfied.*

6.2. LEMMA. *Let $p \in [1,\infty]$ and $\gamma, \delta \in \mathbb{R}$, $\gamma \neq 0$, $\delta \neq 1/p'$, $\sigma \in (0,n)$. Suppose that*

$$f \in L^{n/\sigma,p}(\log L)^\delta(\mathbb{R}^n), \qquad u = g_\sigma * f.$$

Then the inequality

$$\sup_{t\in(0,1)} [(e - \log t)^\gamma u^*(t)] \le c\,\|f\|_{n/\sigma,p,\delta}$$

holds with a constant c which depends only on p, γ, δ, σ and n if, and only if,

$$\gamma < 0 \quad \text{and} \quad \gamma - \delta + 1/p' \le 0. \tag{6.2}$$

6.3. LEMMA. *Let $K \in (0,\infty)$, $\gamma < 0$ and let $\mathfrak{S} \subset \mathfrak{M}^+(0,K)$, $\mathfrak{S} \neq \emptyset$. Then the following statements are equivalent.*

(i) *There exists $B \in (0,\infty)$ such that*

$$\sup_{0<t<K} f^*(t)(e + |\log t|)^\gamma \le B \quad \text{for all } f \in \mathfrak{S}.$$

(ii) *There exist a, $M \in (0,\infty)$ such that*

$$\int_0^K \exp\left(af(t)^{-1/\gamma}\right) dt \le M \quad \text{for all } f \in \mathfrak{S}.$$

6.4. THEOREM. *Let $p \in [1,\infty]$ and $\gamma, \delta \in \mathbb{R}$, $\gamma \neq 0$, $\delta \neq 1/p'$, $\sigma \in (0,n)$. Suppose that $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$.*

(i) *Let the condition (6.2) hold. Then there are constants $a, M \in (0,\infty)$ such that*

$$\int_Q \exp(a|(g_\sigma * f)(x)|^{-1/\gamma})\,dx \le M \tag{6.3}$$

for all f satisfying

$$f \in L^{n/\sigma,p}(\log L)^\delta(\mathbb{R}^n), \qquad \|f\|_{n/\sigma,p,\delta} \le 1. \tag{6.4}$$

(ii) *Let Q have a non-empty interior and let the estimate (6.3) hold for all f satisfying (6.4). Then (6.2) is satisfied.*

6.5. NOTATION. Let $B \in (0,\infty)$ be a fixed number and let Φ_B be a Young function satisfying

$$\Phi_B(t) = \exp t^{1/B} \quad \text{for all } t \ge t_0$$

with some $t_0 \in (0,\infty)$.

6.6. THEOREM. *Let $p \in [1,\infty]$ and $\gamma, \delta \in \mathbb{R}$, $\gamma \neq 0$, $\delta \neq 1/p'$, $\sigma \in (0,n)$. Suppose that $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$.*

(i) *Let the condition (6.2) hold. Then*

$$H^\sigma L^{n/\sigma,p}(\log L)^\delta(\mathbb{R}^n) \hookrightarrow L_{\Phi_{-\gamma}}(Q). \tag{6.5}$$

(ii) *Let Q have a non-empty interior. Then (6.5) implies (6.2).*

6.7. EXAMPLE. Suppose $p \in (1,\infty)$ and $\gamma, \delta \in \mathbb{R}$, $\gamma \neq 0$, $\delta \neq 1/p'$. Let $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$. Then

$$H^{n/p}L^{p,p}(\log L)^\delta(\mathbb{R}^n) \hookrightarrow L_{\Phi_{-\gamma}}(Q) \tag{6.6}$$

provided that (6.2) is satisfied. In particular,

$$H^{n/p}L^{p,p}(\mathbb{R}^n) \hookrightarrow L_{\Phi_{1/p'}}(Q). \tag{6.7}$$

On putting $p = n > 1$, we have from (6.6) and (6.7),

$$H^1L^{n,n}(\log L)^\delta(\mathbb{R}^n) \hookrightarrow L_{\Phi_{-\gamma}}(Q)$$

and

$$H^1L^{n,n}(\mathbb{R}^n) \hookrightarrow L_{\Phi_{1/n'}}(Q),$$

respectively.

6.8. THEOREM. *Suppose $p \in (1,\infty)$, $\alpha < 0$, $\delta \in \mathbb{R}$, $0 \le \delta \neq 1/p'$ and $\alpha - \beta + 1/p' < 0$. Let Ω be a measurable, bounded set in $\mathbb{R}^n$. Then*

$$H^{n/p}L^{p,p}(\log L)^\delta(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{\Phi_{-\alpha}}(\Omega).$$

We conclude this section with two assertions which are analogues of Lemma 6.2 and Theorem 6.4 for the Riesz potential. Note that the latter assertion yields the main result of [8] and extends [5, Remark 3.11(iv)].

6.9. LEMMA. *Let $p \in [1,\infty]$ and $\gamma, \delta \in \mathbb{R}$, $\gamma \neq 0$, $\delta \neq 1/p'$, $\sigma \in (0,n)$. Suppose that*

$$f \in L^{n/\sigma,p}(\log L)^\delta(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \qquad u = I_\sigma * f.$$

Then the inequality

$$\sup_{t\in(0,1)} [(e - \log t)^\gamma u^*(t)] \le c(\|f\|_{n/\sigma,p,\delta} + \|f\|_1)$$

holds with a constant c which depends only on p, γ, δ, σ and n if, and only if, condition (6.2) is satisfied.

6.10. THEOREM. *Let $p \in [1,\infty]$ and $\gamma, \delta \in \mathbb{R}$, $\gamma \neq 0$, $\delta \neq 1/p'$, $\sigma \in (0,n)$. Suppose that $Q \subset \mathbb{R}^n$, $|Q|_n < \infty$.*

(i) *Let the condition (6.2) hold. Then there are constants $a, M \in (0,\infty)$ such that*

$$\int_Q \exp(a|(I_\sigma * f)(x)|^{-1/\gamma})\,dx \le M \tag{6.8}$$

for all f satisfying

$$f \in L^{n/\sigma,p}(\log L)^\delta(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \qquad \|f\|_{n/\sigma,p,\delta} + \|f\|_1 \le 1. \tag{6.9}$$

(ii) *Let Q have non-empty interior and let the estimate (6.8) hold for all f satisfying (6.9). Then (6.2) is satisfied.*

Concluding remark. Note that using a higher-order limiting form of Hardy's inequality and generalized Lorentz–Zygmund spaces with corresponding higher-order logarithmic terms, one can obtain higher-order exponential integrability of the Bessel and Riesz potentials.

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