[11] J. O. Strömberg and R. L. Wheeden, *Fractional integrals on weighted H^p and L^p spaces*, ibid. 287 (1985), 293–321.

DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907-1395
U.S.A.
E-mail: CRUZ@MATH.PURDUE.EDU
NEUG@MATH.PURDUE.EDU
OLESEN@MATH.PURDUE.EDU

Received December 14, 1994
Revised version May 31, 1995 (3389)

STUDIA MATHEMATICA 116 (3) (1995)

Weak Cauchy sequences in $L_\infty(\mu, X)$

by

GEORG SCHLÜCHTERMANN (München)

Abstract. For a finite and positive measure space (Ω, Σ, μ) characterizations of weak Cauchy sequences in $L_\infty(\mu, X)$, the space of μ-essentially bounded vector-valued functions $f : \Omega \to X$, are presented. The fine distinction between Asplund and conditionally weakly compact subsets of $L_\infty(\mu, X)$ is discussed.

1. Introduction and preliminaries. In his celebrated paper [Ta, Th. 1] M. Talagrand gave a parametric Rosenthal ℓ_1-dichotomy. With the help of this result conditionally weakly compact subsets of $L_p(\mu, X)$, $1 \le p < \infty$, the space of Bochner integrable functions, can be characterized. A characterization for $p = \infty$ has not been found yet. The relatively weakly compact subsets of $L_\infty(\mu, X)$ were considered in special cases by K. T. Andrews and J. J. Uhl [AU] and in general by the author [S3]. A basic tool in both papers is the celebrated factorization lemma of Davis, Figiel, Johnson and Pełczyński.

Here, in a modified version, this method will be applied to give a complete (i.e. for all Banach spaces X) characterization of conditionally weakly compact subsets and weak Cauchy sequences of $L_\infty(\mu, X)$. It is mainly based on a result on parametrizing operators $T : X \to L_1(\mu, Y)^*$ (see the definition below). In Section 3 a fine distinction between Asplund sets and conditionally weakly compact sets is sketched for $L_\infty(\mu, X)$. In the survey article of L. H. Riddle and J. J. Uhl [AU], this was given for arbitrary Banach spaces by means of topology, vector measures and geometry. Here, this will be illustrated in the particular situation of $L_\infty(\mu, X)$.

First we fix some notations and definitions which are used in the paper. X and Y denote Banach spaces; $B(X)$ resp. $S(X)$ is the unit ball resp. the unit sphere of the Banach space X. If not indicated otherwise, we consider a positive and finite measure space, which will be denoted by (Ω, Σ, μ). Then $L_p(\mu, X) := L_p(\Omega, \Sigma, \mu, X)$ for $1 \le p \le \infty$ is the usual Bochner space. $L_\infty(\mu, X^*, X)$ is the set of equivalence classes of w^*-measurable and

1991 *Mathematics Subject Classification*: 46E10, 46G10.

essentially bounded functions $f : \Omega \to X^*$, where a function $f : \Omega \to Y^*$ is called *w^*-measurable* if $\langle y, f(\cdot)\rangle$ is measurable for all $y \in Y$. If $U : \Omega \to L(X,Y)$ is such that for all $x \in X$, $U(\cdot)(x) \in L_\infty(\mu, Y)$ (resp. $U(\cdot)(x) \in L_\infty(\mu, Y^*, Y)$), then U is called *strongly measurable* (resp. *w^*-measurable*). Define $\overline{U} : X \to L_\infty(\mu, Y)$ (resp. $\overline{U} : X \to L_\infty(\mu, Y^*, Y)$) by $\overline{U}(x) = U(\cdot)(x), x \in X$. Then U is a *parametric version* for a bounded linear operator $T : X \to L_\infty(\mu, Y)$ (resp. for a $T : X \to L_\infty(\mu, Y^*, Y)$) if $\overline{U} = T$.

A bounded set in a Banach space is called *conditionally weakly compact* if every sequence admits a weak Cauchy subsequence. Rosenthal's dichotomy theorem says that this is equivalent to no sequence in the set being equivalent to the ℓ_1-basis.

A bounded linear operator $T : X \to Y$ is called *conditionally weakly compact* if $T(B(X))$ is conditionally weakly compact.

A convex and closed subset D of a Banach space X has the *complete continuity property* (*CCP*) if every bounded linear operator $V : L_1([0,1]) \to X$ such that $V(\{\chi_A/\lambda(A) : \lambda(A) > 0\}) \subset D$ is a Dunford–Pettis operator, i.e. V maps weak Cauchy sequences into norm Cauchy sequences.

Here are some facts about conditionally weakly compact operators which are used in the sequel:

(1.1) *Let $T : X \to Y$ be linear and bounded. Then the following conditions are equivalent*:

(a) *T is conditionally weakly compact.*

(b) *$T^*(B(Y^*))$ has the CCP.*

(c) *For all bounded linear $V : Y \to L_\infty([0,1])$ the composition $V \circ T$ maps bounded sequences into almost everywhere convergent subsequences.*

(d) *T factors through a space not containing a copy of ℓ_1.*

(a)⇔(b) is done in [Bo, p. 309, Th. 7.4.12], (a)⇔(c) is quoted from [RSU, p. 528, Th. 1] and (a)⇔(d) is in [Di, p. 237, 2(iii)].

A bounded set $K \subset X$ is an *Asplund set* if for each countable subset $A \subset K$ the dual $(\overline{\operatorname{span}} A)^*$ is norm separable. A bounded linear operator $T : X \to Y$ is called an *Asplund operator* if $T(B(X))$ is an Asplund set.

Next we formulate some facts about Asplund operators which are needed:

(1.2) *Let $T : X \to Y$ be linear and bounded. Then the following conditions are equivalent*:

(a) *T is an Asplund operator.*

(b) *$T^*(B(Y^*))$ is a set with the RNP.*

(c) *For each bounded linear $V : Y \to L_\infty([0,1])$ and each $\varepsilon > 0$ there is a set $E \subset [0,1]$ with $\lambda([0,1] \setminus E) < \varepsilon$ such that $(V \circ T)(B(X))\chi_E$ is relatively compact in $L_\infty([0,1])$.*

(d) *T factors through an Asplund space Z (i.e. where $B(Z)$ is an Asplund set).*

A reference for (a)⇔(b)⇔(d) is [Bo, p. 135, Th. 5.3.5, Th. 5.3.7], and [RU, p. 150, Th. A] for (a) ⇔(c).

2. Weak Cauchy sequences in $L_\infty(\mu, X)$.

We start with a dual version for the representation of operators $S : L_1(\mu, Y) \to X^*$, which is stated e.g. in [Din, p. 279, Th. 8] and which also occurs for special cases of X in [AU, p. 908, Lemma 3]. All measure spaces in this section are assumed to be separable. This is not an essential restriction, since the result can be generalized by the usual technique of a long sequence of expectation operators (see e.g. [S2]). The proof of the following lemma is omitted. A proof can be found in [S3].

LEMMA 2.1. *Let $T : X \to L_1(\mu, Y)^*$ be a bounded linear operator. Then there exists a w^*-measurable parametric version $U : \Omega \to L(X, Y^*)$ of T. In particular, if $T : X \to L_\infty(\mu, Y)$, then there exists a strongly measurable parametric version $U : \Omega \to L(X,Y)$ of T.*

The following proposition is a parametric version of the factorization of conditionally weakly compact operators (see (1.1)(b), (d); [S3, Prop. 2.2]).

PROPOSITION 2.2. *Let $T : X \to L_1(\mu, Y)^*$ be bounded and linear. Then T is conditionally weakly compact if and only if there exist a w^*-measurable parametric version $U : \Omega \to L(X, Y^*)$ of T and an absolutely convex, w^*-closed $K \subset X^*$ with the CCP such that $U(\omega)^*(B(Y^{**})) \subset K$ for a.a. $\omega \in \Omega$.*

Proof. ⇒: By (1.1)(d), T factors through a space R which does not contain a copy of ℓ_1, i.e. there exist bounded linear operators $T_1 : X \to R$ and $T_2 : R \to L_1(\mu, Y)^*$ such that $T = T_2 \circ T_1$. By Lemma 2.1 there exists an operator-valued map $\widetilde{U} : \Omega \to L(R, Y^*)$ such that $\overline{\widetilde{U}} = T_2$. Define $U(\omega) := \widetilde{U}(\omega) \circ T_1$ for $\omega \in \Omega$. Then $U : \Omega \to L(X, Y^*)$ has the property that $U(\cdot)(x)$ is w^*-measurable for all $x \in X$, since $\widetilde{U}$ enjoys this property and T_1 is a bounded operator. Evidently $\overline{U} = T$ and $U(\omega)^* = T_1^* \circ \widetilde{U}(\omega)^*$ for $\omega \in \Omega$. Assume now without loss of generality that $\|T_2\| \le 1$. Then $\|\widetilde{U}(\omega)^*\| \le 1$ for a.a. $\omega \in \Omega$. Define $K := T_1^*(B(R^*))$. Then for a.a. $\omega \in \Omega$,

$$U(\omega)^*(B(Y^{**})) \subset K$$

and K is absolutely convex, w^*-closed and has the CCP according to (1.1)(b).

⇐: Define

$$\widetilde{T} : L_1(\mu, Y) \to X^*, \quad f \mapsto \int_\Omega U(\omega)^*(f(\omega))\, d\mu(\omega).$$

Let K be as assumed. For each $y \in B(Y)$ the map $\Omega \ni \omega \mapsto U(\omega)^*(y)$ is w^*-measurable, since U is w^*-measurable. Hence, $\omega \mapsto U(\omega)^*(f(\omega))$ is w^*-measurable for all $f \in L_1(\mu, Y)$. Let $f \in B(L_1(\mu, Y))$, $f = \sum_{i=1}^n y_i \chi_{A_i}$. Then

$$\widetilde{T}(f) = \underbrace{\sum_{i=1}^{n} \underbrace{\int_{A_i} \underbrace{U(\omega)^*(y_i)}_{\in \|y_i\| K} d\mu(\omega)}_{\in \|y_i\| \mu(A_i) K}}_{\in \sum_{i=1}^n \|y_i\| \mu(A_i) K \subset K}.$$

It is easy to see that $T^*|_{L_1(\mu,Y)} = \widetilde{T}$. Hence, by the Goldstine argument (see [Di, p. 13]) and the w^*-w^*-continuity of $\widetilde{T}|_{B(L_1(\mu,Y))}$, $T^*(B(L_1(\mu,Y)^{**})) \subset K$ is absolutely convex, w^*-compact and has the CCP. By (1.1)(b), T is conditionally weakly compact. ∎

The $L_\infty(\mu, Y)$-version is an easy consequence of the preceding proposition.

COROLLARY 2.3. *Let $T : X \to L_\infty(\mu, Y)$ be bounded and linear. Then T is conditionally weakly compact if and only if there exist a strongly measurable version $U : \Omega \to L(X, Y)$ of T and an absolutely convex, w^*-closed subset $K \subset X^*$ with the CCP such that $U(\omega)^*(B(Y^*)) \subset K$ for a.a. $\omega \in \Omega$.*

Proof. $\Rightarrow$: Transform the proof of the preceding proposition literally, exchange Y^* resp. Y^{**} by Y resp. Y^*, and observe that the operator-valued map $\widetilde{U} : \Omega \to L(X, Y)$ used in the proof is strongly measurable according to Lemma 2.1.

$\Leftarrow$: Embed $L_\infty(\mu, Y)$ in $L_1(\mu, Y^*)^*$ and Y in Y^{**} isometrically. Then this direction follows directly from the above proposition. ∎

The next theorem will give a characterization of weak Cauchy sequences in $L_1(\mu, Y)^*$ and in $L_\infty(\mu, Y)$. It should be seen as an analogue to the characterization of weak zero sequences given in [S3]. For this, set

$$c := \{(x_n) \subset \mathbb{R} : \lim_{n\to\infty} x_n \text{ exists}\}.$$

THEOREM 2.4. (a) *Let $(f_n)_{n\in\mathbb{N}} \subset L_1(\mu, Y)^*$ be a bounded sequence. Then (f_n) is weak Cauchy if and only if there is a closed and absolutely convex set $K \subset c$ with the CCP and a set $N \in \Sigma$ with $\mu(N) = 0$ such that $(|\langle f_n(\omega), y^{**}\rangle|)_{n\in\mathbb{N}} \in K$ for all $\omega \in \Omega \setminus N$ and $y^{**} \in B(Y^{**})$* (see [S3, Cor. 2.3]).

(b) *Let $(f_n)_{n\in\mathbb{N}} \subset L_\infty(\mu, Y)$ be a bounded sequence. Then (f_n) is weak Cauchy if and only if there is a closed and absolutely convex set $K \subset c$ with the CCP and a set $N \in \Sigma$ with $\mu(N) = 0$ such that $(|\langle f_n(\omega), y^*\rangle|)_{n\in\mathbb{N}} \in K$ for all $\omega \in \Omega \setminus N$ and $y^* \in B(Y^*)$.*

Proof. (a) Define $T : \ell_1 \to L_1(\mu, Y)^*$ by $T(e_n) = f_n$. Then by (1.1)(b), (f_n) is weak Cauchy if and only if $T^* : L_1(\mu, Y)^{**} \to c$ and $T^*(B(L_1(\mu, Y)^*))$ has the CCP. By Proposition 2.2 this is equivalent to the existence of an operator-valued map $U : \Omega \to L(\ell_1, Y^*)$ such that $\overline{U} = T$, and of an absolutely convex, w^*-closed set $K \subset \ell_\infty$ with the CCP such that $U(\omega)^*(B(Y^{**})) \subset K$ for a.a. $\omega \in \Omega$. This last condition is successively equivalent to:

- $(|\langle e_n, U(\omega)^*(y^{**})\rangle|)_{n\in\mathbb{N}} \in K$ for a.a. $\omega \in \Omega$ and $y^{**} \in B(Y^{**})$,
- $(|\langle U(\omega)(e_n), y^{**}\rangle|)_{n\in\mathbb{N}} \in K$ for a.a. $\omega \in \Omega$ and $y^{**} \in B(Y^{**})$,
- $(|\langle f_n(\omega), y^{**}\rangle|)_{n\in\mathbb{N}} \in K$ for a.a. $\omega \in \Omega$ and $y^{**} \in B(Y^{**})$.

Since $(\langle f_n(\omega), y^{**}\rangle) \in c$ for all $y^{**} \in B(Y^{**})$ and a.a. $\omega \in \Omega$ the set K can be chosen in c.

(b) As in part (a) but use Corollary 2.3 instead of Proposition 2.2. ∎

For a further characterization of weak Cauchy sequences the following characterization of weak zero sequences in $L_\infty(\mu, X)$ is useful. It is an immediate consequence of [S3, Th. 2.7].

PROPOSITION 2.5. *Let $(f_n) \subset L_\infty(\mu, X)$ be bounded. Then the following conditions are equivalent:*

(a) *(f_n) is a weak zero sequence.*

(b) *There is a set $N \subset \Omega$ with $\mu(N) = 0$ such that:*

  (i) *For all $\omega \in \Omega \setminus N$, $(f_n(\omega))$ is weak zero.*

  (ii) *For all sequences $(x_j^*) \subset B(X^*)$ and $(\omega_j) \subset \Omega \setminus N$ there are subsequences $(x_{j_k}^*)$ and (ω_{j_k}) with $\lim_{n\to\infty} \lim_{k\to\infty} \langle f_n(\omega_{j_k}), x_{j_k}^*\rangle = 0$.*

The following corollary follows from the previous proposition.

COROLLARY 2.6. *Let $(f_n) \subset L_\infty(\mu, X)$ be bounded. Then the following conditions are equivalent:*

(a) *(f_n) is weak Cauchy.*

(b) *For each subsequence (f_{n_m}) there is a set $N \in \Sigma$ with $\mu(N) = 0$ such that all sequences $(x_j^*) \subset B(X^*)$ and $(\omega_j) \subset \Omega \setminus N$ admit subsequences $(x_{j_k}^*)$ and (ω_{j_k}) such that $\lim_{m\to\infty} \lim_{k\to\infty} \langle (f_{n_{m+1}} - f_{n_m})(\omega_{j_k}), x_{j_k}^*\rangle = 0$.*

The following theorem gives the final characterization of weak Cauchy sequences.

THEOREM 2.7. *Let $(f_n) \subset L_\infty(\mu, X)$ be bounded. Then the following conditions are equivalent:*

(a) *(f_n) is weak Cauchy.*

(b) *There is a set $N \in \Sigma$ with $\mu(N) = 0$ such that for all sequences $(x_j^*) \subset B(X^*)$ and $(\omega_j) \subset \Omega \setminus N$ there exist subsequences $(x_{j_k}^*)$ and (ω_{j_k}) such that* $\lim_{m,n\to\infty} \lim_{k\to\infty} \langle (f_m - f_n)(\omega_{j_k}), x_{j_k}^* \rangle = 0$.

Proof. (a)⇒(b). Define $T : \ell_1 \to L_\infty(\mu, X)$ by $T(e_n) := f_n$. Let $U : \Omega \to L(\ell_1, X)$ be an operator-valued version of T with $\sup_{\omega\in\Omega} \|U(\omega)\| = \sup_{n\in\mathbb{N}} \|f_n\|$. For $n \in \mathbb{N}$ define $\widetilde{f}_n : \Omega \to X$ by $\widetilde{f}_n(\omega) := U(\omega)(e_n)$. Then for all choices of representatives $(\bar{f}_n)$ of (f_n) we have

$$\exists N \in \Sigma,\ \mu(N) = 0\ \forall \omega \in \Omega \setminus N : \quad \widetilde{f}_n(\omega) = \bar{f}_n(\omega) \quad (n \in \mathbb{N}).$$

Let $(\omega_j) \subset \Omega \setminus N$ and $(x_j^*) \subset B(X^*)$ be given. As $(\bar{f}_n)$ are uniformly bounded, there exist subsequences (ω_{j_k}) and $(x_{j_k}^*)$ such that $\lim_{k\to\infty} \langle \bar{f}(\omega_{j_k}), x_{j_k}^* \rangle$ exists for all $n \in \mathbb{N}$. Define a functional $F : \operatorname{span}(f_n) \to \mathbb{R}$ by $F(f) := \lim_{k\to\infty} \langle \bar{f}(\omega_{j_k}), x_{j_k}^* \rangle$. Since the sequence $(x_{j_k}^*)$ is bounded and the limit exists by the above, F can be seen as an element of $(\overline{\operatorname{span}}(f_n))^*$. But this implies (b).

(b)⇒(a) follows by Corollary 2.6. ■

Remark 2.8. Let $K \subset L_\infty(\mu, X)$ be bounded and separable. Select a dense sequence $(f_n) \subset K$ and choose a corresponding sequence of representatives $(\bar{f}_n)$, $\bar{f}_n : \Omega \to X$. Define now $\mathcal{C}_K := \{(\overline{\omega}, \overline{x^*}) = (\omega_k, x_k^*) \in \Omega^{\mathbb{N}} \times B(X^*)^{\mathbb{N}} : \lim_{k\to\infty} \langle \bar{f}_n(\omega_k), x_k^* \rangle$ exists for all $n \in \mathbb{N}\}$. Then according to the proof above, for all $(\overline{\omega}, \overline{x^*})$, $F_{(\overline{\omega},\overline{x^*})}(f) := \lim_{k\to\infty} \langle \bar{f}(\omega_k), x_k^* \rangle$ defines an element of $(\overline{\operatorname{span}}\, K)^*$. By Proposition 2.5 and Theorem 2.7 a sequence $(f_n) \subset K$ is convergent to $f \in L_\infty(\mu, X)$ resp. is weak Cauchy if and only if $\lim_{n\to\infty} F_{(\overline{\omega},\overline{x^*})}(f - f_n) = 0$ resp. $\lim_{n,m\to\infty} F_{(\overline{\omega},\overline{x^*})}(f_n - f_m) = 0$.

One might suspect that weak Cauchy in $L_\infty(\mu, X)$ may imply a strong condition on the collective image set of the sequence such as the following one:

> Let $(f_n)_{n\in\mathbb{N}} \subset L_\infty(\mu, X)$ be weak Cauchy. Then for each $\varepsilon > 0$ there are sets $\Omega_\varepsilon \subset \Omega$ with $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and $L_\varepsilon \subset X$ conditionally weakly compact such that $\bigcup_{\omega\in\Omega_\varepsilon} (f_n(\omega))_{n\in\mathbb{N}} \subset L_\varepsilon$.

But this assumption fails, as Example 2.6 in [S3] demonstrates. Note that the Banach space X in this particular example is weakly sequentially complete. Thus the notions of weak Cauchy and weak convergence resp. conditionally weakly compact and weakly compact coincide.

In the following, some special conditionally weakly compact sets will be given. But first a simple (and well-known) example will demonstrate that a result similar to that for weak convergent sequences, namely that $(x_n) \subset X$ weak zero and $(f_n) \subset L_\infty(\mu)$ bounded imply $(f_n x_n)$ weak zero in $L_\infty(\mu, X)$, does not hold for weak Cauchy sequences.

EXAMPLE 2.9. Let $X := \mathbb{R}$, $f_n := r_n$ the Rademacher functions and $x_n := 1$. Then (f_n) is bounded and (x_n) is Cauchy but $(f_n x_n) = (r_n)$ is not conditionally weakly compact by Corollary 2.6 ((r_n) also spans a copy of ℓ_1 in $L_\infty([0,1])$ [Di, p. 223]).

Nevertheless a certain subclass of conditionally weakly compact sets can be characterized.

COROLLARY 2.10. (a) *Let $(f_n) \subset L_\infty(\mu)$ and $(x_n) \subset X$ be weak Cauchy. Then $(f_n x_n) \subset L_\infty(\mu, X)$ is weak Cauchy.*

(b) *Let $C \subset L_\infty(\mu)$ and $L \subset X$ be conditionally weakly compact. Then $K := C \otimes L := \{fx : f \in C,\ x \in L\} \subset L_\infty(\mu, X)$ is conditionally weakly compact.*

Proof. We use Theorem 2.7. Since $(f_n) \subset L_\infty(\mu)$ is weak Cauchy, it is bounded (assume without loss of generality $(f_n) \subset B(L_\infty(\mu))$), and there exists $N \in \Sigma$ with $\mu(N) = 0$ such that each sequence $(\omega_j) \subset \Omega \setminus N$ has a subsequence (ω_{j_k}) satisfying

$$\lim_{n,m\to\infty} \lim_{k\to\infty} |f_n(\omega_{j_k}) - f_m(\omega_{j_k})| = 0. \tag{1}$$

Let $\varepsilon > 0$, $(\omega_j) \subset \Omega \setminus N$ and $(x_j^*) \subset B(X^*)$ be given. Then there are subsequences (ω_{j_k}) and $(x_{j_k}^*)$ such that (1) is satisfied and there is an $x^* \in B(X^*)$ with

$$\forall n \in \mathbb{N} : \quad \lim_{k\to\infty} \langle x_n, x_{j_k}^* - x^* \rangle = 0. \tag{2}$$

Assume further without loss of generality that $(x_n) \subset B(X)$, since it is a weak Cauchy sequence. According to (1) and since (x_n) is weak Cauchy, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \ge n_0$,

$$\lim_{k\to\infty} |f_n(\omega_{j_k}) - f_m(\omega_{j_k})| < \varepsilon/4 \quad \text{and} \quad |\langle x_n - x_m, x^* \rangle| < \varepsilon/2.$$

Then for $n, m \ge n_0$,

$$\begin{aligned}
&\lim_{k\to\infty} |\langle f_n(\omega_{j_k}) x_n - f_m(\omega_{j_k}) x_m, x_{j_k}^* \rangle| \\
&\qquad \le \lim_{k\to\infty} |\langle f_n(\omega_{j_k}) x_n - f_n(\omega_{j_k}) x_m, x_{j_k}^* \rangle| + \varepsilon/2 \\
&\qquad = \lim_{k\to\infty} |\langle x_n - x_m, x_{j_k}^* \rangle| \cdot |f_n(\omega_{j_k})| + \varepsilon/2 \\
&\qquad \le \lim_{k\to\infty} |\langle x_n - x_m, x_{j_k}^* - x^* + x^* \rangle| + \varepsilon/2 \\
&\qquad \le_{(2)} \lim_{k\to\infty} |\langle x_n - x_m, x^* \rangle| + \varepsilon/2 < \varepsilon
\end{aligned}$$

by the above.

Part (b) follows directly from (a). ■

3. Asplund sets and conditionally weakly compact sets. In [RU], L. H. Riddle and J. J. Uhl discussed the difference between Asplund sets and conditionally weakly compact sets. Here a parametric version is given. Namely, it will be shown how the characterization of conditionally weakly compact subsets derived in Section 2 and a corresponding one for Asplund sets are separated. But first the operator results of Section 2 will be transferred to the Asplund property. The proofs will be omitted, since they run parallel to those of Proposition 2.2 and Corollary 2.3, upon replacing "conditionally weakly compact" by "Asplund", "CCP" by "RNP" and applying (1.2) instead of (1.1).

PROPOSITION 3.1. *Let $T : X \to L_1(\mu, Y)^*$ be bounded and linear. Then T is an Asplund operator if and only if there exist a w^*-measurable version $U : \Omega \to L(X, Y^*)$ of T and an absolutely convex and w^*-closed subset $K \subset X^*$ with the RNP such that $U(\omega)^*(B(Y^{**})) \subset K$ for a.a. $\omega \in \Omega$.*

COROLLARY 3.2. *Let $T : X \to L_\infty(\mu, Y)$ be bounded and linear. Then T is an Asplund operator if and only if there exist a strongly measurable parametric version $U : \Omega \to L(X, Y)$ of T and an absolutely convex, w^*-closed set $K \subset X^*$ with the RNP such that $U(\omega)^*(B(Y^*)) \subset K$ for a.a. $\omega \in \Omega$.*

The characterization in Corollary 2.3 of conditionally weakly compact sets is not very useful in practice. So, an equivalent version of 2.3 is presented, which gives the analogue to the characterization of the relatively weakly compact sets in [S3] and in [DRS]. The result follows directly from Theorem 2.7.

PROPOSITION 3.3. *For a bounded subset $K \subset L_\infty(\mu, X)$ the following conditions are equivalent*:

(a) *K is conditionally weakly compact.*

(b) *For each sequence $(f_n)_{n\in\mathbb{N}} \subset K$ there are a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ and a set $N \subset \Omega$ with $\mu(N) = 0$ such that*:

   (i) *For all $\omega \in \Omega \setminus N$, $(f_{n_m}(\omega))$ is weak Cauchy.*

   (ii) *For all sequences $(x_j^*) \subset B(X^*)$ and $(\omega_j) \subset \Omega \setminus N$ there are subsequences $(x_{j_k}^*)$ and (ω_{j_k}) with*

$$\lim_{l,m\to\infty} \lim_{k\to\infty} \langle (f_{n_m} - f_{n_l})(\omega_{j_k}), x_{j_k}^* \rangle = 0.$$

Remark 2.8 exhibits certain functionals which are characteristic for the determination of weak sequential convergence. Let $\mathcal{C}_K$ be defined as in 2.8. The next two theorems will demonstrate the "parametric" fine line between Asplund sets and conditionally weakly compact sets. But first an auxiliary result is needed.

LEMMA 3.4. *Let $K \subset L_\infty(\mu, X)$ be separable and $L_K := \overline{\operatorname{span}}\, K$.*

(a) *$\mathcal{F}_K := \{F_{(\overline{\omega}, \overline{x^*})} : (\overline{\omega}, \overline{x^*}) \in \mathcal{C}_K\}$ is w^*-compact in $B(L_K^*)$.*

(b) *There exists a finite Borel measure ν on $(\mathcal{F}_K, w^*)$ such that $L_\infty([0,1])$ and $L_\infty(\nu)$ are isometrically w^*-homeomorphic.*

Proof. (a) Let $(f_n) \subset K$ be dense. For simplicity let (f_n) denote the sequence of representatives. Further, let $(F^k) \subset \mathcal{F}_K$ be w^*-converging to $F \in B(L_K^*)$. Select $(\overline{\omega}^k, \overline{x^*}^k) = (w_j^k, x_j^{*k}) \in \mathcal{C}_K$ such that $F_{(\overline{\omega}^k, \overline{x^*}^k)} = F^k$. By definition,

$$\forall k \in \mathbb{N}\ \forall n \in \mathbb{N}\ \exists j_{k,n} \in \mathbb{N}\ \forall j \geq j_{k,n} : \quad |F^k(f_n) - \langle f_n(\omega_j^k), x_j^{*k} \rangle| < 1/k.$$

One may assume that

$$\forall k \in \mathbb{N}\ \forall n \in \mathbb{N} : \quad j_{k,n} \leq j_{k,n+1}.$$

Define $(\overline{\omega}, \overline{x^*}) := (\omega_l, x_l^*)$, where $\omega_l := \omega_{j_{l,l}}^l$ and $x_l^* := x_{j_{l,l}}^{*l}$.

CLAIM. *For all $n \in \mathbb{N}$, $\lim_{l\to\infty} \langle f_n(\omega_l), x_l^* \rangle = F(f_n)$.*

To prove this, let $\varepsilon > 0$ and $n \in \mathbb{N}$ be given. Choose $k_0 \in \mathbb{N}$ such that $1/k_0 < \varepsilon/2$ and

$$\forall k \geq k_0 : \quad |F(f_n) - F^k(f_n)| < \varepsilon/2. \tag{1}$$

For $k \geq k_0$ there are $j_{k,n} \in \mathbb{N}$ such that for all $j \geq j_{k,n}$,

$$|F^k(f_n) - \langle f_n(\omega_j^k), x_j^{*k} \rangle| < 1/k. \tag{2}$$

Let $k \geq n$. Then for $l = j_{k,k} \geq j_{k,n}$, by (2),

$$|F^k(f_n) - \langle f_n(\omega_l), x_l^* \rangle| < 1/k < \varepsilon/2.$$

Application of (1) gives the claim.

(b) According to (a), $\mathcal{F}_K$ is w^*-compact in $B(L_K^*)$. Hence, $(\mathcal{F}_K, w^*)$ is a compact metrizable space, since L_K is separable. Thus, by a result of Kuratowski's [Ku, p. 227] there is a regular Borel measure ν defined on $(\mathcal{F}_K, w^*)$ such that $([0,1], \mathcal{L}, \lambda)$ is measure isomorphic to $(\mathcal{F}_K, \mathcal{B}(w^*), \nu)$. Hence by a result of Carathéodory there is an isometry between $L_1([0,1])$ and $L_1(\nu)$ (see e.g. [La, p. 128, Cor.]). But this gives (b). ∎

In the following ν will denote the above regular Borel measure on $\mathcal{F}_K$.

THEOREM 3.5. *For a bounded separable subset $K \subset L_\infty(\mu, X)$ the following conditions are equivalent*:

(a) *K is an Asplund set.*

(b) *There is an increasing sequence $(\mathcal{F}_l)$ of w^*-compact subsets of $\mathcal{F}_K$ such that $\lim_{l\to\infty} \nu(\mathcal{F}_l) = 1$ and for each sequence $(f_n)_{n\in\mathbb{N}} \subset K$ there exists a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ such that*

$$\lim_{k,m\to\infty} \sup_{F\in\mathcal{F}_l} |F(f_{n_m} - f_{n_k})| = 0 \quad \textit{for all } l \in \mathbb{N}.$$

Proof. According to (1.2)(c), K is an Asplund set if and only if for all $T : L_K \to L_\infty([0,1])$ and for all $\varepsilon > 0$ there is a set $E \subset [0,1]$ with $\lambda([0,1] \setminus E) < \varepsilon$ so that $\chi_E \circ T$ is compact.

(a)⇒(b). Let K be a separable Asplund set. Consider

$$T : L_K \to L_\infty(\nu)\ (\cong L_\infty([0,1])), \qquad f \mapsto (F \mapsto F(f)).$$

By (1.2)(c) and since ν is a regular Borel measue on $(\mathcal{F}_K, w^*)$ we have

$$\forall l \in \mathbb{N}\ \exists \mathcal{F}_l \subset \mathcal{F}_K,\ \nu(\mathcal{F}_l) \geq 1 - 1/l : \qquad \chi_{\mathcal{F}_l} \circ T \text{ is compact on } K.$$

Since $(\mathcal{F}_l)$ may be chosen to be increasing, (b) follows immediately.

(b)⇒(a). Let $T : L_K \to L_\infty([0,1])$ be given. Then, according to Lemma 3.4(b), let $I : L_\infty([0,1]) \to L_\infty(\nu)$ be an isometry which is a w^*-w^*-homeomorphism. By (b) we conclude that

$$\exists\ (\mathcal{F}_l) \text{ increasing}, \lim_{l\to\infty} \nu(\mathcal{F}_l) = 1\ \forall\ l \in \mathbb{N} : \qquad \chi_{\mathcal{F}_l} \circ I \circ T \text{ is compact on } K.$$

The property of I in combination with (1.2)(c) shows immediately that K is an Asplund set. ■

THEOREM 3.6. *For a bounded separable subset $K \subset L_\infty(\mu, X)$ the following conditions are equivalent*:

(a) *K is conditionally weakly compact.*

(b) *For each sequence $(f_n)_{n\in\mathbb{N}} \subset K$ there exist a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ and an increasing sequence $(\mathcal{F}_l)$ of w^*-compact subsets of $\mathcal{F}_K$ such that $\lim_{l\to\infty} \nu(\mathcal{F}_l) = 1$ and*

$$\lim_{k,m\to\infty} \sup_{F\in\mathcal{F}_l} |F(f_{n_m} - f_{n_k})| = 0 \quad \textit{for all } l \in \mathbb{N}.$$

Proof. (a)⇒(b). Let K be conditionally weakly compact. Let T be defined as in the proof of Theorem 3.5(a)⇒(b). Then, by (1.1)(c), for each sequence $(f_n) \subset K$ there is a subsequence (f_{n_m}) so that $T(f_{n_m})$ converges pointwise. By Egorov's theorem and the regularity of ν, there is an $l \in \mathbb{N}$ and a w^*-compact set $\mathcal{F}_l$ with $\nu(\mathcal{F}_l) \geq 1 - 1/l$ so that $(\chi_{\mathcal{F}_l} \circ T)(f_{n_m})$ converges uniformly. Since $(\mathcal{F}_l)$ may be chosen to be increasing, (b) follows.

(b)⇒(a). Use (1.1)(c). For this purpose suppose $T : L_K \to L_\infty([0,1])$. Let $(f_n) \subset K$ be a sequence and I be the isometry between $L_\infty([0,1])$ and $L_\infty(\nu)$ (see the proof of Theorem 3.5 (b)⇒(a)). Then according to (b) there is a subsequence (f_{n_m}) such that $(I \circ T)(f_{n_m})$ converges a.e. The property of I and the regularity of ν imply the a.e. convergence of $T(f_{n_m})$ and (a) is proved. ■

Remark 3.7. (a) It should be mentioned that Corollary 2.6, Theorem 2.7 and the results of Section 3 can be formulated for $L_1(\mu, X)^*$, where the functions $f \in L_1(\mu, X^*, X)$ depend heavily on the parametrization formulated e.g. in Lemma 2.1.

(b) In addition, all the results can be extended to some measure spaces which are not necessarily finite. The key Lemma 2.1, which stated the parametrization, is true as long as (Ω, Σ, μ) is localizable (see e.g. [Din] for the definition of *localizable* and [S1] for the technique to extend the result).

References

[AU] K. T. Andrews and J. J. Uhl, *On weak compactness in $L_\infty(\mu, X)$*, Indiana Univ. Math. J. 30 (1981), 907–915.

[BH] J. Batt and W. Hiermeyer, *On compactness in $L_p(\mu, X)$ in the weak topology and in the topology $\sigma(L_p(\mu, X), L_q(\mu, X^*))$*, Math. Z. 182 (1983), 409–432.

[Bo] R. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property*, Lecture Notes in Math. 993, Springer, 1983.

[Di] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, Berlin, 1984.

[Din] N. Dinculeanu, *Vector Measures*, Deutscher Verlag Wiss., Berlin, 1966.

[DRS] J. Diestel, W. Ruess and W. Schachermayer, *On weak compactness in $L_1(\mu, X)$*, Proc. Amer. Math. Soc. 118 (1993), 447–453.

[DU] J. Diestel and J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.

[Ku] C. Kuratowski, *Topologie I*, Monograf. Mat., Warszawa, 1933.

[La] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer, Berlin, 1974.

[RSU] L. H. Riddle, E. Saab and J. J. Uhl, *Sets with the weak Radon–Nikodým property in dual Banach spaces*, Indiana Univ. Math. J. 32 (1983), 527–541.

[RU] L. H. Riddle and J. J. Uhl, *Martingales and the fine line between Asplund spaces and spaces not containing a copy of ℓ_1*, in: Martingale Theory in Harmonic Analysis and Banach Spaces, Lecture Notes in Math. 939, Springer, 1983, 145–156.

[S1] G. Schlüchtermann, *Renorming in the space of Bochner integrable functions $L_1(\mu, X)$*, Manuscripta Math. 73 (1991), 397–409.

[S2] —, *On weakly compact operators*, Math. Ann. 292 (1992), 263–266.

[S3] —, *Weak compactness in $L_\infty(\mu, X)$*, J. Funct. Anal. 125, (1994), 379–388.

[Ta] M. Talagrand, *Weak Cauchy sequences in $L^1(E)$*, Amer. J. Math. 106 (1984), 703–724.

MATHEMATISCHES INSTITUT DER
LUDWIG-MAXIMILIANS-UNIVERSITÄT
THERESIENSTRASSE 39
80333 MÜNCHEN, GERMANY
E-mail: SCHLUECH@RZ.MATHEMATIK.UNI-MUENCHEN.DE

Received December 20, 1994
Revised version June 29, 1995 (3395)