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Plurisubharmonic saddles

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Abstract. A certain linear growth of the pluricomplex Green function of a bounded convex domain of $\mathbb{C}^N$ at a given boundary point is related to the existence of a certain plurisubharmonic function called a "plurisubharmonic saddle". In view of classical results on the existence of angular derivatives of conformal mappings, for the case of a single complex variable, this allows us to deduce a criterion for the existence of subharmonic saddles.

Introduction. If $\varphi : [-\delta, \delta] \to [0, \infty[$ $(\delta > 0)$ is a convex function with $\varphi(0) = 0$, a *subharmonic saddle* for φ is a subharmonic function u on $\{z \in \mathbb{C} : |z| \le \delta'\}$ $(0 < \delta' \le \delta)$ with $u(z) \le \varphi(\operatorname{Im} z)$ for all $|z| \le \delta'$, $u(0) = 0$, and $u(x) < 0$ for all $x \in [-\delta', \delta']\setminus\{0\}$. In complex analysis the existence of subharmonic saddles for $\varphi(y) = |y|$, $y \in \mathbb{R}$, like the harmonic function $u(z) = -\operatorname{Re} z^2$, is sometimes applied as a technical tool. There are harmonic saddles also for $\varphi(y) = |y|^d$ $(d \ge 1)$. Of course, there is no subharmonic saddle for $\varphi \equiv 0$. We prove

Theorem. *Let $\varphi : [-\delta, \delta] \to [0, \infty[$ be convex with $\varphi(0) = 0$ and with $\varphi(y) = \varphi(-y)$, $|y| \le \delta$. A subharmonic saddle for φ exists if and only if*

$$\int_0^\delta \log \varphi(t)\, dt > -\infty.$$

This result will be deduced from a theorem of Warschawski and Tsuji on the existence of angular derivatives of conformal mappings. The key of this reduction is an observation which we prove for several complex variables: Let φ be a nonnegative convex function defined on a zero neighborhood in $\mathbb{C}^{N-1} \times \mathbb{R}$, with $\varphi(z) = 0$ if and only if $z = 0$, and with $\lim_{z\to 0} \varphi(z)/|z| = 0$. If $\varphi(z) = \varphi(-z)$ for all z, a plurisubharmonic saddle for φ exists if and only if the pluricomplex Green function of every bounded convex domain Ω of

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$\mathbb{C}^N$ has a certain linear growth at each point of the boundary of Ω at which $\partial\Omega$ can be represented as the graph of the Legendre conjugate function φ^* over the supporting hyperplane of $\partial\Omega$ (Proposition 11). This is a type of *local* version of a result of [6], for which the results of Kiselman [1], Lempert [4], and Zakharyuta [10] have been applied. In the course of the proof, for every bounded convex domain $\Omega \subset \mathbb{C}$, we prove that two cones $\mathbb{R}_+ P_H$ and $\mathbb{R}_+ P_H^*$ in $\mathbb{C}$ coincide, where the first is related to the boundary behavior of the complex Green function of Ω and the second is related to the complex Green function of $\mathbb{C}\backslash\overline{\Omega}$ (Proposition 6). The several-variable analogue of this identity does not hold. For convex polyhedra, in general, $\mathbb{R}_+ P_H \subset \mathbb{R}_+ P_H^*$. We give an example of a convex polyhedron in $\mathbb{C}^2$ for which this inclusion is in fact strict (Example 8).

Notations. For $z, w \in \mathbb{C}^N$, we write $\langle z, w\rangle := \sum_{i=1}^N z_i \overline{w}_i$ and $|z| := \langle z, z\rangle^{1/2}$. We put $B_R(a) := \{z \in \mathbb{C}^N : |z - a| \le R\}$ for $R > 0$ and $a \in \mathbb{C}^N$, $S := \{z \in \mathbb{C}^N : |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \ge 0\}$. For each set $F \subset \mathbb{C}^N$ we write $\mathbb{R}_+ F := \{ta : t \ge 0,\ a \in F\}$. Throughout this paper, we identify $\mathbb{C}^N$ and $\mathbb{C}^{N-1}$ with $\mathbb{R}^{2N}$ and $\mathbb{R}^{2N-2}$, respectively. We refer to Schneider [8] for notions from convex analysis.

1. Definition. For $\delta > 0$ let $\varphi : (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0) \to \mathbb{R}_+$ be a convex function with $\varphi(0) = 0$. A plurisubharmonic function u on $B_{\delta'}(0)$ $(0 < \delta' \le \delta)$ is called a *plurisubharmonic saddle* for φ if

(i) $u(0) = 0$,
(ii) $u(0, x_N) < 0$ for all $x_N \in [-\delta', \delta']\backslash\{0\}$,
(iii) $u(z', z_N) \le \varphi(z', \mathrm{Im} z_N)$ for all $(z', z_N) \in (\mathbb{C}^{N-1} \times \mathbb{C}) \cap B_{\delta'}(0)$.

2. Remark. (a) Let $C_1, C_2 > 0$. There is a plurisubharmonic saddle for φ if and only if there is a plurisubharmonic saddle for $C_1\varphi(\cdot/C_2)$.

(b) If $\varphi > 0$ outside the origin and if φ admits a plurisubharmonic saddle u then we may assume that $u < \varphi$ outside the origin (otherwise consider $u/2$).

3. Example. Let $N = 1$. For each $d \ge 1$ there is a (sub)harmonic saddle $u : B_1(0) \to \mathbb{R}$ for $\varphi(y) = |y|^d$. Just choose an even integer $l \ge d$, a sufficiently small $\varepsilon > 0$ and put $u(z) := -\varepsilon \,\mathrm{Re}\, z^l = -\varepsilon r^l \cos(l\theta)$ for all $z = re^{i\theta} \in B_1(0)$.

4. Definition. Let Ω be a bounded convex domain of $\mathbb{C}^N$ with $0 \in \Omega$. By $H : \mathbb{C}^N \to \mathbb{R}_+$ we denote its *support function*, i.e.

$$H(z) := \sup_{w\in\Omega} \mathrm{Re}\langle z, w\rangle, \qquad z \in \mathbb{C}^N.$$

(a) Let $v_H : \mathbb{C}^N \to \mathbb{R}_+$ be the largest plurisubharmonic function on $\mathbb{C}^N$ with $v_H \le H$ and for which $v_H(z) - \log|z|$ remains bounded if $z \in \mathbb{C}^N$ tends to infinity (for the existence see [6]). Since H is positively homogeneous, there is a lower semicontinuous function $C_H : S \to]0,\infty]$ such that

$$P_H := \{z \in \mathbb{C}^N : v_H(z) = H(z)\} = \{\lambda a : a \in S,\ 0 \le \lambda \le 1/C_H(a)\}.$$

(b) Let $v_H^* : \mathbb{C}^N \to \mathbb{R}_+$ be the largest plurisubharmonic function on $\mathbb{C}^N$ with $v_H^* \le H$ and for which $v_H^*(z) - \log|z|$ remains bounded if $z \in \mathbb{C}^N$ tends to zero (for the existence see [7]). Since H is positively homogeneous, there is an upper semicontinuous function $C_H^* : S \to [0,\infty[$ such that

$$P_H^* := \{z \in \mathbb{C}^N : v_H^*(z) = H(z)\} = \{\lambda a : a \in S,\ 1/C_H^*(a) \le \lambda\}.$$

If $N = 1$, and $\psi : \mathbb{D} \to \Omega$ and $\varphi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{\Omega}$ are biholomorphic mappings, then the numbers $C_H(a)$ and $C_H^*(a)$ are closely related to the angular derivatives of ψ and φ, respectively (see [5]–[7]).

NOTATION. If Ω is a bounded convex domain with $0 \in \Omega$, we consider its polar set

$$\Omega^\circ := \{w \in \mathbb{C}^N : \mathrm{Re}\langle z, w\rangle \le 1 \text{ for all } z \in \Omega\}.$$

Ω° is a compact convex set with 0 in its interior. Since we deal with polar sets, we use the following normalization of Ω:

5. PROPOSITION. *Let Ω be a bounded convex domain in $\{z \in \mathbb{C}^N : \mathrm{Re}\, z_N \le 1\}$, such that $0 \in \Omega$ and $(0,1) := (0,\ldots,0,1) \in \partial\Omega$. There are $\varepsilon > 0$ and a continuous convex function $h : (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\varepsilon(0) \to \mathbb{R}_+$ with $h(0) = 0$ and such that*

$$(1)\quad \partial\Omega \cap B_\varepsilon(0,1) = \{(z', 1 - h(z',t) + it) \mid (z',t) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cup B_\varepsilon(0)\}.$$

The polar set Ω° is contained in $\{w \in \mathbb{C}^N : \mathrm{Re}\, w_N \le 1\}$ and $(0,1) \in \partial\Omega^\circ$. There are $\delta > 0$, and a continuous convex function $\varphi : (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0) \to \mathbb{R}_+$ with $\varphi(0) = 0$, such that

$$(2)\quad \partial\Omega^\circ \cap B_\delta(0,1) = \{(w', 1 - \varphi(w',s) + is) \mid (w',s) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0)\}.$$

If $\varphi > 0$ outside the origin, i.e. $\lim_{(z',t)\to 0} h(z',t)/|(z',t)| = 0$ (see Schneider [8], Lemma 2.2.3), *and if in addition $\varphi(w',s) = \varphi(-w',-s)$, or, what is the same, $h(z',t) = h(-z',-t)$, then the following assertions are equivalent*:

(i) *There is a plurisubharmonic saddle for φ.*
(ii) $C_H(0,1) < \infty$.
(iii) $C_H^*(0,1) > 0$.

Proof. Choose $0 < \varepsilon, \delta < 1$ with $B_\varepsilon(0) \subset \Omega \subset B_{1/\delta}(0)$. Then $B_\delta(0) \subset \Omega^\circ$. By the convexity of Ω and Ω°, $\partial\Omega \cap B_\varepsilon(0,1)$ is a graph over $\{(z', 1 + it) : (z',t) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\varepsilon(0)\}$, and $\partial\Omega^\circ \cap B_\delta(0,1)$ is a graph over $\{(w', 1 + is) : (w',s) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0)\}$. Thus h and φ exist.

We may assume that ε and δ are chosen so small that h and φ are bounded from above by $1/2$ on $B_\varepsilon(0)$ and $B_\delta(0)$, respectively.

Since $\Omega^\circ = \{w \in \mathbb{C}^N : H(w) \le 1\}$, it follows that

$$\Gamma := \{\lambda z : \lambda \ge 0,\ z \in \partial\Omega^\circ \times \{1\}\}$$

is the graph of H. Let $G := \Gamma \cap E$ be its intersection with the hyperplane $E := \mathbb{C}^{N-1} \times (1 + i\mathbb{R}) \times \mathbb{R}$. Then there is a convex function $\psi : (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0) \to \mathbb{R}_+$ with $\psi(0) = 0$ and

$$G \cap B_\delta((0,1),1) = \{(z', 1 + is, 1 + \psi(z', s)) \mid (z', s) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0)\}.$$

Let $(z', s) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0)$. The ray from the origin of $\mathbb{C}^N \times \mathbb{R}$ through the point $(z', 1 - \varphi(z', s) + is, 1)$ hits the plane E at the point

$$\lambda(z', 1 - \varphi(z', s) + is, 1) = (\lambda z', 1 + i\lambda s, 1 + \psi(\lambda(z', s)))$$

for some $\lambda \ge 1$. This shows that

$$\frac{\varphi(z', s)}{1 - \varphi(z', s)} = \frac{1}{1 - \varphi(z', s)} - 1 = \lambda - 1 = \psi\bigg(\frac{(z', s)}{1 - \varphi(z', s)}\bigg).$$

Since φ is bounded by $1/2$, for all $(z', s) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0)$ we obtain

$$\varphi(z', s) \le \psi(2(z', s)) \quad \text{and} \quad \psi(z', s) \le 2\varphi(z', s). \tag{3}$$

For the sequel we note that $\{(z, \operatorname{Re} z_N) : z \in \mathbb{C}^N\}$ is a supporting hyperplane for Γ at $((0,1),1)$.

(i)$\Rightarrow$(ii). If there is a plurisubharmonic saddle for φ, then by (3) and Remark 2(a), there is also a saddle u for $\psi((1-\delta)\cdot)/(1+\delta)$. By the hypothesis, (3), and Remark 2(b), we may assume that $u < \psi(\cdot/(1+\delta))(1-\delta)$ outside the origin. We consider the plurisubharmonic function

$$v(z) := u(z - (0,1)) + \operatorname{Re} z_N, \qquad z \in B_\delta(0,1).$$

Then $v(0,1) = 1 = H(0,1)$. If $z = (z', z_N) \in B_\delta(0,1)$, put $\lambda := \operatorname{Re} z_N$ and $w := z/\lambda$. Since $1 - \delta \le \lambda \le 1 + \delta$, it follows that

$$\begin{aligned} v(z) &\le \psi((z', \operatorname{Im} z_N)/(1+\delta))(1-\delta) + \operatorname{Re} z_N \\ &= \psi(\lambda(w', \operatorname{Im} w_N)/(1+\delta))(1-\delta) + \operatorname{Re} z_N \\ &\le \lambda\psi(w', \operatorname{Im} w_N) + \operatorname{Re} z_N = \lambda(H(w) - \operatorname{Re} w_N) + \operatorname{Re} z_N = H(z) \end{aligned}$$

and $v(z) < H(z)$ if $z \ne (0,1)$. By [6], Prop. 1.13, there is $C > 0$ such that $Cv_H(z/C) > v(z)$ for all $z \in \partial B_\delta(0,1)$. Then

$$\widetilde{v}(z) := \begin{cases} Cv_H(z/C) & \text{if } z \in \mathbb{C}^N \setminus B_\delta(0,1), \\ \max\{Cv_H(z/C), v(z)\} & \text{if } z \in B_\delta(0,1), \end{cases}$$

is plurisubharmonic on $\mathbb{C}^N$ with $\widetilde{v} \le H$ and $\widetilde{v}(0,1) = H(0,1)$, and such that $\widetilde{v}(z) - C\log|z|$ remains bounded if $z \in \mathbb{C}^N$ tends to infinity. This shows that $C_H(0,1) \le C$.

(ii)⇒(i). Put $C := C_H(0,1)$ and $v_H(\cdot\,;C) := Cv_H(\cdot/C)$. Then $v_H(\cdot\,;C) \le H$, $v_H(0,1;C) = H(0,1)$ and $v_H(0,s;C) < H(0,s)$ if $s > 1$. Hence the function

$$v(z) := v_H(z;C) - \operatorname{Re} z_N, \qquad z \in \mathbb{C}^N,$$

is plurisubharmonic with $v(0,1) = 0$ and with $v(0,s) < 0$ for all $s > 1$. Moreover, as in "(i)⇒(ii)" we obtain

$$v(z) \le H(z) - \operatorname{Re} z_N = \lambda\psi(w', \operatorname{Im} w_N) \le (1+\delta)\psi((z', \operatorname{Im} z_N)/(1-\delta))$$

for all $z \in B_\delta((0,1))$. This shows that

$$u(z) := v(z+(0,1)) + v(-z+(0,1)), \qquad z = (z', z_N) \in \mathbb{C}^N,$$

is a plurisubharmonic saddle for $2(1+\delta)\psi(\cdot/(1-\delta))$. Hence by Remark 2(a) and by (3), there is a plurisubharmonic saddle for φ.

(i)⇔(iii). As (i)⇔(ii). Just apply [7] instead of [6].

A corollary to the proof of Proposition 5 is the following:

6. Proposition. *For $N = 1$ let Ω be a bounded convex domain in $\mathbb{C}$ which contains the origin. Then for each $a \in S$, $C_H(a) < \infty$ if and only if $C_H^*(a) > 0$, i.e. $\mathbb{R}_+P_H = \mathbb{R}_+P_H^*$.*

Proof. For $N = 1$, in the proof of (ii)⇒(i) of Proposition 5, we may replace u by

$$u(z) := v(z+1) + v(-\operatorname{Re} z + i \operatorname{Im} z + 1), \qquad z \in \mathbb{C},$$

which is a subharmonic saddle for $2(1+\delta)\psi(\cdot/(1-\delta))$. This shows that for $N = 1$, we need no assumption on the symmetry of φ.

Furthermore, since a nonnegative subharmonic function u on a domain is negative everywhere if it is negative at some point, for $N = 1$ each subharmonic saddle u for φ is proper in the sense that $u < \varphi$ outside the origin (see Remark 2(b)). This shows that for $N = 1$, we need no assumption on the smoothness of $\partial\Omega$.

For $N > 1$, the assertion of Proposition 6 does not hold. To give an example, first we recall from [6], Thm. 2.11 (see also Krivosheev [3]), and from [7] a result which compares the cones $\mathbb{R}_+P_H$, $\mathbb{R}_+P_H^*$, and $\operatorname{supp}(dd^cH)^N$, which is defined to be the smallest closed subset of $\mathbb{C}^N$ for which H is a maximal plurisubharmonic function on its complement (see Klimek [2]). By the homogeneity of H, this is a cone.

7. Proposition. *Let $\Omega \subset \mathbb{C}^N$ be an open bounded convex polyhedron which contains the origin. Then*

$$\operatorname{supp}(dd^cH)^N \subset \mathbb{R}_+ P_H \subset \mathbb{R}_+ P_H^*,$$

where equalities hold for $N = 1$. More precisely: Let $a \in S$ belong to the relative interior of the cone $\mathbb{R}_+ F$ for some face F of $\partial\Omega^\circ$. Let $L(F)$ denote *the* $\mathbb{R}$*-linear span of* $\mathbb{R}_+ F$*. Then*

$$a \in \operatorname{supp}(dd^c H)^N \Leftrightarrow L(F) \cap iL(F) = \{0\};$$
$$C_H(a) < \infty \Leftrightarrow \mathbb{R}_+ F \cap (L(F) \cap iL(F)) = \{0\},$$
$$C_H^*(a) > 0 \Leftrightarrow \mathbb{R}_+ a \cap (L(F) \cap iL(F)) = \{0\}.$$

As the following example shows, for $N \ge 2$, in general, both inclusions of Proposition 7 are strict.

8. Example. Let

$$\Omega := \Big\{ z = (z_1, z_2) = (x_1, x_2, x_3, x_4) \in \mathbb{C}^2 : \sum_{j=1}^{4} |x_j| < 1 \Big\}.$$

Its support function $H : \mathbb{C}^2 \to \mathbb{R}_+$ is given by

$$H(z) = H(x_1, \ldots, x_4) = \max_{j=1,\ldots,4} |x_j|, \qquad z \in \mathbb{C}^2.$$

It has been proved in [6], 2.13, that $\operatorname{supp}(dd^c H)^N \neq \mathbb{R}_+ P_H$. Moreover, for the face $F := \{z \in \mathbb{C}^2 : x_1 = x_4 = 1\} \cap \partial\Omega^\circ$ of $\partial\Omega^\circ = \{z \in \mathbb{C}^2 : H(z) = 1\}$, it has been calculated that $L(F) \cap iL(F) = \mathbb{R}(1,0,0,1) + \mathbb{R}(0,1,1,0)$ and that $\mathbb{R}_+ F \cap (L(F) \cap iL(F)) \neq \{0\}$. Since $\mathbb{R}_+ F$ has dimension 3, there exists a in the relative interior of $\mathbb{R}_+ F$ with $a \notin L(F) \cap iL(F)$ and $a \in S$. Thus $\mathbb{R}_+ a \cap (L(F) \cap iL(F)) = \{0\}$, and by Proposition 7, this gives $a \in \mathbb{R}_+ P_H^* \setminus \mathbb{R}_+ P_H$.

Notation. Let $\varepsilon > 0$ and $h : (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\varepsilon(0) \to \mathbb{R}_+$ be a continuous convex function with $h(0) = 0$. We extend h to a convex function on $\mathbb{C}^{N-1} \times \mathbb{R}$ by $h(w', t) := \infty$ whenever $(w', t) \notin B_\varepsilon(0)$. Its *conjugate function* $h^* : \mathbb{C}^{N-1} \times \mathbb{R} \to \mathbb{R}_+$ is defined by

$$h^*(z', s) := \sup_{(w',t) \in \mathbb{C}^{N-1} \times \mathbb{R}} (\operatorname{Re}\langle z', w' \rangle + st - h(w', t)), \quad (z', s) \in \mathbb{C}^{N-1} \times \mathbb{R}.$$

h^* is again a convex function with $h^*(0) = 0$. Moreover, $h^{**} = h$ (see Schneider [8], Thm. 1.6.5).

9. Remark. If h_j, $j = 1, 2$, are two convex functions which coincide on $(\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\varepsilon(0)$ for some $\varepsilon > 0$, vanish in 0 and are positive outside 0, then there is $\delta > 0$ such that h_j^*, $j = 1, 2$, coincide on $(\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0)$.

Proof. Since $h_j > 0$ outside the origin, we may choose $0 < \delta \le \min_{|a|=\varepsilon} h_j(a)/\varepsilon$, $j = 1, 2$. Fix $j = 1, 2$ and let $(z', s) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0)$. If $(w', t) \in (\mathbb{C}^{N-1} \times \mathbb{R}) \setminus B_\varepsilon(0)$, we put

$$(\widetilde{w}', \widetilde{t}) := \varepsilon (w', t)/|(w', t)| \in B_\varepsilon(0)$$

and get $|(\widetilde{w}', \widetilde{t})| \le |(w', t)|$. By the convexity of h_j we obtain

$$h_j(\widetilde{w}', \widetilde{t})/|(\widetilde{w}', \widetilde{t})| \le h_j(w', t)/|(w', t)|.$$

Since
$$|\mathrm{Re}\langle (z',s),(\widetilde{w}',\widetilde{t})\rangle/\varepsilon| \le \delta \le h_j(\widetilde{w}',\widetilde{t})/|(\widetilde{w}',\widetilde{t})|,$$
we get
$$\begin{aligned}\mathrm{Re}\langle z',w'\rangle + st - h_j(w',t) &= |(w',t)|(\mathrm{Re}\langle (z',s),(\widetilde{w}',\widetilde{t})\rangle/\varepsilon - h_j(w',t)/|(w',t)|)\\ &\le \varepsilon(\mathrm{Re}\langle (z',s),(\widetilde{w}',\widetilde{t})\rangle/\varepsilon - h_j(\widetilde{w}',\widetilde{t})/|(\widetilde{w}',\widetilde{t})|)\\ &= \mathrm{Re}\langle z',\widetilde{w}'\rangle + s\widetilde{t} - h_j(\widetilde{w}',\widetilde{t}).\end{aligned}$$
This shows that
$$\begin{aligned}h_j^*(z',s) &= \sup_{(w',t)\in\mathbb{C}^{N-1}\times\mathbb{R}} (\mathrm{Re}\langle z',w'\rangle + st - h_j(w',t))\\ &= \sup_{(w',t)\in(\mathbb{C}^{N-1}\times\mathbb{R})\cap B_\varepsilon(0)} (\mathrm{Re}\langle z',w'\rangle + st - h_j(w',t)).\end{aligned}$$
Hence $h_1^* = h_2^*$ on $(\mathbb{C}^{N-1}\times\mathbb{R})\cap B_\delta(0)$.

10. Lemma. *Let h, φ, ε, and δ be as in Proposition* 5. *Assume that $h > 0$ outside the origin. Then there is $0 < \delta' \le \delta$ such that for all $(z',s) \in (\mathbb{C}^{N-1}\times\mathbb{R})\cap B_\delta(0)$,*
$$h^*(z',s) \le \varphi(z',s) \le 2h^*(z',s).$$

Proof. Since $h > 0$ outside the origin, we can choose $0 < \delta' \le \delta$ such that

(4) $$\begin{aligned}&\Omega^\circ \cap B_{\delta'}(0,1)\\ &\quad= \{z\in\mathbb{C}^N : \mathrm{Re}\langle w,z\rangle \le 1 \text{ for all } w \in \partial\Omega\cap B_\varepsilon(0,1)\}\cap B_{\delta'}(0,1).\end{aligned}$$

Let $(z',s) \in (\mathbb{C}^{N-1}\times\mathbb{R})\cap B_{\delta'}(0)$. Then $a := (z', 1-\varphi(z',s)+is) \in \partial\Omega^\circ$, by (2). Thus by the definition of Ω°, by (4) and (1), we have
$$\begin{aligned}1 &= \sup_{w\in\partial\Omega\cap B_\varepsilon(0,1)} \mathrm{Re}\langle w,a\rangle\\ &= \sup_{(w',t)\in(\mathbb{C}^{N-1}\times\mathbb{R})\cap B_\varepsilon(0)} (\mathrm{Re}\langle w',z'\rangle + (1-h(w',t))(1-\varphi(z',s)) + ts).\end{aligned}$$
Hence
$$\begin{aligned}1-\varphi(z',s) &= \inf_{(w',t)\in(\mathbb{C}^{N-1}\times\mathbb{R})\cap B_\varepsilon(0)} \frac{1-\mathrm{Re}\langle w',z'\rangle - ts}{1-h(w',t)}\\ &= 1 - \sup_{(w',t)\in(\mathbb{C}^{N-1}\times\mathbb{R})\cap B_\varepsilon(0)} \frac{\mathrm{Re}\langle w',z'\rangle + ts - h(w',t)}{1-h(w',t)}.\end{aligned}$$
Since we may assume that $0 \le h(w',t) \le 1/2$, we obtain $h^*(z',s) \le \varphi(z',s) \le 2h^*(z',s)$.

Notation. Let Ω be a bounded convex domain of $\mathbb{C}^N$ and fix $w_0 \in \Omega$. By g_Ω we denote the *pluricomplex Green function* of Ω with pole at w_0, i.e. g_Ω is the largest negative plurisubharmonic function on Ω for which

$g_\Omega(z) - \log|z - w_0|$, $z \in \Omega\backslash\{w_0\}$, is bounded (for the existence see Klimek [2]). We consider the level sets $\Omega_x := \{z \in \Omega : g_\Omega(z) < x\}$, $x < 0$, which are convex by a result of Lempert (see [6], Lemma 1.2). By $H_x : \mathbb{C}^N \to \mathbb{R}$ we denote their support functions

$$H_x(z) := \sup_{w\in\Omega_x} \mathrm{Re}\langle z, w\rangle, \quad z \in \mathbb{C}^N,\ x < 0.$$

Then (see [6], Prop. 1.3) the limits

$$D_\Omega(a) := \lim_{x\uparrow 0} \frac{H(a) - H_x(a)}{-x} \in\ ]0,\infty], \quad a \in S,$$

exist. By [6], Thms. 1.14 and 1.20, there is $C > 0$ with $C_H \le D_\Omega \le CC_H$.

11. Proposition. *Let $\Omega \subset \mathbb{C}^N$ be a bounded convex domain normalized as in Proposition* 5. *Let h and φ be convex functions as there. In addition, assume that also $h > 0$ outside the origin. If g_Ω is the pluricomplex Green function of Ω with pole at* 0, *the following are equivalent*:

(i) *There is a plurisubharmonic saddle for h^*.*
(ii) *There is a plurisubharmonic saddle for φ.*
(iii) $D_\Omega(0,\ldots,0,1) < \infty$.

Proof. (i)⇔(ii). Since $h > 0$ outside the origin, this follows from the remark in Proposition 5, Lemma 10, and Remark 2(a).

(ii)⇔(iii). By the hypothesis and by the remark in Proposition 5, we have $\varphi > 0$ outside the origin. Hence we deduce from Proposition 5 that (ii) holds if and only if $C_H(0,\ldots,0,1) < \infty$. By [6], Thms. 1.14 and 1.20, this is equivalent to (iii).

For $N = 1$ there is a close relation between the limits $D_\Omega(a)$ and the angular derivatives of the Riemann conformal mappings from the unit disc $\mathbb{D}$ onto Ω. This relationship is applied in the proof of the following lemma.

12. Lemma. *Let Ω be a bounded convex domain of $\mathbb{C}^N$. Let $w \in \partial\Omega$ and let $a \in S$ be an outer normal to $\partial\Omega$ at w. Put $\Omega^1 := \{z \in \mathbb{C} : zia + w \in \Omega\}$ and let Ω^2 be the set of all $z \in \mathbb{C}$ such that $zia + w$ is contained in the image of the orthogonal projection of Ω onto $\mathbb{C}a + w$. Then $\Omega^1 \subset \Omega^2 \subset \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. Assume that there are $\varepsilon > 0$ and convex functions $h_j : [-\varepsilon,\varepsilon] \to \mathbb{R}_+$ with $h_j(0) = 0$ and*

$$\partial\Omega^j \cap B_\varepsilon(0) = \{t + ih_j(t) : t \in [-\varepsilon,\varepsilon]\}, \quad j = 1, 2.$$

Let g_Ω be the pluricomplex Green function of Ω with pole at some fixed $w_0 \in \Omega$. If $\int_{-\varepsilon}^{\varepsilon}(h_1(t)/t^2)\,dt < \infty$ then $D_\Omega(a) < \infty$. If $D_\Omega(a) < \infty$ then $\int_{-\varepsilon}^{\varepsilon}(\widetilde{h}_2(t)/t^2)\,dt < \infty$, where $\widetilde{h}_2(t) := \min\{h_2(t), h_2(-t)\}$, $|t| \le \varepsilon$.

Proof. After a translation followed by a unitary transformation of $\mathbb{C}^N$, we may assume that $a = (0,\ldots,0,-i)$ and $w = 0$. Since the finiteness of $D_\Omega(a)$ does not depend on the choice of the pole, we may assume that $w_0 \in \mathbb{C}a+w = \{0\}\times\mathbb{C}$, i.e. $w_0 = (0, w_0')$. Since $\{0\}\times\Omega^1 \subset \Omega \subset \mathbb{C}^{N-1}\times\Omega^2$, for the complex Green functions g_{Ω^j} of Ω^j, $j = 1, 2$, with pole at w_0', the following holds:

$$g_{\Omega^1}(z_N) \geq g_\Omega(0, z_N), \qquad z_N \in \Omega^1,$$

and

$$g_\Omega(z', z_N) \geq g_{\Omega^2}(z_N), \qquad (z', z_N) \in \mathbb{C}^{N-1}\times\Omega^2.$$

Hence for the corresponding level sets Ω_x^j, $x < 0$, $j = 1, 2$, we obtain

$$\{0\}\times\Omega_x^1 \subset \Omega_x \subset \mathbb{C}^{N-1}\times\Omega_x^2.$$

This shows that $H_x^1(-i) \leq H(a) \leq H_x^2(-i)$, $x < 0$, for the corresponding support functions. Hence $D_{\Omega^2}(-i) \leq D_\Omega(a) \leq D_{\Omega^1}(-i)$.

Now the assertion follows from [5], Lemma 3.3, Ex. 4.2, [6], Lemma 2.3, and a result of Warschawski and Tsuji (see Tsuji [9], Thm. IX.10).

13. Lemma. *Let $\varepsilon > 0$ and let $h : [-\varepsilon, \varepsilon] \to \mathbb{R}_+$ be a convex function with $h(0) = 0$.*

(a) *If $h(t) = tq(t)$, $t > 0$, with $\lim_{t\downarrow 0} q(t) = 0$, such that $q(t)$ is strictly increasing for $t > 0$, then there is $\delta > 0$ such that for all $0 < s \leq \delta$,*

$$2q^{-1}(s)s \geq h^*(s) \geq q^{-1}(s/2)s/2.$$

(b) *$\int_{-\varepsilon}^{\varepsilon}(h(t)/t^2)\,dt < \infty$ if and only if $\int_{-\delta}^{\delta}\log h^*(s)\,ds > -\infty$ for some $\delta > 0$.*

Proof. (a) Since $q(t)$ is strictly increasing, we have $h(t) > 0$ for all $0 < t \leq \varepsilon$. Fix $0 < s \leq \delta$. Since $st - tq(t) \leq 0$ for all $q^{-1}(s) \leq t \leq \varepsilon$, we obtain

$$h^*(s) = \sup_{0\leq t\leq q^{-1}(s)}(st - h(t)) \leq \sup_{0\leq t\leq q^{-1}(s)} st \leq sq^{-1}(s).$$

Let $0 < s \leq \delta := 2q(\varepsilon)$. Then $q^{-1}(s/2) \leq \varepsilon$ and

$$h^*(s) \geq sq^{-1}(s/2) - h(q^{-1}(s/2)) = sq^{-1}(s/2)/2.$$

(b) For the proof we have to consider the integrals over negative and positive numbers separately. It is no restriction to consider the positive ones only. If $q(t) := h(t)/t = c$ is constant for all $t > 0$ in a neighborhood of 0, then the assertion obviously holds (we have to distinguish the cases $c = 0$ and $c > 0$). Otherwise the map $t \mapsto q(t)$ is strictly increasing for $0 < t \leq \varepsilon$, with $\lim_{t\downarrow 0} q(t) = 0$. We claim that

$$(5)\qquad \int_\eta^\delta \log(sq^{-1}(s))\,ds + \int_\eta^{q^{-1}(\delta)} \frac{q(t)}{t}\,dt = \delta\log(\delta q^{-1}(\delta)) - q^{-1}(\delta) - q(\eta)\log(q(\eta)\eta)$$

for all $0 < \eta < \delta$. Fix η. Since we may approximate the continuous function q uniformly on $[\eta, \delta]$ by strictly increasing C^1-functions, we may assume that q itself is of class C^1. We obtain

$$\begin{aligned}\int_\eta^\delta \log(sq^{-1}(s))\,ds &= \int_0^{q^{-1}(\delta)} \log(q(t)t)q'(t)\,dt \\ &= [\log(q(t)t)q(t)]_\eta^{q^{-1}(\delta)} - \Big(\int_\eta^{q^{-1}(\delta)} q'(t)\,dt + \int_\eta^{q^{-1}(\delta)} \frac{q(t)\,dt}{t}\Big).\end{aligned}$$

This proves (5).

Let $\int_0^\varepsilon (h(t)/t^2)\,dt < \infty$. Since $\limsup_{t\downarrow 0} q(t)\log(q(t)t) \le 0$, we deduce from (5) letting $\eta \downarrow 0$ that $\int_0^\delta \log(sq^{-1}(s))\,ds > -\infty$. This proves $\int_0^\delta \log h^*(s)\,ds > -\infty$.

Let $\int_0^\delta \log h^*(s)\,ds > -\infty$, i.e. $\int_0^\delta \log(sq^{-1}(s))\,ds > -\infty$. Since for all $0 < t < \varepsilon$ we have $q(t) \le \delta$ and

$$-\infty < \int_0^\delta \log(sq^{-1}(s))\,ds \le \int_0^{q(t)} \log(sq^{-1}(s))\,ds \le q(t)\log(q(t)t),$$

we get $\liminf_{t\downarrow 0} q(t)\log(q(t)t) > -\infty$. Hence by (5), we get $\int_0^\varepsilon (h(t)/t^2)\,dt < \infty$.

14. Proposition. *Let $N = 1$, $\delta > 0$, and let $\varphi : [-\delta, \delta] \to \mathbb{R}_+$ be convex with $\varphi(0) = 0$ and with $\varphi(y) = \varphi(-y)$, $|y| \le \delta$. There is a subharmonic saddle for φ if and only if*

$$\int_0^\delta \log\varphi(t)\,dt > -\infty.$$

Proof. If $\varphi = 0$ in a neighborhood of 0, the integral equals $-\infty$, and by the maximum principle, there is no subharmonic saddle for φ. If $\varphi(y) = c|y|$ in a neighborhood of 0, the integral converges, and by Example 3, there is a subharmonic saddle for φ. Thus we may assume that $\varphi > 0$ outside the origin and that $\lim_{y\to 0} \varphi(y)/|y| = 0$. We choose a bounded convex domain Ω in $\mathbb{C}$ such that (2) holds. By Proposition 11 (and the remark in Proposition 5), there is a subharmonic saddle for φ if and only if $D_\Omega(1) < \infty$. By Lemmas 12, 10 and 13, this is equivalent to $\int_0^\delta \log\varphi(t)\,dt > -\infty$.

15. Remark. Let $\delta > 0$ and let $\varphi : (\mathbb{C}^{N-1} \times \mathbb{R}) \cap B_\delta(0) \to \mathbb{R}_+$ be a convex function with $\varphi(0) = 0$, $\varphi > 0$ outside the origin, and $\varphi(y) = \varphi(-y)$ for all y. If there is no plurisubharmonic saddle for φ, then the following holds:

Each plurisubharmonic function u on $B_{\delta'}(0)$ $(0 < \delta' \le \delta)$ which satisfies

$$u(z', z_N) \le \varphi(z', \operatorname{Im} z_N), \qquad z = (z', z_N) \in \mathbb{C}^{N-1} \times \mathbb{C},$$

vanishes on $\{0\} \times \mathbb{R}$ if $u(0) = 0$.

Proof. Consider $[a, b] := \{s \in [\delta', \delta'] : u(0, s) = 0\}$. If for example $b \ne \delta'$, then

$$v(z) := \tfrac{1}{2}(u(z + (0, b)) + u(-z + (0, b)), \qquad z = (z', z_N) \in \mathbb{C}^{N-1} \times \mathbb{C},$$

would be a plurisubharmonic saddle for φ.

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