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On the diophantine equation $D_1x^4 - D_2y^2 = 1$

by

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ be the sets of integers, positive integers, rational numbers and real numbers respectively. Let $D_1, D_2 \in \mathbb{N}$ with $\gcd(D_1, D_2) = 1$. There were many papers concerned with the equation

$$D_1x^4 - D_2y^2 = 1, \quad x, y \in \mathbb{N}, \tag{1}$$

written by Ljunggren, Bumby, Cohn, Ke and Sun. Concerning the solvability of (1), Zhu [7] and Le [2] proved independently that if $D_1 = 1$, then (1) has solutions (x, y) if and only if the fundamental solution $u_1 + v_1\sqrt{D_2}$ of Pell's equation

$$u^2 - D_2v^2 = 1, \quad u, v \in \mathbb{Z},$$

satisfies either $u_1 = x_1^2$ or $u_1^2 + D_2v_1^2 = x_1^2$, where $x_1 \in \mathbb{N}$. In addition, Zhu [7] showed that if $D_2 = 1$, then (1) has solutions (x, y) if and only if the equation

$$u'^2 - D_1v'^2 = -1, \quad u', v' \in \mathbb{Z},$$

has solutions (u', v') and its least positive integer solution (u_1', v_1') satisfies $v_1' = x_1^2$, where $x_1 \in \mathbb{N}$. In this paper we prove a general result as follows.

THEOREM 1. *If* $\min(D_1, D_2) > 1$, *then* (1) *has solutions* (x, y) *if and only if the equation*

$$D_1U^2 - D_2V^2 = 1, \quad U, V \in \mathbb{Z}, \tag{2}$$

has solutions (U, V) *and its least positive integer solution* (U_1, V_1) *satisfies* $U_1 = x_1^2$, *where* $x_1 \in \mathbb{N}$.

Let $N(D_1, D_2)$ denote the number of solutions (x, y) of (1). Ljunggren [4] showed that $N(1, D_2) \le 2$. In [3], Le proved that if $D_2 > e^{64}$, then $N(1, D_2) \le 1$. Recently, Wu [6] relaxed the condition $D_2 > e^{64}$ to $D_2 > e^{37}$. In this paper we prove the following result.

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THEOREM 2. *If* D_1 *is a square and*

$$\max(D_1, D_2) \geq \begin{cases} 9.379 \cdot 10^8 & \text{if } \min(D_1, D_2) = 1, \\ 2.374 \cdot 10^{10} & \text{if } \min(D_1, D_2) > 1, \end{cases}$$

then $N(D_1, D_2) \leq 1$.

2. Preliminaries

LEMMA 1. *For any odd prime* p *with* $p \equiv 1 \pmod 4$, *there exists* $a_1 \in \mathbb{N}$ *such that* $p > a_1 > 1$, $2 \nmid a_1$ *and* $(a_1/p) = -1$, *where* (a_1/p) *is Legendre's symbol.*

Proof. It is a well known fact that there exists $a \in \mathbb{N}$ with $p > a > 1$ and $(a/p) = -1$. Since $(a/p) = ((p-a)/p)$ for $p \equiv 1 \pmod 4$, we get

$$a_1 = \begin{cases} a & \text{if } 2 \nmid a, \\ p - a & \text{if } 2 \mid a. \end{cases}$$

The lemma is proved.

LEMMA 2 ([3, Lemma 3]). *Let* $d \in \mathbb{N}$ *be square-free. If* (u, v) *and* (u', v') *are solutions of the equation*

$$u^2 - dv^2 = 1, \quad u, v \in \mathbb{N}, \tag{3}$$

with $u' \equiv 0 \pmod u$, *then there exist fixed* $d_1, d_2 \in \mathbb{N}$ *such that*

$$\begin{gathered} d_1 d_2 = d, \quad u + 1 = \delta d_1 v_1^2, \quad u - 1 = \delta d_2 v_2^2, \\ u' + 1 = \delta d_1 v_1'^2, \quad u' - 1 = \delta d_2 v_2'^2, \end{gathered} \tag{4}$$

where $\delta, v_1, v_2, v_1', v_2' \in \mathbb{N}$ *satisfy*

$$\delta v_1 v_2 = v, \quad \delta v_1' v_2' = v', \quad \delta = \begin{cases} 1 & \text{if } 2 \nmid v, \\ 2 & \text{if } 2 \mid v. \end{cases} \tag{5}$$

LEMMA 3 ([5]). *For* $\min(D_1, D_2) > 1$, *if* (2) *has solutions* (U, V), *then it has a unique positive integer solution* (U_1, V_1) *such that* $U_1\sqrt{D_1} + V_1\sqrt{D_2} \leq U\sqrt{D_1} + V\sqrt{D_2}$ *for all positive integer solutions* (U, V) *of* (2). (U_1, V_1) *is called the least solution of* (2). *Moreover, all positive integer solutions* (U, V) *of* (2) *are given by*

$$U\sqrt{D_1} + V\sqrt{D_2} = (U_1\sqrt{D_1} + V_1\sqrt{D_2})^t, \quad t \in \mathbb{N}, \ 2 \nmid t.$$

LEMMA 4. *For* $\min(D_1, D_2) > 1$, *let* (U, V) *be a solution of* (2), *and let*

$$\varepsilon = U\sqrt{D_1} + V\sqrt{D_2}, \quad \overline{\varepsilon} = U\sqrt{D_1} - V\sqrt{D_2}. \tag{6}$$

Further, for any $m \in \mathbb{Z}$ *with* $2 \nmid m$, *let*

$$E(m) = \frac{\varepsilon^m + \overline{\varepsilon}^m}{\varepsilon + \overline{\varepsilon}}. \tag{7}$$

Then $E(m) \in \mathbb{N}$ *and*:

(i) $E(m) = E(-m)$.

(ii) $E(m) \equiv 1 \pmod 4$, $E(m) \equiv (-1)^{(m-1)/2} m \pmod U$.

(iii) *For any* $m, m' \in \mathbb{Z}$ *with* $2 \nmid mm'$, $E(m) \equiv -E(m-2m') \pmod{E(m')}$.

(iv) *For any* $m, m' \in \mathbb{Z}$ *with* $2 \nmid mm'$ *and* $\gcd(m, m') = 1$, $(E(m)/E(m')) = 1$, *where* $(*/*)$ *is the Jacobi symbol.*

P r o o f. Since $\varepsilon\overline{\varepsilon} = 1$ by (2) and (6), we get (i) by (7). Since we have $E(m+4)+E(m) = (\varepsilon^2+\overline{\varepsilon}^2)E(m+2) \equiv -2E(m+2) \pmod{4U}$, (ii) follows by induction on m in view of the fact that $E(-1) = E(1) = 1$.

Notice that

$$E(m) + E(m - 2m') = (\varepsilon^{m-m'} + \overline{\varepsilon}^{m-m'})E(m'), \tag{8}$$

where $\varepsilon^{m-m'} + \overline{\varepsilon}^{m-m'} \in \mathbb{Z}$, since $m - m'$ is even. So we have (iii). Moreover, using (8) and (ii), we obtain (iv) by induction. The lemma is proved.

By much the same argument as in the proof of Lemma 4, we can prove the following lemma.

LEMMA 5. *For* $\min(D_1, D_2) > 1$, *let* $U, V, \varepsilon, \overline{\varepsilon}$ *be defined as in Lemma* 4, *and let*

$$F(m) = \frac{\varepsilon^m - \overline{\varepsilon}^m}{\varepsilon - \overline{\varepsilon}}$$

for any $m \in \mathbb{Z}$ *with* $2 \nmid m$. *Then* $F(m) \in \mathbb{Z}$ *and*:

(i) $F(m) = -F(-m)$ *and* $F(m) > 0$ *if* $m > 0$.

(ii) *For any* $m, m' \in \mathbb{Z}$ *with* $2 \nmid mm'$, $F(m) \equiv F(m-2m') \pmod{F(m)}$.

LEMMA 6. *If* (U, V) *and* (U', V') *are positive integer solutions of* (2) *satisfying* $U' \equiv 0 \pmod U$ *or* $V' \equiv 0 \pmod V$, *then there exists* $t' \in \mathbb{N}$ *such that*

$$U'\sqrt{D_1} + V'\sqrt{D_2} = (U\sqrt{D_1} + V\sqrt{D_2})^{t'}, \quad 2 \nmid t'. \tag{9}$$

P r o o f. Let $\varepsilon = U_1\sqrt{D_1} + V_1\sqrt{D_2}$ and $\overline{\varepsilon} = U_1\sqrt{D_1} - V_1\sqrt{D_2}$, where (U_1, V_1) is the least solution of (2). By Lemma 3, there exist $m, m' \in \mathbb{N}$ such that

$$U\sqrt{D_1} + V\sqrt{D_2} = \varepsilon^m, \quad U'\sqrt{D_1} + V'\sqrt{D_2} = \varepsilon^{m'}, \quad 2 \nmid mm'. \tag{10}$$

Then we have $U = U_1E(m)$, $V = V_1F(m)$, $U' = U_1E(m')$ and $V' = V_1F(m')$.

By Lemma 4(iii), if $U' \equiv 0 \pmod U$, then we have

$$0 \equiv U' = U_1E(m') \equiv -U_1E(m' - 2m) \equiv \ldots \equiv \pm U_1E(s) \pmod U, \tag{11}$$

where $s \in \mathbb{Z}$ satisfies $2 \nmid s$ and $-m < s \le m$. Since $0 < E(s) < E(m)$ if $|s| < m$, we obtain $s = m$ by (11). Therefore, $m \mid m'$ and (9) holds by (10).

The proof in the case $V' \equiv 0 \pmod V$ is analogous. By Lemma 5(ii), we then have

$$0 \equiv V' = V_1 F(m') \equiv V_1 F(m'-2m) \equiv \ldots \equiv V_1 F(s) \pmod V,$$

where $s \in \mathbb{Z}$ satisfies $2 \nmid s$ and $-m < s \le m$. Since $0 < |F(s)| < F(m)$ if $|s| < m$, we get $s = m$, and hence, $m \mid m'$. Thus (9) holds in this case. The lemma is proved.

Let α be an algebraic number with the minimal polynomial

$$a_0 z^d + \ldots + a_d = a_0 \prod_{i=1}^{d} (z - \sigma_i \alpha), \qquad a_0 > 0,$$

where $\sigma_1\alpha, \ldots, \sigma_d\alpha$ are all conjugates of α. Then

$$h(\alpha) = \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d} \log\max(1, |\sigma_i\alpha|)\Big)$$

is called the *logarithmic absolute height* of α.

Lemma 7 ([1, Corollary 2]). *Let α_1, α_2 be real algebraic numbers with $\alpha_1 > 1$ and $\alpha_2 > 1$ which are multiplicatively independent, and let $\log A_j \ge \max(h(\alpha_j), |\log\alpha_j|/r, 1/r)$ for $j = 1, 2$, where $r = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$. If $\Lambda = b_1 \log\alpha_1 - b_2 \log\alpha_2 \ne 0$ for some $b_1, b_2 \in \mathbb{N}$, then*

$$|\Lambda| \ge \exp(-24.34 r^4 (\log A_1)(\log A_2)(\max(\log b' + 0.69, 21/r, 1/2))^2),$$

where $b' = b_1/(r\log A_2) + b_2/(r\log A_1)$.

3. Proof of Theorem 1. The sufficiency of the theorem is clear; it suffices to prove the necessity. Now we assume that (1) has solutions (x, y). Then (1) has a unique solution (x_1, y_1) such that

$$x_1^2\sqrt{D_1} + y_1\sqrt{D_2} \le x^2\sqrt{D_1} + y\sqrt{D_2} \tag{12}$$

for all solutions (x, y) of (1). Clearly, (x_1^2, y_1) is a positive integer solution of (2). Let (U_1, V_1) be the least solution of (2). By Lemma 3, we have

$$x_1^2\sqrt{D_1} + y_1\sqrt{D_2} = (U_1\sqrt{D_1} + V_1\sqrt{D_2})^t, \qquad t \in \mathbb{N},\ 2 \nmid t. \tag{13}$$

If $t = 1$, then the theorem is proved. Otherwise, t has an odd prime factor p. By Lemma 3, (2) has a positive integer solution (U, V) which satisfies

$$U\sqrt{D_1} + V\sqrt{D_2} = (U_1\sqrt{D_1} + V_1\sqrt{D_2})^{t/p}. \tag{14}$$

From (13) and (14), we get

$$x_1^2\sqrt{D_1} + y_1\sqrt{D_2} = (U\sqrt{D_1} + V\sqrt{D_2})^p. \tag{15}$$

For any $m \in \mathbb{Z}$ with $2 \nmid m$, let ε, $\overline{\varepsilon}$ and $E(m)$ be defined as in (6) and (7), respectively. From (15) we get

$$x_1^2 = \frac{\varepsilon^p + \overline{\varepsilon}^p}{2\sqrt{D_1}} = UE(p). \tag{16}$$

By Lemma 4(ii), $E(p) \in \mathbb{N}$ with $E(p) \equiv (-1)^{(p-1)/2}p \pmod{U}$. This implies that $\gcd(U, E(p)) = 1$ or p.

If $\gcd(U, E(p)) = 1$, then from (16) we get $U = x_{11}^2$ and $E(p) = x_{12}^2$, where $x_{11}, x_{12} \in \mathbb{N}$ with $x_{11}x_{12} = x_1$. It follows that (x_{11}, V) is a solution of (1) satisfying

$$x_{11}^2\sqrt{D_1} + V\sqrt{D_2} = U\sqrt{D_1} + V\sqrt{D_2} = \varepsilon < \varepsilon^p = x_1^2\sqrt{D_1} + y_1\sqrt{D_2},$$

which contradicts (12).

If $\gcd(U, E(p)) = p$, then we have

$$U = px_{11}^2, \qquad E(p) = px_{12}^2, \tag{17}$$

where $x_{11}, x_{12} \in \mathbb{N}$ with $px_{11}x_{12} = x_1$. Since $E(p) \equiv 1 \pmod 4$ by Lemma 4(ii), we see from (17) that $p \equiv 1 \pmod 4$. Therefore, by Lemma 1, there exists $a_1 \in \mathbb{N}$ such that $2 \nmid a_1$, $p > a_1 > 1$ and $(a_1/p) = -1$. Further, since $p \mid U$, by Lemma 4(ii), we get $E(a_1) \equiv (-1)^{(a_1-1)/2}a_1 \pmod{p}$. So we have

$$\begin{aligned}\left(\frac{E(p)}{E(a_1)}\right) &= \left(\frac{px_{12}^2}{E(a_1)}\right) = \left(\frac{p}{E(a_1)}\right) = \left(\frac{E(a_1)}{p}\right) \\ &= \left(\frac{(-1)^{(a_1-1)/2}a_1}{p}\right) = \left(\frac{a_1}{p}\right) = -1,\end{aligned} \tag{18}$$

by (17). However, by Lemma 4(iv), (18) is impossible. The theorem is proved.

4. Proof of Theorem 2. First we consider the case where $\min(D_1, D_2) > 1$. By Theorem 1, if (1) has solutions (x, y), then $(x_1, y_1) = (\sqrt{U_1}, V_1)$ is a solution of (1), where (U_1, V_1) is the least solution of (2). Further, by Lemma 3, if $N(D_1, D_2) > 1$, then (1) has another solution (x_2, y_2) which satisfies $x_2 > x_1$ and

$$x_2 \equiv 0 \pmod{x_1}. \tag{19}$$

Since D_1 is a square, D_2 cannot be such, therefore we may also assume, without loss of generality, that D_2 is square-free. Let $D_1 = a^2$, where $a \in \mathbb{N}$ with $a > 1$. Then (ax_1^2, y_1) and (ax_2^2, y_2) are solutions of the equation

$$u^2 - D_2v^2 = 1, \qquad u, v \in \mathbb{N}.$$

Notice that $ax_2^2 \equiv 0 \pmod{ax_1^2}$ by (19). We have

$$ax_1^2 + 1 = \delta D_{21}y_{11}^2, \quad ax_1^2 - 1 = \delta D_{22}y_{12}^2, \tag{20}$$

$$ax_2^2 + 1 = \delta D_{21}y_{21}^2, \quad ax_2^2 - 1 = \delta D_{22}y_{22}^2, \tag{21}$$

by Lemma 2, where $\delta, D_{21}, D_{22}, y_{11}, y_{12}, y_{21}, y_{22} \in \mathbb{N}$ satisfy

$$D_{21}D_{22} = D_2, \quad \gcd(D_{21}, D_{22}) = 1, \tag{22}$$

$$\delta y_{11}y_{12} = y_1, \quad \delta y_{21}y_{22} = y_2, \quad \delta = \begin{cases} 1 & \text{if } 2 \nmid y_1, \\ 2 & \text{if } 2 \mid y_1. \end{cases} \tag{23}$$

We see from (20) and (21) that (y_{11}, x_1) and (y_{21}, x_2) are solutions of the equation

$$\delta D_{21}X^2 - aY^2 = 1, \quad X, Y \in \mathbb{N},$$

while (x_1, y_{12}) and (x_2, y_{22}) are solutions of the equation

$$aX'^2 - \delta D_{22}Y'^2 = 1, \quad X', Y' \in \mathbb{N}.$$

Let

$$\varepsilon_1 = x_1\sqrt{a} + y_{11}\sqrt{\delta D_{21}}, \quad \overline{\varepsilon}_1 = x_1\sqrt{a} - y_{11}\sqrt{\delta D_{21}}, \tag{24}$$

$$\varepsilon_2 = x_1\sqrt{a} + y_{12}\sqrt{\delta D_{22}}, \quad \overline{\varepsilon}_2 = x_1\sqrt{a} - y_{12}\sqrt{\delta D_{22}}. \tag{25}$$

Recall that $x_2 \equiv 0 \pmod{x_1}$ by (19). Using Lemma 6, we have

$$x_2\sqrt{a} + y_{21}\sqrt{\delta D_{21}} = \varepsilon_1^{t_1}, \quad x_2\sqrt{a} - y_{21}\sqrt{\delta D_{21}} = \overline{\varepsilon}_1^{t_1}, \tag{26}$$

$$x_2\sqrt{a} + y_{22}\sqrt{\delta D_{22}} = \varepsilon_2^{t_2}, \quad x_2\sqrt{a} - y_{22}\sqrt{\delta D_{22}} = \overline{\varepsilon}_2^{t_2}, \tag{27}$$

where $t_1, t_2 \in \mathbb{N}$ satisfy $t_1 > 1$, $t_2 > 1$ and $2 \nmid t_1t_2$. From (24)–(27), we obtain

$$\varepsilon_1 + \overline{\varepsilon}_1 = \varepsilon_2 + \overline{\varepsilon}_2, \tag{28}$$

$$\varepsilon_1^{t_1} + \overline{\varepsilon}_1^{t_1} = \varepsilon_2^{t_2} + \overline{\varepsilon}_2^{t_2}. \tag{29}$$

Let $\Delta = \overline{\varepsilon}_2 - \overline{\varepsilon}_1$ and $\Delta' = \overline{\varepsilon}_2^{t_2} - \overline{\varepsilon}_1^{t_1}$. Since $\varepsilon_1\overline{\varepsilon}_1 = -1$ and $\varepsilon_2\overline{\varepsilon}_2 = 1$, from (28) and (29) we get

$$\begin{aligned} \log\varepsilon_1 &= \log\varepsilon_2 + \frac{2\Delta}{\varepsilon_1+\varepsilon_2}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{\Delta}{\varepsilon_1+\varepsilon_2}\right)^{2i} \\ &= \log\varepsilon_2 + \frac{2}{\varepsilon_1\varepsilon_2}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{1}{\varepsilon_1\varepsilon_2}\right)^{2i} \\ &= \log\varepsilon_2 + \frac{2}{\varepsilon_2^2}\left(\frac{\varepsilon_2}{\varepsilon_1}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{1}{\varepsilon_1\varepsilon_2}\right)^{2i}\right) \\ &= \log\varepsilon_2 + \frac{2}{\varepsilon_2^2}\left(\frac{1}{1+1/(\varepsilon_1\varepsilon_2)+1/\varepsilon_2^2}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{1}{\varepsilon_1\varepsilon_2}\right)^{2i}\right) \\ &< \log\varepsilon_2 + \frac{2}{\varepsilon_2^2}, \end{aligned} \tag{30}$$

$$t_1\log\varepsilon_1 = t_2\log\varepsilon_2 + \frac{2\Delta'}{\varepsilon_1^{t_1}+\varepsilon_2^{t_2}}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{\Delta'}{\varepsilon_1^{t_1}+\varepsilon_2^{t_2}}\right)^{2i} \tag{31}$$

$$\begin{aligned}
&= t_2 \log \varepsilon_2 + \frac{2}{\varepsilon_1^{t_1}\varepsilon_2^{t_2}} \sum_{i=0}^{\infty} \frac{1}{2i+1}\left(\frac{1}{\varepsilon_1^{t_1}\varepsilon_2^{t_2}}\right)^{2i} \\
&= t_2 \log \varepsilon_2 + \frac{2}{\varepsilon_2^{2t_2}}\left(\frac{\varepsilon_2^{t_2}}{\varepsilon_1^{t_1}} \sum_{i=0}^{\infty} \frac{1}{2i+1}\left(\frac{1}{\varepsilon_1^{t_1}\varepsilon_2^{t_2}}\right)^{2i}\right) \\
&= t_2 \log \varepsilon_2 + \frac{2}{\varepsilon_2^{2t_2}}\left(\frac{1}{1+1/(\varepsilon_1^{t_1}\varepsilon_2^{t_2})+1/\varepsilon_2^{2t_2}} \sum_{i=0}^{\infty} \frac{1}{2i+1}\left(\frac{1}{\varepsilon_1^{t_1}\varepsilon_2^{t_2}}\right)^{2i}\right) \\
&< t_2 \log \varepsilon_2 + \frac{2}{\varepsilon_2^{2t_2}},
\end{aligned}$$

respectively. By (30) and (31), we get $\log \varepsilon_1 - \log \varepsilon_2 > t_1 \log \varepsilon_1 - t_2 \log \varepsilon_2 > 0$. This implies that $(t_2-1)\log \varepsilon_2 > (t_1-1)\log \varepsilon_1$. Since $\varepsilon_1 > \varepsilon_2 > 1$ by (28), we obtain $t_2 > t_1$. Since $2 \nmid t_1t_2$, we get

$$t_2 \geq t_1 + 2. \tag{32}$$

Therefore, we find from (30)–(32) that

$$t_2 > \frac{(t_2-t_1)\log \varepsilon_1}{\log \varepsilon_1 - \log \varepsilon_2} > \varepsilon_2^2 \log \varepsilon_1 > \varepsilon_1^2 (\log \varepsilon_1) e^{-4/\varepsilon_2^2}. \tag{33}$$

Let $K_1 = \mathbb{Q}(\sqrt{\delta D_{21} a})$ and $K_2 = \mathbb{Q}(\sqrt{\delta D_{22} a})$. Since D_2 is not a square, we see from (22) that $K_1 \setminus \mathbb{Q} \cap K_2 \setminus \mathbb{Q} = \emptyset$. If there exist $k_1, k_2 \in \mathbb{Q}$ such that $\varepsilon_1^{k_1}\varepsilon_2^{k_2} = 1$, then $\varepsilon_1^{m_1}\varepsilon_2^{m_2} = 1$ for some $m_1, m_2 \in \mathbb{Z}$ with $2 \mid m_1$ and $2 \mid m_2$. Notice that $\varepsilon_1^m \in K_1 \setminus \mathbb{Q}$ and $\varepsilon_2^m \in K_2 \setminus \mathbb{Q}$ for any $m \in \mathbb{Z} \setminus \{0\}$ with $2 \mid m$. We get $m_1 = m_2 = 0$ and $k_1 = k_2 = 0$. This implies that ε_1 and ε_2 are multiplicatively independent.

Let $h(\varepsilon_1), h(\varepsilon_2)$ denote the logarithmic absolute heights of ε_1, ε_2 respectively, and let r denote the degree of $\mathbb{Q}(\varepsilon_1, \varepsilon_2)$. Then

$$4 \leq r \leq 8, \quad h(\varepsilon_1) = \frac{\log \varepsilon_1}{r/2}, \quad h(\varepsilon_2) = \frac{\log \varepsilon_2}{r/2}. \tag{34}$$

Further, let $\Lambda = t_1 \log \varepsilon_1 - t_2 \log \varepsilon_2$ and

$$t = \frac{t_1}{2\log \varepsilon_2} + \frac{t_2}{2 \log \varepsilon_1}. \tag{35}$$

Then we have

$$t = \frac{t_2}{\log \varepsilon_1} + \frac{\Lambda}{2(\log \varepsilon_1)(\log \varepsilon_2)} < \frac{t_2}{\log \varepsilon_1}\left(1 + \frac{1}{t_2 \varepsilon_2^{2t_2} \log \varepsilon_2}\right), \tag{36}$$

by (31). Using Lemma 7, from (34) we get

$$\Lambda \geq \exp(-6232(\log \varepsilon_1)(\log \varepsilon_2)(\max(\log t + 0.69, 21/4))^2). \tag{37}$$

We now suppose that $\varepsilon_1 \geq 785$. By (28), $\varepsilon_1^2 - (\varepsilon_2 + \overline{\varepsilon}_2)\varepsilon_1 - 1 = 0$. Since $\varepsilon_2 \geq 1+\sqrt{2}$, we get $\varepsilon_1 < 1.745\varepsilon_2$, whence $\varepsilon_2 > 449.856$. If $\log t + 0.69 \leq 21/4$, then $t < 96$ and

$$t_2 < 96 \log \varepsilon_1, \tag{38}$$

by (35). The combination of (33) and (38) yields $96 > \varepsilon_1^2 e^{-4/\varepsilon_2^2} > 6 \cdot 10^5$, a contradiction. Hence, $\log t + 0.69 > 21/4$, and

$$\Lambda \geq \exp(-6232(\log \varepsilon_1)(\log \varepsilon_2)(\log t + 0.69)^2), \tag{39}$$

by (37). The combination of (31) and (39) yields

$$\log 2 + 6232(\log \varepsilon_1)(\log \varepsilon_2)(\log t + 0.69)^2 > 2t_2 \log \varepsilon_2. \tag{40}$$

Further, by (31), (36) and (40), we get

$$\begin{aligned} 1 + 3116&\left(\log\left(\frac{t_2}{\log \varepsilon_1}\right) + 0.7\right)^2 \\ &> \frac{\log 2}{\log \varepsilon_2} + 3116\left(\log\left(\frac{t_2}{\log \varepsilon_1} + \frac{\Lambda}{2(\log \varepsilon_1)(\log \varepsilon_2)}\right) + 0.69\right)^2 > \frac{t_2}{\log \varepsilon_1}, \end{aligned}$$

whence we conclude that

$$\frac{t_2}{\log \varepsilon_1} < 615000. \tag{41}$$

Therefore, from (33) and (41) we get $616000 < \varepsilon_1^2 e^{-4/\varepsilon_2^2} < 615000$, a contradiction. So we have

$$\varepsilon_1 < 785. \tag{42}$$

From (22), (24), (25) and (28), we get

$$\varepsilon_1 = \varepsilon_2 - \overline{\varepsilon}_1 + \overline{\varepsilon}_2 > \varepsilon_2 + \overline{\varepsilon}_2 = 2x_1\sqrt{a} \geq 2\sqrt{a} = 2D_1^{1/4} \tag{43}$$

and

$$\begin{aligned} \varepsilon_1^2 &> \varepsilon_1\varepsilon_2 - \varepsilon_1/\varepsilon_2 + \varepsilon_2/\varepsilon_1 - 1/(\varepsilon_1\varepsilon_2) = (\varepsilon_1 - \overline{\varepsilon}_1)(\varepsilon_2 - \overline{\varepsilon}_2) \\ &= (2y_{11}\sqrt{\delta D_{21}})(2y_{12}\sqrt{\delta D_{22}}) \geq 4\sqrt{D_{21}D_{22}} = 4\sqrt{D_2}. \end{aligned} \tag{44}$$

Therefore, by (42)–(44), we obtain $\max(D_1, D_2) < \varepsilon_1^4/16 < 2.374 \cdot 10^{10}$. Thus, if $\min(D_1, D_2) > 1$ and $\max(D_1, D_2) \geq 2.374 \cdot 10^{10}$, then $N(D_1, D_2) \leq 1$.

Next we consider the case where $\min(D_1, D_2) = 1$. Since D_1 is a square, we have $D_1 = 1$ and $D_2 > 1$. By much the same argument as in the proof of the case $\min(D_1, D_2) > 1$, we can find from the proof of [3, Theorem 1] that if $D_2 \neq 1785$ and $N(1, D_2) > 1$, then there exist $t_1, t_2 \in \mathbb{N}$ and real quadratic algebraic numbers ϱ_1, ϱ_2 satisfying $1 < t_1 < t_2$, $1 < \varrho_2 < \varrho_1$, $D_2 < \varrho_1^4/16$, $h(\varrho_1) = (\log \varrho_1)/2$, $h(\varrho_2) = (\log \varrho_2)/2$, $[\mathbb{Q}(\varrho_1, \varrho_2) : \mathbb{Q}] = 4$,

$$0 < \log \varrho_1 - \log \varrho_2 < 2/\varrho_2^2, \quad 0 < t_1 \log \varrho_1 - t_2 \log \varrho_2 < 2/\varrho_2^{2t_2}, \tag{45}$$

$$t_2 > \varrho_1^2(\log\varrho_1)e^{-4/\varrho_2^2}. \tag{46}$$

Using Lemma 7, we get either $\varrho_1^2 e^{-4/\varrho_2^2} < 96$ or

$$t_1\log\varrho_1 - t_2\log\varrho_2 \ge \exp(-1558(\log\varrho_1)(\log\varrho_2)(\log t + 0.69)^2), \tag{47}$$

where

$$\begin{aligned} t &= \frac{t_1}{2\log\varrho_2} + \frac{t_2}{2\log\varrho_1} = \frac{t_2}{\log\varrho_1} + \frac{t_1\log\varrho_1 - t_2\log\varrho_2}{2(\log\varrho_1)(\log\varrho_2)} \\ &< \frac{t_2}{\log\varrho_1} + \frac{1}{2\varrho_2^{2t_2}(\log\varrho_1)(\log\varrho_2)}. \end{aligned} \tag{48}$$

We now suppose that $\varrho_1 \ge 350$. Then from (45), (47) and (48) we get

$$t_2/\log\varrho_1 < 122000. \tag{49}$$

The combination of (46) and (49) yields $122400 < \varrho_1^2 e^{-4/\varrho_2^2} < 122000$, a contradiction. So we have $\varrho_1 < 350$ and $D_2 < 9.379\cdot 10^8$. This implies that if $\min(D_1, D_2) = 1$ and $\max(D_1, D_2) \ge 9.379 \cdot 10^8$, then $N(D_1, D_2) \le 1$. The proof is complete.

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