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A selection theorem of Helly type and its applications

by

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Abstract. We prove an abstract selection theorem for set-valued mappings with compact convex values in a normed space. Some special cases of this result as well as its applications to separation theory and Hyers–Ulam stability of affine functions are also given.

1. Introduction. A starting point of our investigations is the classical Helly theorem stating that a family $\mathcal{K}$ of convex compact subsets of an l-dimensional Minkowski space has a non-empty intersection iff any $l+1$ elements of $\mathcal{K}$ have a non-empty intersection; if the number of elements of $\mathcal{K}$ is finite, the assumption that the sets are compact can be omitted (cf. [4, Thm. 6.1]). Using this result we obtain an abstract selection theorem for set-valued mappings with convex compact values in a normed space. Our theorem is in fact a generalization of Helly's (cf. Corollary 1). It also generalizes the known result on common transversals for parallel segments in $\mathbb{R}^2$ (cf. Corollary 4). As an application of our result we also get a theorem on separation of two real functions defined on a subset of $\mathbb{R}^n$ by an affine one. It generalizes the sandwich theorem obtained recently by Nikodem and Wąsowicz [3] and corresponds to the sandwich theorem with a convex function proved by Baron, Matkowski and Nikodem [1]. Finally, as another consequence of our main result, we get a stability theorem of Hyers–Ulam type for affine functions.

2. An abstract selection theorem. The main result of this paper is the following theorem.

THEOREM 1. *Let D be a non-empty subset of a set X, $(Y,\|\cdot\|)$ be a normed space and $\Phi : D \to \mathrm{cc}(Y)$ (the family of all convex compact non-empty subsets of Y) be a set-valued mapping. Assume that $\mathcal{W}$ is an l-dimensional subspace of the vector space of all mappings from X to Y and*

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D has enough points for $f|_D = 0$ to imply $f = 0$, for each $f \in \mathcal{W}$. Then the following conditions are equivalent:

(i) *there exists an $h \in \mathcal{W}$ such that $h(x) \in \Phi(x)$ for all $x \in D$;*

(ii) *for every $\widetilde{D} \subset D$ with* $\operatorname{card} \widetilde{D} = l + 1$ *there exists an $h \in \mathcal{W}$ such that $h(x) \in \Phi(x)$ for all $x \in \widetilde{D}$.*

Proof. The implication (i)$\Rightarrow$(ii) is obvious. To prove that (ii)$\Rightarrow$(i) fix a base $h_1, \ldots, h_l$ of $\mathcal{W}$. With the norm $|||\cdot|||_1$ defined by

$$|||f|||_1 := |\alpha_1| + \ldots + |\alpha_l| \quad \text{for } f = \alpha_1 h_1 + \ldots + \alpha_l h_l,$$

$\mathcal{W}$ is an l-dimensional Minkowski space. Consider the sets

$$K_x = \{h \in \mathcal{W} : h(x) \in \Phi(x)\}, \quad x \in D.$$

Clearly, these sets are convex and, by assumption, the intersection of any $l+1$ of them is non-empty. Hence, by Helly's theorem, the intersection of any finite number of them is non-empty. This finishes the proof in the case where $\operatorname{card} D < \infty$. In the case of infinite D note that the sets K_x are closed and to prove that $\bigcap_{x \in D} K_x \neq \emptyset$ it is enough to show that $K_{x_1} \cap \ldots \cap K_{x_s}$ is compact for some $x_1, \ldots, x_s \in D$. Consider the maps $\omega_x : \mathcal{W} \to Y$ (for $x \in D$) defined by $\omega_x(f) := f(x)$, $f \in \mathcal{W}$. By assumption

$$\bigcap_{x \in D} \operatorname{Ker} \omega_x = \{0\}.$$

Since $\mathcal{W}$ is l-dimensional, there are $x_1, \ldots, x_l \in D$ such that

$$\bigcap_{i=1}^{l} \operatorname{Ker} \omega_{x_i} = \{0\}. \tag{1}$$

Define $|||f|||_2 := \max\{\|f(x_i)\| : i = 1, \ldots, l\}$, $f \in \mathcal{W}$. By (1), $|||\cdot|||_2$ is a norm on $\mathcal{W}$ and thus equivalent to $|||\cdot|||_1$. Since the sets $\Phi(x)$ are compact there exists an M such that $\|y\| \le M$ for all $y \in \Phi(x_i)$, $i = 1, \ldots, l$. Hence $|||f|||_2 \le M$ for $f \in \bigcap_{i=1}^{l} K_{x_i}$ and so the set $\bigcap_{i=1}^{l} K_{x_i}$ is compact. This completes the proof. ■

3. Some special cases.

The following corollary to our main result is equivalent to Helly's theorem.

COROLLARY 1. *Let D be a non-empty set. A set-valued mapping $\Phi : D \to \operatorname{cc}(\mathbb{R}^l)$ has a constant selection iff $\Phi(x_1) \cap \ldots \cap \Phi(x_{l+1}) \neq \emptyset$ for arbitrary $x_1, \ldots, x_{l+1} \in D$. If* $\operatorname{card} D < \infty$, *then compactness of the values of Φ is not necessary.*

Proof. It is enough to notice that the space of all constant functions from D to $\mathbb{R}^l$ is l-dimensional. ■

Using the fact that the space of all polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree not greater than l is $(l+1)$-dimensional, we get the following result due to Rademacher and Schoenberg (cf. [4, Thm. 6.9]).

COROLLARY 2. *Let $D \subset \mathbb{R}$ and* $\operatorname{card} D \ge l + 2$. *A set-valued mapping $\Phi : D \to \operatorname{cc}(\mathbb{R})$ has a selection which is a polynomial of degree not greater than l iff for arbitrary $x_1, \ldots, x_{l+2} \in D$ there exists a polynomial $p : \mathbb{R} \to \mathbb{R}$ of degree not greater than l such that $p(x_i) \in \Phi(x_i)$ for all $i = 1, \ldots, l+2$.*

The next two results follow from the fact that the space of all linear functions $h : \mathbb{R}^k \to \mathbb{R}^r$ is kr-dimensional and the space of all affine functions $h : \mathbb{R}^k \to \mathbb{R}^r$ is $(k+1)r$-dimensional.

COROLLARY 3. *Let $D \subset \mathbb{R}^k$ and* $\operatorname{lin} D = \mathbb{R}^k$. *A set-valued mapping $\Phi : D \to \operatorname{cc}(\mathbb{R}^r)$ has a linear selection iff for arbitrary $x_1, \ldots, x_{kr+1} \in D$ there exists a linear function $h : \mathbb{R}^k \to \mathbb{R}^r$ such that $h(x_i) \in \Phi(x_i)$ for all $i = 1, \ldots, kr+1$.*

COROLLARY 4. *Let $D \subset \mathbb{R}^k$ and* $\operatorname{lin} D = \mathbb{R}^k$. *A set-valued mapping $\Phi : D \to \operatorname{cc}(\mathbb{R}^r)$ has an affine selection iff for arbitrary points $x_1, \ldots, x_{(k+1)r+1} \in D$ there exists an affine function $h : \mathbb{R}^k \to \mathbb{R}^r$ such that $h(x_i) \in \Phi(x_i)$ for all $i = 1, \ldots, (k+1)r+1$.*

Remark 1. The above Corollary 4 generalizes the known result about common transversals stating that for a family $\mathcal{F}$ of parallel compact segments in $\mathbb{R}^2$ there exists a straight line intersecting all members of $\mathcal{F}$ iff any three members of $\mathcal{F}$ are intersected by a straight line (cf. [4, Thm. 6.8]).

Remark 2. In the special case $k = 1$, $r = 2$ we deduce that a set-valued mapping $\Phi : I \to \operatorname{cc}(\mathbb{R}^2)$, where $I \subset \mathbb{R}$ is an interval, has an affine selection iff for each family $x_1, \ldots, x_5$ in I there is an affine f such that $f(x_i) \in \Phi(x_i)$ for $i = 1, \ldots, 5$. Known counterexamples (cf. [5, Remark 3]) show that less points do not suffice. On the other hand, a set-valued mapping $\Phi : D \to \operatorname{cc}(\mathbb{R})$, where D is a convex subset of $\mathbb{R}^2$, has an affine selection iff for any four values of Φ there is a plane intersecting them; clearly, less points are not enough.

4. A sandwich theorem

THEOREM 2. *Let $D \subset \mathbb{R}^k$ be such that* $\operatorname{lin} D = \mathbb{R}^k$ *and let $f, g : D \to \mathbb{R}$, $f \le g$. Then the following conditions are equivalent:*

(i) *there exists an affine function $h : \mathbb{R}^k \to \mathbb{R}$ such that $f \le h|_D \le g$;*

(ii) *for arbitrary points $x_1, \ldots, x_{k+2} \in D$ there is an affine function $h : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_i) \le h(x_i) \le g(x_i)$, $i = 1, \ldots, k+2$;*

$$\text{(iii)} \qquad \sum_{i=1}^{s} \lambda_i f(x_i) \le \sum_{j=s+1}^{k+2} \mu_j g(x_j)$$

for all $x_1, \ldots, x_{k+2} \in D$, $s \in \{1, \ldots, k+1\}$ *and* $\lambda_1, \ldots, \lambda_s, \mu_{s+1}, \ldots, \mu_{k+2} \ge 0$ *such that* $\sum_{i=1}^{s} \lambda_i = \sum_{j=s+1}^{k+2} \mu_j = 1$ *and* $\sum_{i=1}^{s} \lambda_i x_i = \sum_{j=s+1}^{k+2} \mu_j x_j$;

$$\text{(iv)} \qquad \sum_{i=1}^{k+2} \alpha_i f(x_i) \le \sum_{i=1}^{k+2} \beta_i g(x_i)$$

for all $x_1, \ldots, x_{k+2} \in D$ *and* $\alpha_1, \ldots, \alpha_{k+2}, \beta_1, \ldots, \beta_{k+2} \ge 0$ *such that* $\sum_{i=1}^{k+2} \alpha_i = \sum_{i=1}^{k+2} \beta_i = 1$ *and* $\sum_{i=1}^{k+2} \alpha_i x_i = \sum_{i=1}^{k+2} \beta_i x_i$.

Proof. We will show that (i)⇒(iii)⇒(iv)⇒(ii)⇒(i).

The implication (i)⇒(iii) is clear and (ii)⇒(i) follows from Corollary 4. To prove that (iv)⇒(ii) fix points $x_1, \ldots, x_{k+2} \in D$ and consider the sets

$$A = \{(x_i, \lambda) \in \mathbb{R}^k \times \mathbb{R} : i = 1, \ldots, k+2,\ \lambda \le f(x_i)\},$$
$$B = \{(x_i, \lambda) \in \mathbb{R}^k \times \mathbb{R} : i = 1, \ldots, k+2,\ \lambda \ge g(x_i)\}.$$

By assumption, for all $\alpha_1, \ldots, \alpha_{k+2}, \beta_1, \ldots, \beta_{k+2} \ge 0$ such that $\sum_{i=1}^{k+2} \alpha_i = \sum_{i=1}^{k+2} \beta_i = 1$ and $\sum_{i=1}^{k+2} \alpha_i x_i = \sum_{i=1}^{k+2} \beta_i x_i$ we have

$$\sum_{i=1}^{k+2} \alpha_i f(x_i) \le \sum_{i=1}^{k+2} \beta_i g(x_i).$$

This implies that $\operatorname{conv} A$ and $\operatorname{conv} B$ (as subsets of $\operatorname{lin}\{x_1, \ldots, x_{k+2}\} \times \mathbb{R}$) do not intersect at an interior point. Hence, by the Hahn–Banach separation theorem, these sets can be separated by a hyperplane H and (ii) holds with the affine function h with graph H.

To prove that (iii)⇒(iv) assume that $\sum_{i=1}^{k+2} \alpha_i x_i = \sum_{i=1}^{k+2} \beta_i x_i$, where $\alpha_1, \ldots, \alpha_{k+2}, \beta_1, \ldots, \beta_{k+2} \ge 0$ and $\alpha_1 + \ldots + \alpha_{k+2} = \beta_1 + \ldots + \beta_{k+2} = 1$. Suppose that e.g. $\alpha_1 \beta_1 > 0$, say $\alpha_1 \ge \beta_1 > 0$. Then

$$(\alpha_1 - \beta_1)x_1 + \ldots + \alpha_{k+2} x_{k+2} = 0x_1 + \beta_2 x_2 + \ldots + \beta_{k+2} x_{k+2}.$$

This is—up to a multiplicative constant—again a convex combination, since $(\alpha_1 - \beta_1) + \alpha_2 + \ldots + \alpha_{k+2} = \beta_2 + \ldots + \beta_{k+2}$. Repeating this procedure $k+2$ times, we get an expression as in (iii). We apply (iii) and by reversing the above reduction we obtain (iv) (here we also use the fact that $f \le g$). This completes the proof. ∎

Remark 3. Let $k = 1$. Then the equivalence (i)⇔(iii) is just the assertion of the main theorem of [3] stating that two real functions f, g defined on an interval $I \subset \mathbb{R}$ can be separated by an affine function iff for all $x, y \in I$ and $\lambda \in [0,1]$ they satisfy the inequalities

$$f(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda)g(y) \quad \text{and}$$
$$g(\lambda x + (1-\lambda)y) \ge \lambda f(x) + (1-\lambda)f(y).$$

5. Hyers–Ulam stability of affine functions. As another application of our selection theorem we obtain the following result on stability in Hyers–Ulam sense (cf. [2]) of affine functions.

THEOREM 3. *Let ε be a positive constant. If a function $f : \mathbb{R}^k \to \mathbb{R}^r$ satisfies the condition*

$$\text{(2)} \qquad \Big\| f\Big(\sum_{i=1}^{k+1} \lambda_i x_i\Big) - \sum_{i=1}^{k+1} \lambda_i f(x_i) \Big\| \le \varepsilon$$

for all $x_1, \ldots, x_{k+1} \in \mathbb{R}^k$ *and* $\lambda_1, \ldots, \lambda_{k+1} \ge 0$ *with* $\lambda_1 + \ldots + \lambda_{k+1} = 1$, *then there exists an affine function* $h : \mathbb{R}^k \to \mathbb{R}^r$ *such that*

$$\|f(x) - h(x)\| \le \varepsilon, \quad x \in \mathbb{R}^k.$$

Proof. Consider the set-valued mapping $\Phi : \mathbb{R}^k \to \mathrm{cc}(\mathbb{R}^r)$ defined by $\Phi(x) := B(f(x), \varepsilon)$, the closed ball with centre $f(x)$ and radius ε. We will show that Φ has an affine selection. By Corollary 4 it is enough to show that for arbitrary $r(k+1)+1$ points $x_1, \ldots, x_{r(k+1)+1} \in \mathbb{R}^k$ there is an affine function h such that $h(x_j) \in \Phi(x_j)$, $j = 1, \ldots, r(k+1)+1$. Fix $x_1, \ldots, x_{r(k+1)+1}$ and choose affinely independent $y_1, \ldots, y_{k+1} \in \mathbb{R}^k$ such that $x_j \in \operatorname{conv}\{y_1, \ldots, y_{k+1}\}$ for all $j = 1, \ldots, r(k+1)+1$. Define $h : \mathbb{R}^k \to \mathbb{R}^r$ to be affine and such that $h(y_i) = f(y_i)$, $i = 1, \ldots, k+1$. Then for every x_j we have $x_j = \lambda_{j,1} y_1 + \ldots + \lambda_{j,k+1} y_{k+1}$ with suitable $\lambda_{j,1}, \ldots, \lambda_{j,k+1} \ge 0$ such that $\lambda_{j,1} + \ldots + \lambda_{j,k+1} = 1$, and hence

$$\begin{aligned} &\|f(x_j) - h(x_j)\| \\ &= \Big\| f\Big(\sum_{i=1}^{k+1} \lambda_{j,i} y_i\Big) - \sum_{i=1}^{k+1} \lambda_{j,i} f(y_i) + \sum_{i=1}^{k+1} \lambda_{j,i} h(y_i) - h\Big(\sum_{i=1}^{k+1} \lambda_{j,i} y_i\Big) \Big\| \\ &= \Big\| f\Big(\sum_{i=1}^{k+1} \lambda_{j,i} y_i\Big) - \sum_{i=1}^{k+1} \lambda_{j,i} f(y_i) \Big\| \le \varepsilon. \end{aligned}$$

Thus $h(x_j) \in \Phi(x_j)$, which was to be proved. ∎

Remark 4. For $k = r = 1$ an analogous result is proved in [3]. In our approach expressions of the form (2) are appropriate. One can show by induction that if a function $f : \mathbb{R}^k \to \mathbb{R}^r$ is ε-affine in the sense that

$$\|f(\lambda x + (1-\lambda)y) - \lambda f(x) - (1-\lambda) f(y)\| \le \varepsilon \quad x, y \in \mathbb{R}^k,\ \lambda \in [0,1],$$

then it satisfies the condition (2) with the constant $c_k \varepsilon$ (instead of ε), where $c_k = k(k+3)/(2k+2)$. Thus, by Theorem 3, for every ε-affine function $f : \mathbb{R}^k \to \mathbb{R}^r$ there exists an affine function $h : \mathbb{R}^k \to \mathbb{R}^r$ such that $\|f(x) - h(x)\| \le c_k \varepsilon$, $x \in \mathbb{R}^k$.

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The functor $\sigma^2 X$

by

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Abstract. We disprove the existence of a universal object in several classes of spaces including the class of weakly Lindelöf Banach spaces.

It is well known that the Tikhonov cube is an injectively universal compact space in a given weight. Surjectively universal compact spaces can also be constructed for certain weights (see [5]) but frequently one would like to know whether there can be universal objects in some more restrictive classes of compacta such as, for example, the class of first countable compacta or one of the classes of compacta which naturally occur in functional analysis (see [14; p. 620]). The purpose of this note is to answer a number of questions of this sort by introducing a new topological functor which might be of independent interest. The following is an example of a result which can be obtained by the new method.

THEOREM. *For every compact countably tight space X of weight continuum there is a first countable retractive* (1) *Corson compact space Y which is not a continuous image of any closed subspace of X.*

It follows that a number of natural classes of compact spaces mentioned in Question 10.6 of [14] have neither injectively nor surjectively universal objects. Similarly, this shows that there are neither injectively nor surjectively universal objects in the class of Corson compacta of weight continuum. In particular, the class of first countable compacta does not have such universal objects. The fact that there is no injectively universal first countable space follows from an earlier result of Filippov [6] (see also [20]) who showed that there exist more separable perfectly normal compacta than closed separable subsets of a given first countable space.

To state the dual form of our result let us recall that a Banach space E

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(1) A space X is *retractive* if every closed subset of X is a retract of X.