ANNALES
POLONICI MATHEMATICI
LXIII.2 (1996)

On some elliptic transmission problems

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Abstract. Boundary value problems for second order linear elliptic equations with coefficients having discontinuities of the first kind on an infinite number of smooth surfaces are studied. Existence, uniqueness and regularity results are furnished for the diffraction problem in such a bounded domain, and for the corresponding transmission problem in all of $\mathbb{R}^N$. The transmission problem corresponding to the scattering of acoustic plane waves by an infinitely stratified scatterer, consisting of layers with physically different materials, is also studied.

0. Introduction. In this work we study boundary value problems for linear equations of elliptic type whose coefficients have discontinuities of the first kind on an infinite number of smooth surfaces that divide a bounded domain in $\mathbb{R}^N$ into nested layers. On those surfaces, the so-called "transmission (conjugacy, matching, linking) conditions" are imposed, that express the continuity of the medium and the equilibrium of the forces acting on it. The discontinuity of the coefficients of the equations corresponds to the fact that the medium consists of several physically different materials.

From the point of view of the theory of generalized solutions—which we employ in our approach—such problems can be considered as special cases of usual boundary value problems. On the contrary, the investigation of these problems by classical methods requires the theory of integral equations, and in this context they differ essentially from the usual boundary value problems where the medium has smoothly varying characteristics.

Boundary value problems with discontinuous coefficients (also known as diffraction problems) have been treated by many authors, employing a variety of approaches. In [16], Stampacchia introduced a general theory for second order linear elliptic equations with discontinuous coefficients; it is

1991 *Mathematics Subject Classification*: 35R05, 35P25, 76Q05.
Key words and phrases: transmission condition, Dirichlet problem, Robin problem, diffraction problem, acoustic scattering.

closer to potential theory than to the theory of differential equations with continuous coefficients. He also considered some non-linear problems. In his general setting, the essential difference between operators with continuous and discontinuous coefficients is that the former can be considered locally as small perturbations of operators with constant coefficients; this is not true for the latter. In [15], Oleĭnik studied elliptic and parabolic diffraction problems, obtaining their solutions as limits of solutions of corresponding problems for equations with smooth coefficients that approximate the given discontinuous ones. In [13], [14], Ladyzhenskaya studied diffraction problems by a weak solutions approach; she established regularity results as well. She also used difference schemes for approximation of the solutions, observing that the presence of discontinuities on the interfaces causes the convergence of the approximation not to be uniform but almost everywhere, still, though, being sufficient for passage to the limit. Finally, in [3], there are several applications of diffraction problems.

In Section 1, we study the existence and uniqueness of generalized solutions for the Dirichlet, Robin and the oblique derivative diffraction boundary value problems for second order, linear, elliptic equations with discontinuous coefficients on an infinite number of smooth surfaces in bounded domains. Moreover, we consider the regularity of these solutions. These questions are then investigated for the corresponding transmission problem in all of $\mathbb{R}^N$.

In Section 2, we study the scattering of a plane acoustic wave by an infinitely stratified scatterer, consisting of homogeneous layers of physically different media. We first prove that the only classical solution of the homogeneous transmission problem for the Helmholtz equation is the trivial solution, thus extending a result of Kress and Roach referring to one interface [11] to our infinitely stratified structure. For the existence of solutions of the non-homogeneous transmission problem, we apply the theory of generalized solutions, in the spirit of Section 1.

1. Elliptic equations. We consider elliptic boundary value problems of the form

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$Lu := \sum_{i,j=1}^{N} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{N} b_i(x)u_{x_i} + d(x)u \tag{1.2}$$

and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let a_{ij}, b_i

and d be measurable functions satisfying

$$\text{(E1)} \qquad \begin{cases} \mu_1|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \le \mu_2|\xi|^2, \quad \mu_1,\mu_2 > 0, \\ a_{ij} = a_{ji}, \end{cases}$$

$$\text{(E2)} \qquad \Big(\sum_{i=1}^{N} b_i^2\Big)^{1/2} \le \mu_3,$$

$$\text{(E3)} \qquad \mu_4 \le d(x) \le \mu_5,$$

$$\text{(E4)} \qquad f \in L^2(\Omega).$$

The quantities μ_1 to μ_5 above are constants.

Let c_Ω be the constant in the Poincaré–Friedrichs inequality ([8], [14]) and let

$$\delta := \max_{0<\varepsilon\le\mu_1} \Big\{(\mu_1-\varepsilon)c_\Omega^{-2} - \mu_5 - \frac{\mu_3^2}{4\varepsilon}\Big\}. \tag{1.3}$$

Then, as is well known, we have ([14], Thm. 2.1, p. 50):

Theorem A. *If* (E1) *to* (E4) *are satisfied, and if, moreover,*

$$\delta > 0, \tag{1.4}$$

then the Dirichlet problem (1.1) *has a unique generalized solution, and the following a priori bound holds*:

$$\|u\|_{H_0^1(\Omega)} \le c\|f\|_{L^2(\Omega)}. \tag{1.5}$$

As far as the differentiability of the generalized solution is concerned, the following is also well known ([8], Thm. 8.12, p. 176):

Theorem B. *If, in addition to the hypotheses of Theorem* A, *we assume that $a_{ij} \in C^{0,1}(\overline{\Omega})$ and that $\partial\Omega$ is of class C^2 then the unique solution of* (1.1) *belongs to $H_0^2(\Omega)$.*

Remark 1.1. As is well known, Theorem A (resp. Theorem B) applies also to the case of Dirichlet problems with non-homogeneous boundary data φ, provided $\varphi \in H^1(\Omega)$ is such that $u - \varphi \in H_0^1(\Omega)$ (resp. $\varphi \in H^2(\Omega)$ is such that $u - \varphi \in H_0^2(\Omega)$).

Remark 1.2. Similar considerations hold for the homogeneous (and, in view of the above remark, for the non-homogeneous) Robin problem

$$\begin{aligned} Lu &= f && \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \sigma u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where σ is a bounded non-negative function on $\partial\Omega$ and $\partial u/\partial n$ denotes the co-normal derivative

$$\frac{\partial u}{\partial n} = \sum_{i,j=1}^{N} a_{ij}(x)\cos(n, x_j)\, u_{x_i}, \tag{1.7}$$

n being the outward normal to $\partial\Omega$.

Having listed the necessary preliminary concepts, we now proceed to our results. We start with the description of the domain we shall be dealing with. Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$, with boundary S_0; a *core* Ω_{c} is contained in $\widetilde{\Omega}$, and we work actually in $\widetilde{\Omega} - \Omega_{\mathrm{c}}$, which will be denoted by Ω in the sequel. The boundary of Ω is $S_0 \cup S_{\mathrm{c}}$, and both S_0 and S_{c} are supposed to be $(N-1)$-dimensional C^2 surfaces. Ω is tessellated in the following way: let S_k, $k = 1, 2, \ldots$, be $(N-1)$-dimensional C^2 surfaces. S_k surrounds S_{k+1}, $k = 0, 1, 2, \ldots$, and S_{c}. We assume that $\operatorname{dist}(S_{k-1}, S_k) > 0$ for all $k = 1, 2, \ldots$, and moreover that $\lim_{k\to\infty} S_k = S_{\mathrm{c}}$. The S_k divide Ω into "annuli-like" domains Ω_k with $\partial\Omega_k = S_{k-1} \cup S_k$.

We shall study the question of the solvability of a boundary value problem in such an Ω when the coefficients of the equation are allowed to have discontinuities of the first kind on the $S_k, k = 1, 2, \ldots$ On the surfaces of discontinuity, S_k, certain conditions must be imposed, known as "transmission conditions".

As will be clear from the formulation of the problem, the transmission conditions are not necessarily uniquely determined.

We consider this specific geometry for Ω, because its nature is compatible with approximating inhomogeneous media by piecewise homogeneous ones ([3], [9]). The continuous variations of the material parameters are subdivided into regions of homogeneous media; provided these subdivisions are fine enough, the exact solution to this problem might be expected to be a reasonable approximation to the answer for continuous variation. Our results are not restricted to the above special tessellation of Ω. As in [15], Ω may be partitioned in an arbitrary fashion, provided the interfaces of the subdomains Ω_j are sufficiently smooth, and that the transmission conditions are satisfied on all these interfaces.

The symbol $[w]|_{S_k}$ denotes the difference between the limiting values of $w(x)$ on S_k, calculated for approach to S_k from Ω_k and Ω_{k+1} (i.e. the jump in the function w as it crosses S_k).

In the first part of this section we shall study the solvability of the following problems, with L as in (1.2), and Ω as in the above description.

The *Dirichlet problem* is

$$
(1.8)\qquad
\begin{aligned}
&\text{(i)} && Lu = f && \text{in } \Omega,\\
&\text{(ii)} && u = 0 && \text{on } \partial\Omega,\\
&\text{(iii)} && [u]|_{S_k} = 0, &&\\
&\text{(iv)} && \left[q\frac{\partial u}{\partial n}\right]\Big|_{S_k} = 0, && k = 1,2,\ldots,
\end{aligned}
$$

where q is a positive, piecewise constant function ($q = q_k > 0$ in Ω_k), while the *Robin problem* is

$$
(1.9)\qquad
\begin{aligned}
&\text{(i)} && Lu = f && \text{in } \Omega,\\
&\text{(ii)} && \frac{\partial u}{\partial n} + \sigma u = 0 && \text{on } \partial\Omega,\\
&\text{(iii)} && [u]|_{S_k} = 0, &&\\
&\text{(iv)} && \left[q\frac{\partial u}{\partial n} + \sigma u\right]\Big|_{S_k} = 0, && k = 1,2,\ldots,
\end{aligned}
$$

where σ is a given continuous non-negative function defined on each S_k, and q is as above.

Let us define the generalized solutions of (1.8) and (1.9).

Definition 1.1. A function $u \in H_0^1(\Omega)$ is called a *generalized* (*weak*) *solution of* (1.8) iff for all $v \in H_0^1(\Omega)$ we have

$$
(1.10)\qquad \int_\Omega \Big(\sum_{i,j=1}^N q a_{ij} u_{x_i} v_{x_j} - \sum_{i=1}^N q b_i u_{x_i} v - q d u v\Big)\, dx = -\int_\Omega q f v\, dx.
$$

Definition 1.2. A function $u \in H^1(\Omega)$ is called a *generalized* (*weak*) *solution of* (1.9) iff for all $v \in H^1(\Omega)$ we have

$$
(1.11)\qquad \int_\Omega \Big(\sum_{i,j=1}^N q a_{ij} u_{x_i} v_{x_j} - \sum_{i=1}^N q b_i u_{x_i} v - q d u v\Big)\, dx + \sum_{k=0}^\infty \int_{S_k} \sigma u v\, ds = \int_\Omega q f v\, dx.
$$

We now prove the following result:

Theorem 1.1. *Let the data of* (1.8.i) *satisfy* (E1) *to* (E4) *and* (1.4) *and*, *suppose*, *moreover*, *that* $a_{ij} \in C^{0,1}(\Omega_k)$, $k = 1,2,\ldots$ *Then* (1.8) *has a unique solution* $u \in H_0^1(\Omega) \cap H^2(\Omega_k)$, $k = 1,2,\ldots$

Proof. The existence and uniqueness of a generalized solution for (1.8) is ascertained by Theorem A. By the previous definition it is clear that (1.8.ii) and (1.8.iii) are satisfied.

Now consider an arbitrary discontinuity surface S_k, and work in the two adjacent layers Ω_k and Ω_{k+1} separated by S_k. In what follows, the superscript (k) will denote the restriction of the quantity involved to Ω_k.

By our assumptions, we have $u \in H^2(\Omega_k),\ k = 1, 2, \ldots$ It follows from the embedding theorems ([8], [14]) that not only $u(x)$, but also its derivatives u_{x_j} have traces on S_k: $u_{x_j} \in L^2(S_k),\ k = 1, 2, \ldots$ However, the traces of u_{x_j} admit jumps as x passes through S_k, in such a way as to have $\left[q\frac{\partial u}{\partial n}\right]|_{S_k} = 0$. Let us also point out that in the sequel n is to be understood as the outward normal to the boundary of the set over which each integration is performed.

Let $v \in H_0^1(\Omega_k \cup S_k \cup \Omega_{k+1})$, defined to be zero outside $\Omega_k \cup S_k \cup \Omega_{k+1}$, be arbitrary. Then

$$
\begin{aligned}
(1.12)\qquad & -\int_{\Omega_k\cup\Omega_{k+1}} qfv\,dx \\
&= \int_{\Omega_k\cup\Omega_{k+1}} q\Big\{\sum_{i,j=1}^N a_{ij}u_{x_i}v_{x_j} - \sum_{i=1}^N b_i u_{x_i} v - duv\Big\}\,dx \\
&= \int_{\Omega_k}\Big(\sum_{i,j=1}^N q^{(k)}a_{ij}^{(k)}u_{x_i}^{(k)}v_{x_j} - \sum_{i=1}^N q^{(k)}b_i^{(k)}u_{x_i}^{(k)}v - q^{(k)}d^{(k)}u^{(k)}v\Big)\,dx \\
&\quad + \int_{\Omega_{k+1}}\Big(\sum_{i,j=1}^N q^{(k+1)}a_{ij}^{(k+1)}u_{x_i}^{(k+1)}v_{x_j} \\
&\quad - \sum_{i=1}^N q^{(k+1)}b_i^{(k+1)}u_{x_i}^{(k+1)}v - q^{(k+1)}d^{(k+1)}u^{(k+1)}v\Big)\,dx
\end{aligned}
$$

and using integration by parts we get

$$
\begin{aligned}
&\int_{S_k}\sum_{i,j=1}^N q^{(k)}a_{ij}^{(k)}u_{x_i}^{(k)}\cos(n,x_j)\,v\,ds \\
&\quad - \int_{\Omega_k} q^{(k)}\Big\{\sum_{i,j=1}^N (a_{ij}^{(k)}u_{x_i}^{(k)})_{x_j} + \sum_{i=1}^N b_i^{(k)}u_{x_i}^{(k)} + d^{(k)}u^{(k)} - f^{(k)}\Big\}v\,dx \\
&\quad - \int_{S_k}\sum_{i,j=1}^N q^{(k+1)}a_{ij}^{(k+1)}u_{x_i}^{(k+1)}\cos(n,x_j)\,v\,ds
\end{aligned}
$$

$$- \int\limits_{\Omega_{k+1}} q^{(k+1)} \Big\{ \sum_{i,j=1}^{N} (a_{ij}^{(k+1)} u_{x_i}^{(k+1)})_{x_j} + \sum_{i=1}^{N} b_i^{(k+1)} u_{x_i}^{(k+1)}$$
$$+ d^{(k+1)} u^{(k+1)} - f^{(k+1)} \Big\} v\, dx = 0.$$

The previous relation can be written as

$$\begin{aligned} 0 = \int\limits_{S_k} &\Big\{ q^{(k)} \frac{\partial u^{(k)}}{\partial n} - q^{(k+1)} \frac{\partial u^{(k+1)}}{\partial n} \Big\} v\, ds \\ &- \int\limits_{\Omega_k} q^{(k)} \Big\{ L^{(k)} u^{(k)} - f^{(k)} \Big\} v\, dx \\ &- \int\limits_{\Omega_{k+1}} q^{(k+1)} \Big\{ L^{(k+1)} u^{(k+1)} - f^{(k+1)} \Big\} v\, dx. \end{aligned}$$

Since $v \in H_0^1(\Omega_k \cup S_k \cup \Omega_{k+1})$ is arbitrary, we may take it to be equal to zero in $\Omega_k \cup \Omega_{k+1}$, $S_k \cup \Omega_{k+1}$, and $\Omega_k \cup S_k$, whereby we, respectively, have

$$\begin{aligned} &q^{(k)} \frac{\partial u^{(k)}}{\partial n} - q^{(k+1)} \frac{\partial u^{(k+1)}}{\partial n} = 0 && \text{on } S_k, \\ &L^{(k)} u^{(k)} = f^{(k)} && \text{a.e. in } \Omega_k, \\ &L^{(k+1)} u^{(k+1)} = f^{(k+1)} && \text{a.e. in } \Omega_{k+1}, \end{aligned}$$

thus proving that (1.8.iv) and (1.8.i) are satisfied.

Remark 1.3. The analogue of Theorem 1.1 can be proved for the non-homogeneous problem

$$\begin{aligned} &Lu = f && \text{in } \Omega, \\ &u = \varphi && \text{on } \partial\Omega, \\ &[u]|_{S_k} = \psi_1^{(k)}, && k = 1, 2, \ldots, \\ &\Big[q \frac{\partial u}{\partial n} \Big] \Big|_{S_k} = \psi_2^{(k)}, && k = 1, 2, \ldots, \end{aligned}$$

by replacing u by a new unknown function $w(x) = u(x) - \theta(x)$, where $\theta(x)$ can be appropriately chosen so that the boundary and transmission conditions become homogeneous.

As far as the Robin problem is concerned, we need the following well-known trace estimate ([14], Thm. 6.5):

$$\|u\|_{L^2(\partial\Omega)}^2 \le \widetilde{c}_\Omega \|u\|_{H^1(\Omega)}^2, \qquad u \in H^1(\Omega), \tag{1.13}$$

where $\widetilde{c}_\Omega$ is a constant independent of u. Then we have

THEOREM 1.2. *Assume that the hypotheses of Theorem* 1.1 *are valid.*

Suppose, moreover, that for all $k = 0, 1, 2, \ldots$ *we have*

$$\sup_k M_k c_{k+1} \le c < \infty, \tag{1.14}$$

where $M_k = \max\{\sigma(x) : x \in S_k\}$ *and* c_k *is the constant for* Ω_k *as in* (1.13). *Then* (1.9) *has a unique solution in* $H^1(\Omega) \cap H^2(\Omega_k)$, $k = 1, 2, \ldots$

Proof. The proof goes along the same lines as that of Theorem 1.1 with the exception of the following subtle point: here we need to establish the convergence of the series

$$\sum_{k=0}^{\infty} \int_{S_k} \sigma u v \, ds, \qquad v \in H^1(\Omega).$$

It, therefore, suffices to show that the series

$$\sum_{k=0}^{\infty} \int_{S_k} \sigma w^2 \, ds$$

converges for $w \in H^1(\Omega)$. But, using (1.13) and (1.14) we get

$$\begin{aligned}\int_{S_k} \sigma w^2 \, ds \le M_k \int_{S_k} w^2 \, ds = M_k \|w\|^2_{L^2(S_k)} &\le M_k c_{k+1} \|w\|^2_{H^1(\Omega_{k+1})} \\ &\le (\sup_k M_k c_{k+1}) \|w\|^2_{H^1(\Omega_{k+1})} \le c \|w\|^2_{H^1(\Omega_{k+1})},\end{aligned}$$

whereby, since $\sum_{k=0}^{\infty} \|w\|^2_{H^1(\Omega_{k+1})} = \|w\|^2_{H^1(\Omega)}$, the proof is complete.

It is possible to consider more general transmission conditions, in the form of an oblique derivative; we have

THEOREM 1.3. *Consider the problem*

$$\begin{aligned} &Lu = f && \text{in } \Omega, \\ &\frac{\partial u}{\partial m} + \sigma u = 0 && \text{on } \partial\Omega, \\ &[u]|_{S_k} = 0, && k = 1, 2, \ldots, \\ &\left[q \frac{\partial u}{\partial m} + \sigma u \right]\Big|_{S_k} = 0, && k = 1, 2, \ldots, \end{aligned} \tag{1.15}$$

and suppose that the hypotheses of Theorem 1.1 *are valid, while* σ *is as in Theorem* 1.2. *Let* $m = (m_1, \ldots, m_N)$, *with* $|m| = 1$, *be a smooth vector field on* S_k *which is non-tangential to* $\partial\Omega$ *and* S_k, $k = 1, 2, \ldots$ *Then* (1.15) *has a unique solution in* $H^1(\Omega) \cap H^2(\Omega_k)$, $k = 1, 2, \ldots$

The proof of Theorem 1.3 follows from

LEMMA 1.1. *Let Ω^* be a bounded domain with smooth boundary S^*. Then the problem*

$$(1.16)\qquad \begin{aligned} &\sum_{i,j=1}^{N}(a_{ij}u_{x_i})_{x_j}+\sum_{i=1}^{N}b_iu_{x_i}+du=f && \text{in } \Omega^*,\\ &\frac{\partial u}{\partial m}+\sigma u=0 && \text{on } S^*, \end{aligned}$$

has a generalized solution.

Proof. It can be shown ([6], p. 376) that (1.16) is equivalent to the Robin problem

$$(1.17)\qquad \begin{aligned} &\sum_{i,j=1}^{N}(A_{ij}u_{x_i})_{x_j}+\sum_{i=1}^{N}B_iu_{x_i}+du=f && \text{in } \Omega^*,\\ &\frac{\partial u}{\partial m}+\widehat{\sigma}u=0 && \text{on } S^*, \end{aligned}$$

with

$$\frac{\partial u}{\partial m}=\sum_{i,j=1}^{N}A_{ij}u_{x_i}n_j,$$

where

$$A_{ij}=a_{ij}+\mu_{ij},\quad B_i=b_i+\sum_{j=1}^{N}\frac{\partial\mu_{ij}}{\partial x_j},\quad \widehat{\sigma}=\sigma\sigma_0,$$

$$\mu_{ij}=\frac{1}{m\cdot n}(n_im_j-n_jm_i)\sum_{r,p=1}^{N}a_{rp}n_rn_p-\sum_{r=1}^{N}(a_{rj}n_rn_i-a_{ri}n_rn_j),$$

$$\sigma_0=\frac{1}{m\cdot n}\sum_{r,p=1}^{N}a_{rp}n_rn_p.$$

But, by standard theory, (1.17) is solvable, and hence the proof of the lemma is complete.

We also have the following regularity result; its proof may be performed in the spirit of [13], or [3], p. 592, and is omitted for the sake of brevity.

THEOREM 1.4. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. If, additionally, $a_{ij},\partial a_{ij}/\partial x_r,d,f\in C^{0,a}(\overline{\Omega}_k)$, $a\in(0,1)$ and if $\partial\Omega$ and S_k, $k=1,2,\ldots$, are C^2-surfaces, then the generalized solution of* (1.8) *belongs to $C(\overline{\Omega})\cap C^{2,a}(\Omega_k)$.*

To study the solutions in unbounded domains we need to resort to weighted Sobolev spaces. Let Ω_0 be the exterior of a bounded domain Ω in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$. Let w be a non-negative function on

Ω_0 which is locally Lebesgue integrable on Ω_0. Let $L^2(\Omega_0, w)$ be the linear space of functions u on Ω_0 which are measurable with respect to the measure $w(x)\,dx$, and which satisfy

$$\|u\|_{0,w} \equiv \Big(\int\limits_{\Omega_0} |u(x)|^2 w(x)\,dx\Big)^{1/2} < \infty.$$

By $H_0^n(\Omega_0, w)$, where n is any positive integer, we shall mean the completion of $C_0^n(\Omega_0)$ endowed with the norm

$$\|u\|_{n,w} \equiv \sum_{i=0}^{n} \|D^i u\|_{0,w}.$$

Here $|D^i u(x)|^2 = \sum_{|a|=i} |D^a u(x)|^2$, where the summation extends over all multiindices a of length i, and

$$D^a u = \frac{\partial^{|a|} u}{\partial x_1^{a_1} \ldots \partial x_N^{a_N}}.$$

Given suitable inner products, $L^2(\Omega_0, w)$ and $H_0^n(\Omega_0, w)$ become Hilbert spaces. When $w(x) = 1$ and Ω_0 is a bounded domain, these spaces coincide with the usual Sobolev spaces.

The notion of a generalized solution of the Dirichlet problem

$$\begin{aligned} Lu &= f \quad &&\text{in } \Omega_0, \\ u &= 0 \quad &&\text{on } \partial\Omega_0, \end{aligned} \tag{1.18}$$

where L is given by (1.2), can be defined in accordance to the situation of a bounded domain; see [5].

Let

$$N_\sigma(w, x) := \Big[\int\limits_{\Omega_0 \cap B_x} w(y)^{-\sigma} dy\Big]^{1/\sigma}, \qquad \sigma > N/2,$$

and

$$M_\varrho(g(x)) := \int\limits_{\Omega_0 \cap B_x} |g(y)| N_\sigma(w, y) |x - y|^{\varrho - N} dy, \qquad 0 < \varrho < 2 - N/\sigma,$$

where B_x is the closed ball in $\mathbb{R}^N$ with centre x and radius 1.

We make the following assumptions on the data of (1.18):

(E5) $$\mu_1 w(x)|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \quad \forall x \in \Omega_0,\ \forall \xi \in \mathbb{R}^N,\ \mu_1 > 0,$$

(E6) $$a_{ij}(x) w(x)^{-1} \in L^\infty(\Omega_0), \qquad i, j = 1, \ldots, N,$$

(E7) the Poincaré inequality holds in $H_0^1(\Omega_0, w)$, with constant c_{Ω_0},

(E8) $$\sup_{x \in \Omega_0} N_\sigma(w, x) < \infty,$$

(E9)
$$\sup_{x\in\Omega_0} M_\varrho(|d(x)|)<\infty,$$

(E10)
$$\max_{1\le i\le N}\sup_{x\in\Omega_0} M_\varrho(|b_i(x)|^2 w(x)^{-1})<\infty,$$

(E11)
$$\widetilde{\delta}>0,\quad \text{where}\quad \widetilde{\delta}:=\mu_1 c_{\Omega_0}^{-2}-\|d\|_{0,w}-\sum_{i=1}^N \|w^{-1}b_i\|_{0,w}.$$

Then we have the following ([5]):

THEOREM 1.5. *If* (E5)–(E11) *hold*, *and* $f\in L^2(\mathbb{R}^N)$, *then the Dirichlet problem* (1.18) *has a unique generalized solution in* $H_0^1(\Omega_0,w)$.

Now we turn to the transmission problem in $\mathbb{R}^N$; let Ω have the described stratified structure. Recall that $\partial\Omega=S_0\cup S_{\rm c}$. Consider the problem

(1.19)
$$\begin{aligned}
&\text{(i)} && Lu=f && \text{in } \mathbb{R}^N\setminus\Omega_{\rm c},\\
&\text{(ii)} && u=0 && \text{on } S_{\rm c},\\
&\text{(iii)} && [u]|_{S_k}=\left[q\frac{\partial u}{\partial n}\right]\Big|_{S_k}=0, && k=0,1,2,\ldots
\end{aligned}$$

Let
$$w^*(x)=\begin{cases}1, & x\in\overline{\Omega},\\ w(x), & x\in\Omega_0=\mathbb{R}^N\setminus\overline{\Omega},\end{cases}$$
and consider the space $H^1(\mathbb{R}^N,w^*)$ defined as $H_0^1(\Omega_0,w)$ above.

DEFINITION 1.3. A function $u\in H^1(\mathbb{R}^N,w^*)$ is called a *generalized solution of* (1.19) iff for every $v\in H^1(\mathbb{R}^N,w^*)$ we have

(1.20)
$$\int_{\mathbb{R}^N\setminus\Omega_{\rm c}}\Big(\sum_{i,j=1}^N qa_{ij}u_{x_i}v_{x_j}-\sum_{i=1}^N qb_iu_{x_i}v-qduv\Big)\,dx=-\int_{\mathbb{R}^N\setminus\Omega_{\rm c}} qfv\,dx.$$

We can state the following regularity result that can be proved by standard arguments (cf. Theorem 1.4):

THEOREM 1.6. *Suppose that the transmission problem* (1.19) *has a generalized solution* u. *Assume that the regularity hypotheses*, *on the coefficients and the interfaces*, *of Theorem* 1.4 *are satisfied for* $k=0,1,2,\ldots$ *Then* u *is a classical solution* (*behaving at infinity as prescribed by the weight* w).

We conclude this section with the following result establishing the solvability of the transmission problem.

THEOREM 1.7. *Suppose that the data of* (1.19.i) *satisfy* (E1)–(E4) *and* (1.4) *in* Ω, *and* (E5)–(E11) *in* Ω_0. *Additionally*, *assume that* $a_{ij}\in C^{0,1}(\Omega_k)$, $k=0,1,2,\ldots$ *Then*, *for* $f\in L^2(\mathbb{R}^N)$, (1.19) *has a unique solution in* $H^1(\mathbb{R}^N,w^*)\cap H^2(\Omega_k,w^*)$, $k=0,1,2,\ldots$

The proof of this theorem can be performed in a completely analogous manner to that of Theorem 1.1, and is therefore, omitted.

2. Acoustic scattering. In this section we consider the problem of scattering of a plane acoustic wave by an infinitely stratified scatterer; such a scatterer is defined as follows: let $\widetilde{\Omega}$ be a bounded, convex domain of $\mathbb{R}^3$, with boundary S_0. A core Ω_{c}, within which lies the origin of coordinates, is contained in $\widetilde{\Omega}$. We actually work in $\widetilde{\Omega} - \Omega_{\mathrm{c}}$, which will be denoted by Ω in the sequel. The boundary of Ω is $S_0 \cup S_{\mathrm{c}}$, and both S_0 and S_{c} are supposed to be 2-dimensional C^2 surfaces. Ω is divided into annuli-like regions Ω_j by 2-dimensional C^2 surfaces $S_j,\ j = 1, 2, \ldots$ S_j surrounds $S_{j+1},\ j = 1, 2, \ldots,$ and S_{c}. We assume that $\operatorname{dist}(S_{j-1}, S_j) > 0$ for all $j = 1, 2, \ldots,$ and that $\lim_{j\to\infty} S_j = S_{\mathrm{c}}$. The exterior, Ω_0, of $\widetilde{\Omega}$, as well as each Ω_j, are homogeneous isotropic media. The wave number k_j in each region Ω_j is given by

$$k_j^2 = \frac{\omega}{c_j^2}(\omega + id_j), \qquad i^2 = -1,\ j = 0, 1, 2, \ldots,$$

where ω is the angular frequency of the incident wave, c_j is the speed of sound, and d_j is the damping coefficient in Ω_j. We choose the sign of k_j, as usual, such that

$$\operatorname{Im} k_j \geq 0, \qquad j = 0, 1, 2, \ldots$$

It is obvious that $\operatorname{Re} k_j \neq 0,\ j = 0, 1, 2, \ldots$

We assume that a plane acoustic wave $\psi^{\mathrm{inc}}(\mathbf{r})$ is incident upon the infinitely stratified scatterer. Suppressing a harmonic time dependence $\exp(-i\omega t)$, the incident wave takes the form

$$\psi^{\mathrm{inc}}(\mathbf{r}) = \exp(ik_0 \widehat{\mathbf{k}} \cdot \mathbf{r}), \tag{2.1}$$

where $\widehat{\mathbf{k}}$ is the unit vector in the direction of propagation. For more details about the physical problem, we refer to [2], [9].

The total acoustic field u_j in each $\Omega_j,\ j = 1, 2, \ldots,$ must satisfy Helmholtz's equation

$$\Delta u_j(\mathbf{r}) + k_j^2 u_j(\mathbf{r}) = 0. \tag{2.2}$$

For the total exterior field, $\psi_0(\mathbf{r})$, we have

$$\psi_0(\mathbf{r}) = \psi^{\mathrm{inc}}(\mathbf{r}) + u_0(\mathbf{r}), \tag{2.3}$$

where $u_0(\mathbf{r})$ is the scattered field.

Since $\psi^{\mathrm{inc}}(\mathbf{r})$ satisfies the Helmholtz equation in Ω_0, the same is true for $u_0(\mathbf{r})$, whereby (2.2) holds for $j = 0, 1, 2, \ldots$ The scattered field is assumed to satisfy Sommerfelds' radiation condition:

$$\frac{\partial u_0(\mathbf{r})}{\partial n} - ik_0 u_0(\mathbf{r}) = o(1/r), \qquad r \to \infty. \tag{2.4}$$

As is well known ([1], p. 71), by (2.2) and (2.4), u_0 must automatically satisfy

$$u_0(\mathbf{r}) = O(1/r), \qquad r \to \infty. \tag{2.5}$$

On the surface of the core, the desired solution must satisfy the homogeneous Dirichlet boundary condition, corresponding to the core being soft.

On S_0 we have the following transmission conditions:

$$\begin{aligned} &u_1 - u_0 = \psi^{\mathrm{inc}}, \\ &q_1 \frac{\partial u_1}{\partial n} - q_0 \frac{\partial u_0}{\partial n} = q_0 \frac{\partial \psi^{\mathrm{inc}}}{\partial n}. \end{aligned} \tag{2.6}$$

The transmission conditions on $S_j,\ j = 1, 2, \ldots$, are given by

$$\begin{aligned} &u_{j+1} - u_j = 0, \\ &q_{j+1} \frac{\partial u_{j+1}}{\partial n} - q_j \frac{\partial u_j}{\partial n} = 0, \end{aligned} \tag{2.7}$$

where $q_j,\ j = 0, 1, 2, \ldots$, are given non-zero complex constants.

By a standard procedure, the homogeneous equations and non-homogeneous transmission conditions of the above problem can be transformed to

$$\Delta u_j + k_j^2 u_j = f_j \qquad \text{in } \Omega_j, \tag{2.8}$$

$$\begin{aligned} &u_{j+1} - u_j = 0 && \text{on } S_j, \\ &q_{j+1} \frac{\partial u_{j+1}}{\partial n} - q_j \frac{\partial u_j}{\partial n} = 0 && \text{on } S_j, \end{aligned} \tag{2.9}$$

for all $j = 0, 1, 2, \ldots$, where $f_j,\ j = 0, 1, 2, \ldots$, is some known C^2 function depending on ψ^{inc} and q_0.

In the sequel we shall make the following assumptions interrelating the coefficients of (2.8) and (2.9).

(2.10) Let $k_j \in \mathbb{C} - \{0\}$ with $0 \le \arg k_j \le \pi$, and $q_j \in \mathbb{C} - \{0\}$ with $\sup |q_j| < \infty,\ j = 0, 1, 2, \ldots$, be such that

$$\frac{\overline{q}_j}{\overline{q}_0} \cdot \frac{\overline{k_j^2}}{\overline{k_0^2}} = p_j \in \mathbb{R},$$

where $\sup p_j < \infty$, and $p_j \operatorname{Re} k_j \operatorname{Re} k_0 > 0,\ j = 0, 1, 2, \ldots$

Let us denote by (HTP) the homogeneous transmission problem consisting of the equations

$$\Delta u_j + k_j^2 u_j = 0 \quad \text{in } \Omega_j,\ j = 0, 1, 2, \ldots, \tag{2.11}$$

the transmission conditions (2.9), the radiation condition (2.4), the homogeneous Dirichlet boundary condition on the surface of the core of the scatterer, and (2.10).

We can now prove

THEOREM 2.1. (HTP) *has only the trivial solution.*

Proof. Let $\Omega_{0,R} = \{\mathbf{r} \in \Omega_0 : r < R\}$, $R > 0$. Applying Green's first theorem over $\Omega_{0,R}$, we obtain

$$\int\limits_{r=R} u_0 \frac{\partial \overline{u}_0}{\partial n}\, ds = \int\limits_{\Omega_{0,R}} u_0 \Delta \overline{u}_0\, dv + \int\limits_{S_0} u_0 \frac{\partial \overline{u}_0}{\partial n}\, ds + \int\limits_{\Omega_{0,R}} |\operatorname{grad} u_0|^2\, dv,$$

which, again by Green's first theorem over Ω_1, and the transmission conditions (2.9), becomes

$$\begin{aligned}\int\limits_{r=R} u_0 \frac{\partial \overline{u}_0}{\partial n}\, ds = & \int\limits_{\Omega_{0,R}} u_0 \Delta \overline{u}_0\, dv + \int\limits_{\Omega_{0,R}} |\operatorname{grad} u_0|^2\, dv \\ & + \frac{\overline{q}_1}{\overline{q}_0} \int\limits_{\Omega_1} u_1 \Delta \overline{u}_1\, dv + \frac{\overline{q}_1}{\overline{q}_0} \int\limits_{\Omega_1} |\operatorname{grad} u_1|^2\, dv \\ & + \frac{\overline{q}_1}{\overline{q}_0} \int\limits_{S_1} u_1 \frac{\partial \overline{u}_1}{\partial n}\, ds. \end{aligned} \tag{2.12}$$

By repeated use of Green's first theorem, and taking into account (2.11), the transmission conditions (2.9), the boundary behaviour on the surface of the core, and dividing throughout by $\overline{k_0^2}$, we get from (2.12),

$$\begin{aligned} \frac{1}{\overline{k_0^2}} & \int\limits_{r=R} u_0 \frac{\partial \overline{u}_0}{\partial n}\, ds \\ & = - \int\limits_{\Omega_{0,R}} |u_0|^2\, dv + \frac{1}{\overline{k_0^2}} \int\limits_{\Omega_{0,R}} |\operatorname{grad} u_0|^2\, dv \\ & \quad - \sum_{j=1}^{\infty} p_j \int\limits_{\Omega_j} |u_j|^2\, dv + \sum_{j=1}^{\infty} \frac{1}{\overline{k_0^2}} \frac{\overline{q}_j}{\overline{q}_0} \int\limits_{\Omega_j} |\operatorname{grad} u_j|^2\, dv. \end{aligned} \tag{2.13}$$

The convergence of the series in (2.13) follows by (2.10), and by noting that

$$\sum_{j=1}^{\infty} \int\limits_{\Omega_j} |u_j|^2\, dv = \|u\|^2_{L^2(\Omega)} < \infty$$

and

$$\sum_{j=1}^{\infty} \int\limits_{\Omega_j} |\operatorname{grad} u_j|^2\, dv = \|u\|^2_{H^1(\Omega)} < \infty.$$

Taking imaginary parts in (2.13), we get

$$\text{(2.14)}\quad \operatorname{Im}\left(\frac{1}{\overline{k_0^2}}\int_{r=R} u_0\frac{\partial\overline{u}_0}{\partial n}\,ds\right)=\left(\operatorname{Im}\frac{1}{\overline{k_0^2}}\right)\int_{\Omega_{0,R}}|\operatorname{grad}u_0|^2\,dv + \sum_{j=1}^{\infty}\operatorname{Im}\left(\frac{1}{\overline{k_0^2}}\frac{\overline{q}_j}{\overline{q}_0}\right)\int_{\Omega_j}|\operatorname{grad}u_j|^2\,dv.$$

Since u_0 satisfies (2.5), it follows that the LHS of (2.14) tends to zero, and $\Omega_{0,R}$ to Ω_0, as $R\to\infty$. Therefore

$$\text{(2.15)}\qquad \frac{\operatorname{Im}k_0^2}{|k_0|^4}\int_{\Omega_0}|\operatorname{grad}u_0|^2\,dv+\sum_{j=1}^{\infty}\frac{\operatorname{Im}k_j^2}{|k_j|^4}p_j\int_{\Omega_j}|\operatorname{grad}u_j|^2\,dv=0,$$

whereby, since $\operatorname{Im}k_j^2=2\operatorname{Re}k_j\operatorname{Im}k_j$ and $\operatorname{Re}k_j\neq 0,\ j=0,1,2,\dots$, we have

$$\text{(2.16)}\qquad \frac{(\operatorname{Re}k_0)^2\operatorname{Im}k_0}{|k_0|^4}\int_{\Omega_0}|\operatorname{grad}u_0|^2\,dv + \sum_{j=1}^{\infty}\frac{p_j\operatorname{Re}k_j\operatorname{Re}k_0\operatorname{Im}k_j}{|k_j|^4}\int_{\Omega_j}|\operatorname{grad}u_j|^2\,dv=0.$$

If $\operatorname{Im}k_0>0$ and since $\operatorname{Im}k_j\geq 0,\ j=1,2,\dots$, by (2.9), (2.11) and (2.16) it follows that

$$\text{(2.17)}\qquad u_0=0\quad\text{in }\Omega_0.$$

In the case $\operatorname{Im}k_0=0$, we obtain from (2.14), with RHS written as in (2.16),

$$\text{(2.18)}\qquad \operatorname{Im}\left(\int_{r=R}u_0\frac{\partial\overline{u}_0}{\partial n}\,ds\right)\geq 0.$$

From (2.4) it follows that

$$\text{(2.19)}\qquad k_0\int_{r=R}|u_0|^2\,ds+\operatorname{Im}\left(\int_{r=R}u_0\frac{\partial\overline{u}_0}{\partial n}\,ds\right)=o(1)\quad\text{as }R\to\infty.$$

By (2.18) and (2.19) we obtain

$$\text{(2.20)}\qquad \int_{r=R}|u_0|^2\,ds=o(1)\quad\text{as }R\to\infty.$$

Therefore, by Rellich's theorem ([1]), it follows that (2.17) holds. Since $u_0=0$ in Ω_0, it suffices to show that $u_1=0$ in Ω_1. Then by the same argument we can proceed to show that $u_2=0$ in Ω_2, etc. Let

$$\text{(2.21)}\qquad w_1(\mathbf{r})=\begin{cases}u_1(\mathbf{r}), & \mathbf{r}\in\Omega_1,\\ 0, & \mathbf{r}\in\Omega_0.\end{cases}$$

It is obvious that w_1 satisfies

$$\Delta w_1 + k_1^2 w_1 = 0 \quad \text{in } \Omega_1 \cup \Omega_0. \tag{2.22}$$

Let w_1^+, w_1^- denote the values of $w_1(\mathbf{r})$ calculated for approach to S_0 from Ω_0 and Ω_1, respectively. Then $w_1^+ = u_0 = 0$ and $w_1^- = u_1$. By (2.9) we get

$$\begin{aligned} &w_1^- = w_1^+ = 0 && \text{on } S_0,\\ &q_1 \frac{\partial w_1^-}{\partial n} = q_0 \frac{\partial w_1^+}{\partial n} = 0 && \text{on } S_0. \end{aligned} \tag{2.23}$$

By [7], p. 166, we conclude that $w_1 = 0$ in $\Omega_1 \cup S_0 \cup \Omega_0$, whereby

$$u_1 = 0 \quad \text{in } \Omega_1, \tag{2.24}$$

which completes the proof.

Remark 2.1. If $\operatorname{Im} k_j > 0,\ j = 0,1,2,\ldots$, then by (2.11) and (2.16) we arrive at $u_j =$ const. in $\Omega_j,\ j = 1,2,\ldots$ Since $u_0 = 0$ in Ω_0, by (2.9) we conclude that $u_j = 0$ in $\Omega_j,\ j = 1,2,\ldots$ Hence the above procedure can be omitted in the case $k_j^2 \in \mathbb{C} - \mathbb{R},\ j = 1,2,\ldots$

Remark 2.2. Having proved that $u_0 = 0$ in Ω_0, we can proceed to show that (2.24) holds with the following approach as well: By Holmgren's uniqueness theorem ([12]) the solution of the Cauchy problem

$$\begin{aligned} &\Delta u_1 + k_1^2 u_1 = 0 && \text{in } \Omega_1,\\ &u_1 = \frac{\partial u_1}{\partial n} = 0 && \text{on } S_0, \end{aligned}$$

is equal to zero in $\Omega_1 \cap D$, where D is a neighbourhood of any point of S_0. Since u is analytic ([1]) it follows—by the unique continuation principle—that $u_1 = 0$ in Ω_1.

Remark 2.3. Arguing as in the proof of Theorem 2.1, we can prove that the adjoint homogeneous transmission problem, corresponding to (HTP), has only the trivial solution.

Consider now the non-homogeneous transmission problem

$$\begin{aligned} &\Delta u_j + k_j^2 u_j = f_j && \text{in } \Omega_j,\\ &u_{j+1} - u_j = 0 && \text{on } S_j,\\ &q_{j+1} \frac{\partial u_{j+1}}{\partial n} - q_j \frac{\partial u_j}{\partial n} = 0 && \text{on } S_j, \end{aligned}$$

for all $j = 0,1,2,\ldots$, with the homogeneous Dirichlet condition on the surface of the core, and u_0 being assumed to satisfy Sommerfeld's radiation condition (2.4). Suppose also that (2.10) is true. This problem will be denoted by (NH.T.P.).

Let $k(\mathbf{r}) = k_j^2$, $f(\mathbf{r}) = f_j(\mathbf{r})$, $q = q_j$, $u(\mathbf{r}) = u_j(\mathbf{r})$ in Ω_j, $j = 0, 1, 2, \ldots$, and define, as in [4], p. 143,

$$R(\Omega_0) := \{u \in H^1_{\mathrm{loc}}(\overline{\Omega}_0) : u_0 = O(1/r) \quad \text{and } \partial u_0/\partial n - ik_0 u_0 = O(1/r^2),\ r \to \infty\}.$$

As in Section 1, a function $u \in H^1(\Omega) \cap R(\Omega_0)$ will be a generalized solution of (NH.T.P.), for $f \in L^2(\mathbb{R}^3)$, iff

$$\int\limits_{\mathbb{R}^3 - \Omega_{\mathrm{c}}} \Big(\sum_{s=1}^{3} q u_{x_s}(\mathbf{r}) \varphi_{x_s}(\mathbf{r}) - q k(\mathbf{r}) u(\mathbf{r}) \varphi(\mathbf{r}) \Big)\, dv = - \int\limits_{\mathbb{R}^3 - \Omega_{\mathrm{c}}} q f(\mathbf{r}) \varphi(\mathbf{r})\, dv \tag{2.25}$$

for every $\varphi \in H^1(\Omega) \cap R(\Omega_0)$.

Moreover, (NH.T.P.) can be written in the form

$$u + Au = F, \tag{2.26}$$

where, since we are in $H^1(\Omega) \cap R(\Omega_0)$, $A : H^1(\Omega) \cap R(\Omega_0) \to H^1(\Omega) \cap R(\Omega_0)$ is a compact operator ([4]).

We are now in a position to prove

Theorem 2.2. (NH.T.P.) *has a unique* (*classical*) *solution.*

Proof. (HTP) can be written as

$$u + Au = 0. \tag{2.27}$$

The adjoint homogeneous transmission problem can, in turn, be written as

$$w + A^* w = 0. \tag{2.28}$$

By the Fredholm alternative, a necessary and sufficient condition for the existence and uniqueness of a generalized solution of (2.26) is

$$(F, w_m) = 0, \tag{2.29}$$

where w_m, $m = 1, \ldots, s$, are the linearly independent solutions of (2.28). Since the assumptions of the (NH.T.P.) analogue of Theorem 1.6 are valid in our case, the generalized solutions of (2.28) are classical. But then, by Remark 2.3, (2.28) has only the trivial solution, whereby (2.29) is automatically satisfied. Hence, (NH.T.P.) has a unique generalized solution, which, as above, turns to be classical.

Remark 2.4. The results of this section can—in a completely analogous manner—be stated and proved for the corresponding problem with the homogeneous Neumann boundary condition being assumed on the surface of the core, i.e. when the core is rigid.

Acknowledgment. We wish to express our thanks to Professor S. A. Tersenov for valuable discussions.

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Reçu par la Rédaction le 26.9.1994