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Unbounded solutions of positively damped Liénard equations

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Abstract. This paper discusses the asymptotic behavior of solutions of the Liénard equation, especially the global behavior of unbounded solutions, and also gives a class of sufficient and necessary conditions for the orbit of a solution to intersect the vertical isocline.

1. Introduction. In this article we are concerned with the global asymptotic behavior of solutions of the scalar Liénard equation

$$x'' + f(x)x' + g(x) = 0 \qquad ({}' = d/dt), \tag{1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy $f(x) > 0$ for all x and $xg(x) > 0$ for $x \neq 0$. We also assume the regularity for $f(x)$ and $g(x)$ which ensures the existence of a unique solution to the initial value problem.

It is easy to see that the only critical point $(0,0)$ of the equivalent system

$$x' = y, \qquad y' = -f(x)y - g(x) \tag{2}$$

is uniformly asymptotically stable, and is globally uniformly asymptotically stable if $\int_0^x g(s)\,ds \to \infty$ as $x \to \infty$ and $x \to -\infty$, or $\int_0^x f(s)\,ds \to \infty$ $(-\infty)$ as $x \to \infty$ $(-\infty)$.

Seifert [1] gives a class of systems (2) for which there exist unbounded solutions which certainly do not approach $(0,0)$ as $t \to \infty$. If $(x(t), y(t))$ solves (2) with $(x(0), y(0)) = (0, a)$, Seifert's main result [1, Theorem 2] says there exist a_0 and a_1, $0 < a_0 \le a_1 \le \infty$, such that:

(i) $a \ge a_1$ implies $y(t) > 0$ for $t \ge 0$ and $\lim_{t\to\infty}(x(t), y(t)) = (\infty, L(a))$.

(ii) $a_0 \le a < a_1$ implies there exist $t_1(a) > 0$ and $L(a) \le 0$ such that $y(t) > 0$ for $0 \le t < t_1(a)$, $x(t_1(a)) > 0$, $y(t_1(a)) = 0$, $y(t) < 0$ for $t > t_1(a)$, and $\lim_{t\to\infty}(x(t), y(t)) = (-\infty, L(a))$.

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(iii) $0 \le a < a_0$ implies $\lim_{t\to\infty}(x(t), y(t)) = (0,0)$.

Concerning the function $L : \mathbb{R}^+ \to \mathbb{R}$, Seifert [1] proposed the following questions:

(I) Can $L(a_1) > 0$? If so, under what conditions will $L(a_1) = 0$?

(II) Is $L(a)$ strictly increasing for $a \ge a_1$? Again, if not, are there conditions under which it is?

We note that (1) or (2) has another equivalent system

$$x' = y - F(x), \qquad y' = -g(x), \tag{3}$$

where $F(x) = \int_0^x f(s)\,ds$. It is also easy to see that the existence of $a_1 < \infty$ is closely related to the intersection of orbits of (3) and the vertical isocline $y = F(x)$.

In Section 2, we give a simple discussion concerning the relation of systems (2) and (3).

In Section 3 we present a counterexample to a conclusion of Villari [2, Theorem 1], which is also valid for [3, Theorem 2.1]. We give corrections to these theorems and improve the result of [1, Theorem 1].

In Section 4, we answer Seifert's questions completely, i.e., we show that $L(a_1) = 0$ and $L(a)$ is strictly increasing for $a \ge a_1$.

2. Conjugacy. Put $x = u$, $y = v - F(u)$ into (2). We have

$$u' = v - F(u), \qquad v' = -g(u). \tag{4}$$

Define $H : \mathbb{R}^2 \to \mathbb{R}^2$ by $H(x,y) = (x, y + F(x))$. Obviously, H is an isometric homeomorphism and takes orbits of (2) to orbits of (4) (or (3)) preserving their orientation and the parameter t, that is, systems (2) and (3) are conjugate. We note that the y-axis of the phase plane of system (2) stays invariant under H, but the x-axis turns to be the vertical isocline $y = F(x)$ of (3), which we denote by α.

If $(x(t), y(t))$ solves (3) with $(x(0), y(0)) = (0, a)$, let $P = (0, a)$ and denote by γ^+ the positive semi-orbit of $(x(t), y(t))$. The basic condition $f(x) > 0$ for all x implies $F(x) = \int_0^x f(s)\,ds > 0$ for $x > 0$. The monotonicity of solutions of (3) in the variant regions of the phase plane easily leads to the conclusion that $F(\infty) = \int_0^\infty f(s)\,ds < \infty$ is a necessary condition for $\gamma^+(P)$ ($P = (0,a)$, $a > 0$) not to intersect the vertical isocline α.

PROPOSITION 1. *If $f(x) > 0$ for all x, then $a_1 < \infty$ implies*

$$\int_0^\infty f(s)\,ds < \infty \quad \textit{and} \quad \int_0^\infty g(s)\,ds < \infty.$$

Proof. We only need to prove $\int_0^\infty g(s)\,ds < \infty$. Otherwise, it is easy to see that the curves defined by

$$V(x,y) = \tfrac{1}{2}y^2 + G(x) = \text{constant}$$

are closed, and since $V' = -f(x(t))y^2(t)$ along the solution $(x(t), y(t))$ of system (2), the orbits of (2) are bounded by these closed curves and guided to the positive x-axis. ∎

Now by means of H one may thus restate Seifert's questions as follows:

(I) Can $L_1(a_1) > F(\infty)$ $(< \infty)$? $(L_1(a) = L(a) + F(\infty))$.
(II) Is $L_1(a)$ strictly increasing for $a \geq a_1$?

3. An example. For system (3) Villari [2, Theorem 1] proves:

THEOREM A. *Let $F(x) > -c > -\infty$ for $x > 0$. For every (x_0, y) with $x_0 \geq 0$ and $y > F(x_0)$, the orbit of* (3) *which passes through (x_0, y) intersects the curve $y = F(x)$ at $(x, F(x))$ with $x > x_0$ if and only if*

$$\limsup_{x\to\infty}[G(x) + F(x)] = \infty.$$

As a counterexample to the theorem we consider a concrete Liénard system

$$x' = y - (1 - e^{-x}), \qquad y' = e^{-2x}, \tag{5}$$

so $F(x) = 1 - e^{-x}$ and $G(x) = \int_0^x g(s)\,ds = (1 - e^{-2x})/2$. Then

$$\lim_{x\to\infty}[G(x) + F(x)] = \lim_{x\to\infty}[1 - e^{-x} + (1 - e^{-2x})/2] = 3/2 < \infty.$$

But for $P = (0, a)$ with $0 < a < 1$, $\gamma^+(P)$ must intersect the curve $y = F(x)$.

Let

$$K = \sup(F(x) : x \geq 0), \qquad P = \lim_{x\to\infty} G(x),$$
$$K' = \inf(F(x) : x < 0), \qquad P' = \lim_{x\to-\infty} G(x),$$

where K, P and P' may be ∞ and K' may be $-\infty$. We derive the following result as a remedy for Theorem A.

THEOREM 1. *Suppose $F(x) > -c > -\infty$ for $x > 0$ and $F(x) < c < \infty$ for $x < 0$. For every (x_0, y) with $x_0 \geq 0$ and $y > K + (2P)^{1/2}$ the orbit of system* (3) *passing through (x_0, y) intersects the curve $y = F(x)$ at $(x, F(x))$ with $x > x_0$ if and only if $\limsup_{x\to\infty}[G(x) + F(x)] = \infty$.*

For every (x_0, y) with $x_0 < 0$ and $y < K' - (2P')^{1/2}$ the orbit of system (3) *passing through (x_0, y) intersects the curve $y = F(x)$ at $(x, F(x))$ with $x < x_0$ if and only if $\limsup_{x\to-\infty}[G(x) - F(x)] = \infty$.*

Proof. Let α be the curve $y = F(x)$. We only consider the case $y > K + (2P)^{1/2}$ with $x_0 \geq 0$.

Assume that $\limsup_{x\to\infty}[G(x) + F(x)] < \infty$. This implies that $-c < F(x) \leq K < \infty$ and $0 < G(x) < P < \infty$ for $x > x_0$.

Consider the curves defined by

$$V(x,y) = \tfrac{1}{2}(y-K)^2 + G(x) = \text{constant}\,.$$

It is easy to see that if $G(x)$ has no upper bound these curves are closed, but if $G(x) < P$ the curves which intersect the y-axis with $y > K + (2P)^{1/2}$ do not intersect the line $y = K$.

The time rate of change of V along a solution orbit is given by

$$V' = g(x)[K - F(x)].$$

Since $F(x) \le K$, in $x > x_0$ the orbits of system (3) do not cross these curves from their exteriors to their interiors. Thus, if $y > K + (2P)^{1/2}$ the orbit of system (3) which passes through (x_0, y) is bounded away from α.

Now assume that $\limsup_{x\to\infty}[G(x) + F(x)] = \infty$.

If $\limsup_{x\to\infty} F(x) = \infty$, the orbit of system (3) which passes through (x_0, y) with $y_0 > F(x_0)$ obviously intersects α.

If $\limsup_{x\to\infty} G(x) = \infty$, consider the closed nested ovals

$$W(x,y) = \tfrac{1}{2}(y+c)^2 + G(x) = \text{constant}\,.$$

Since $W' = -g(x)[F(x) + c] < 0$ if $x > x_0$, the orbit passing through (x_0, y) is bounded by the same ovals and guided to α.

In exactly the same way we can treat the case $y < K' - (2P')^{1/2}$ with $x_0 < 0$.

R e m a r k 1. The condition $f(x) > 0$ for all x is unnecessary in Theorem 1.

R e m a r k 2. The result of [3, Theorem 2.1] requires a modification as in our Theorem 1.

By Theorem 1, we easily conclude:

THEOREM 2. *Suppose $f(x) > 0$ for all x. Then there exist unbounded solutions of system* (3) *if and only if*

$$\lim_{x\to\infty}[G(x) + F(x)] < \infty \quad \textit{or} \quad \lim_{x\to-\infty}[G(x) - F(x)] < \infty. \tag{6}$$

P r o o f. Sufficiency follows from Theorem 1. We only need to prove the necessity. Assume $\lim_{x\to\infty}[G(x) + F(x)] < \infty$ and $\lim_{x\to-\infty}[G(x) + F(x)] < \infty$. We consider the positive semi-orbit γ^+ of system (3) which passes through (x_0, y_0) with $y_0 > F(x_0)$ and $x_0 \ge 0$. If $\lim_{x\to\infty} F(x) = \infty$, the monotonicity of solutions in the phase plane implies γ^+ intersects the curve α. If $\lim_{x\to\infty} G(x) = \infty$, consider the closed nested ovals

$$V(x,y) = \tfrac{1}{2}y^2 + G(x) = \text{constant}\,.$$

Since $V' = -g(x)F(x) < 0$ for $x > x_0$, γ^+ is bounded by these ovals and guided to α. For (x_0, y_0) with $x_0 < 0$ and $y_0 > F(x_0)$, from $dy/dx = -g(x)/(y - F(x))$, it is easy to see that γ^+ intersects the positive y-axis.

In exactly the same way we can treat the case $y_0 < F(x_0)$. Thus, we conclude that for every $P = (x_0, y_0) \in \mathbb{R}^2$, $\gamma^+(P)$ encircles the origin $(0,0)$. Moreover, $dV(x(t), y(t))/dt = -g(x(t))F(x(t))$ implies that γ^+ tends to $(0,0)$ as $t \to \infty$, that is, all solutions of (3) are bounded. This completes the proof. ∎

Remark 3. Theorem 2 improves [1, Theorem 1].

4. The functions $L_1(a)$ and $L(a)$. To answer Seifert's first question, we use its restatement in Section 2.

THEOREM 3. $L_1(a_1) = F(\infty) = \int_0^\infty f(s)\,ds$ $(< \infty)$.

Proof. By Proposition 1 of Section 2 let $K = F(\infty)$ and $H = G(\infty)$, and suppose $L_1(a_1) > F(\infty)$. We fix $\varepsilon = (L_1(a_1) - K)/2 > 0$. Denote by β the upper component $y = \varphi(x)$ of $(y-K)^2/2 + G(x) = H$. It easily follows that $\lim_{x\to\infty} \varphi(x) = K$, which implies that there exists a sufficiently large x_0 satisfying $\varepsilon^2/2 + G(x_0) = H$. Let $P = (x_0, \varepsilon + K)$. Because the orbits of (3) cross β upwards, the negative semi-orbit $\gamma^-(P)$ passing through P will intersect the y-axis at $Q = (0,k)$ $(k > 0)$, and $\gamma^+(P)$ does not intersect the curve $y = K$. Thus, we easily obtain $k > a_1$. On the other hand, the monotonicity of $\gamma^+(P)$ implies $L_1(k) < K + \varepsilon < L_1(a_1)$, which contradicts the definition of a_1. ∎

To answer Seifert's second question we directly use the system (2).

THEOREM 4. *$L(a)$ is strictly increasing for $a \ge a_1$.*

Proof. Let $e > k \ge a_1$, and denote by $y = y_1(x)$, $y = y_2(x)$ respectively the solutions of system (2) which pass through $(0,e)$ and $(0,k)$, that is,

$$\frac{dy_1(x)}{dx} = -f(x) - \frac{g(x)}{y_1(x)}, \quad \frac{dy_2(x)}{dx} = -f(x) - \frac{g(x)}{y_2(x)}.$$

Therefore

$$\frac{d(y_1(x) - y_2(x))}{dx} = \frac{g(x)}{y_1(x)y_2(x)}(y_1(x) - y_2(x)).$$

Hence $y_1(x) - y_2(x)$ is increasing as x increases, which leads to

$$L(e) - L(k) > e - k > 0. \tag{7}$$

This completes the proof. ∎

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