ANNALES
POLONICI MATHEMATICI
LXIV.2 (1996)

Fueter regular mappings and harmonicity

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Abstract. It is shown that Fueter regular functions appear in connection with the Eells condition for harmonicity. New conditions for mappings from 4-dimensional conformally flat manifolds to be harmonic are obtained.

1. Introduction. There does not exist a "quaternionic analysis" in the same sense as complex analysis. This term stands for a theory investigating properties of regular functions in the sense of Fueter. Let us recall their definition.

Definition 1.1. A function $f : \Omega \to \mathbb{H}$ ($\mathbb{H}$ the quaternions) is said to be *regular* in a domain $\Omega \subset \mathbb{H}$ if f is differentiable in the usual sense as a mapping of Ω, considered as an open set in $\mathbb{R}^4$, with values in $\mathbb{R}^4$ and

$$D^+ f := \frac{1}{4}(\partial_0 + i_1\partial_1 + i_2\partial_2 + i_3\partial_3)(f_0 + i_1 f_1 + i_2 f_2 + i_3 f_3) = 0$$

in Ω, where i_1, i_2, i_3 are the quaternionic units.

These functions are the most natural analog of holomorphic functions. But because of the non-commutativity of quaternions one cannot define a quaternionic manifold with Fueter regular functions as transition functions.

Nevertheless, one does define a "quaternionic manifold" omitting completely the problem of transition functions. This definition is connected with the so-called "Berger list" [1]. Berger proved that the holonomy group of every irreducible Riemannian manifold which is not a symmetric space is a subgroup of one of the following:

$$\begin{gathered} SO(n), \quad U(n/2), \quad \mathrm{Sp}(n/4)\times\mathrm{Sp}(1), \quad \mathrm{Sp}(n/4), \\ G_2\ (n=7), \quad \mathrm{Spin}(7)\ (n=8), \quad \mathrm{Spin}(9)\ (n=16), \end{gathered}$$

where n denotes the dimension of the manifold in question.

1991 *Mathematics Subject Classification*: 53C10, 53C40, 58E20.
Key words and phrases: quaternions, Fueter regular functions, harmonic mappings.

The manifolds whose holonomy group is contained in $SO(n)$ are oriented Riemannian manifolds. One can prove only general theorems about the topology of such manifolds.

The manifolds with holonomy group in $U(n/2)$ are complex Kähler manifolds. They play a very important role in complex analysis.

The next group from the Berger list constitutes a basis of the following definition:

Definition 1.2. A $4n$-dimensional Riemannian manifold is called *quaternionic* (precisely, *quaternionic-Kähler*) if its holonomy group is a subgroup of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ (see e.g. [2]).

Analysis on such manifolds is another kind of "quaternionic analysis". At first sight it has nothing to do with Fueter regular functions.

In this paper it is shown that Fueter regular functions do appear on quaternionic manifolds. It turned out, to our surprise, that the Eells condition for harmonicity [5] is, for some kind of 4-dimensional manifolds, equivalent to the existence of an antiregular function in the sense of Fueter. It is the most significant result of the present paper. This once more points to the importance of Fueter regular functions. In particular, rewriting the Eells condition for harmonicity in quaternionic variables we obtain new results on harmonic mappings from 4-dimensional conformally flat manifolds.

In particular, we get a very interesting new characterization of harmonic mappings from the 4-dimensional torus.

Some of these problems were discussed at the University "La Sapienza" in Rome. The author wants to express his hearty thanks to Prof. Stefano Marchiafava for numerous discussions and encouragement in this work.

The manifolds considered in the paper are assumed to be connected, compact, orientable and without boundary.

2. Elements of quaternionic analysis. Denote by $\mathbb{H}$ the field of quaternions. Let V be a real m-dimensional vector space and V^* the dual space to V. Consider the following vector spaces over $\mathbb{H}$:

$$V^{\mathbb{H}} := V \otimes_{\mathbb{R}} \mathbb{H} \text{ — quaternionification of } V \text{ (a right } \mathbb{H}\text{-vector space)},$$
$$\overline{V^{\mathbb{H}}} := \mathbb{H} \otimes_{\mathbb{R}} V \text{— quaternionic conjugation of } V \text{ (a left } \mathbb{H}\text{-vector space)},$$
$$(V^{\mathbb{H}})^* := \mathbb{H} \otimes_{\mathbb{R}} V^* \text{ — dual of } V^{\mathbb{H}} \text{ (a left } \mathbb{H}\text{-vector space)},$$
$$(\overline{V^{\mathbb{H}}})^* := V^* \otimes_{\mathbb{R}} \mathbb{H} \text{ — dual of } \overline{V^{\mathbb{H}}} \text{ (a right } \mathbb{H}\text{-vector space)}.$$

We use the fact that $\overline{qv} = \overline{v}\overline{q}$, $v \in V^{\mathbb{H}}$, $q \in \mathbb{H}$.

Let $(1, i, j, k)$ be a fixed standard basis of H with the well known properties: $i^2 = j^2 = k^2 = -1$, $ijk = -1$. Consider three involutions of $\mathbb{H}$, denoted by σ_1, σ_2 and σ_3, given by automorphisms corresponding to i, j, k, respectively, which satisfy the following conditions:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathrm{id}, \qquad \sigma_1\sigma_2 = \sigma_2\sigma_1 = \sigma_3,$$
$$\sigma_2\sigma_3 = \sigma_3\sigma_2 = \sigma_1, \qquad \sigma_3\sigma_1 = \sigma_1\sigma_3 = \sigma_2.$$

Using quaternionic units we can write

$$\sigma_1(q) = -iqi, \quad \sigma_2(q) = -jqj, \quad \sigma_3(q) = -kqk, \quad q \in \mathbb{H}.$$

Set

$$q^1 := \sigma_1(q), \quad q^2 := \sigma_2(q), \quad q^3 := \sigma_3(q), \quad q \in \mathbb{H}.$$

Explicitly, if $q = x_0 + ix_1 + jx_2 + kx_3$, then

$$q^1 = x_0 + ix_1 - jx_2 - kx_3,$$
$$q^2 = x_0 - ix_1 + jx_2 - kx_3,$$
$$q^3 = x_0 - ix_1 - jx_2 + kx_3.$$

Right multiplication by i, j and k determines a triple (I_1, I_2, I_3) of complex structures on $\mathbb{R}^{4m} \cong \mathbb{H}^m$ satisfying the following conditions:

$$I_1^2 = I_2^2 = I_3^2 = -\,\mathrm{Id}, \qquad I_1I_2I_3 = -\,\mathrm{Id},$$

where Id stands for the identity mapping of $\mathbb{R}^{4m}$.

Any two such triples (I_1, I_2, I_3) and (I_1', I_2', I_3') are related by a transformation

$$I_h' = \sum_{k=1}^{3} c_{hk} I_k, \qquad h = 1, 2, 3,$$

with $(c_{hk}) \in SO(3)$.

Let (M, g) be a Riemannian manifold.

Definition 2.1 [2]. An *almost quaternionic structure* is defined as a covering $\{U_i\}$ of M with two almost complex structures I_i and J_i on each U_i such that $I_iJ_i = -J_iI_i$ and the 3-dimensional vector spaces of endomorphisms generated by I_i, J_i and $K_i := I_iJ_i$,

$$\mathrm{End}_{U_i} := \{\alpha I_i + \beta J_i + \gamma K_i : \alpha, \beta, \gamma \in \mathbb{R}\}$$

are the same on all of M.

A Riemannian metric g is *quaternionic-Hermitian* if g is Hermitian for each I and J. A manifold (M, g) equipped with an almost quaternionic structure with g quaternionic-Hermitian is called *almost-quaternionic-Hermitian*.

Definition 2.2. The *standard enhanced quaternionic structure* of $\mathbb{H}^m$ is the 3-dimensional subspace Q_0 of the space $\mathrm{End}_{\mathbb{R}}\,\mathbb{H}^m$ spanned by (any) triple (I_1, I_2, I_3) as above, called an *admissible hypercomplex base* of Q_0 (we will also write $(I_1, I_2, I_3) \in Q_0$).

Definition 2.3. Let (M^{4m}, g) and (N^{4n}, h) be two almost-quaternionic-Hermitian manifolds and $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ a smooth map. Then Φ is called *Q-holomorphic* if for every $p \in M^{4m}$ and each hypercomplex base $(I'_1, I'_2, I'_3) \in Q^M_p$ there exists a hypercomplex base $(I_1, I_2, I_3) \in Q^N_{\Phi(p)}$ such that

$$I_a(\Phi_*)_{|p} = (\Phi_*)_{|p} I'_a \quad \text{ for } a = 1, 2, 3.$$

Let $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ be a C^∞-mapping between two almost-quaternionic-Hermitian manifolds of dimensions $4m$ and $4n$, respectively. We can regard the quaternionic extension $d^{\mathbb{H}}\Phi$ of the differential $d\Phi$ as a section of the bundle $(\Phi^{-1}T^{\mathbb{H}}N) \otimes_{\mathbb{H}} (T^{\mathbb{H}}M)^*$ over M.

Let $p \in M$. It is well known (see e.g. [3]) that $T^{\mathbb{H}}_p M := T_pM \otimes_{\mathbb{R}} \mathbb{H}$ can be decomposed in the following way:

$$T^{\mathbb{H}}_p M := U^{\mathbb{H}}_p \oplus \tau_1 U^{\mathbb{H}}_p \oplus \tau_2 U^{\mathbb{H}}_p \oplus \tau_3 U^{\mathbb{H}}_p,$$

where τ_1, τ_2, τ_3 are the semi-involutions defined on $T^{\mathbb{H}}_p M$ as

$$\tau_1 = \mathrm{id} \otimes \sigma_1, \quad \tau_2 = \mathrm{id} \otimes \sigma_2, \quad \tau_3 = \mathrm{id} \otimes \sigma_3$$

and $\sigma_1, \sigma_2, \sigma_3$ are the involutions of the algebra of quaternions $\mathbb{H}$ defined above.

Take $p \in M$ and let U be a small open neighbourhood of p. On U, by Definition 2.1, there are almost complex structures I, J and $K := IJ$. Using I, J and K we can define $U^{\mathbb{H}}_p$ by

$$U^{\mathbb{H}}_p := \{X + iIX + jJX + kKX : X \in T_pM\}.$$

Then we get

$$\begin{aligned}
\tau_1 U^{\mathbb{H}}_p &= \{X + iIX - jJX - kKX : X \in T_pM\},\\
\tau_2 U^{\mathbb{H}}_p &= \{X - iIX + jJX - kKX : X \in T_pM\},\\
\tau_3 U^{\mathbb{H}}_p &= \{X - iIX - jJX + kKX : X \in T_pM\}.
\end{aligned}$$

Remark 2.1. There exist elements $X_1, \ldots, X_m$ of T_pM such that the system $(X_1, \ldots, X_m, IX_1, \ldots, IX_m, JX_1, \ldots, JX_m, KX_1, \ldots, KX_m)$ forms an orthonormal basis for $T^{\mathbb{H}}_p M$ with respect to the metric g.

Proof. It is analogous to that in the complex case [8]. ■

Let $(x^i_0, x^i_1, x^i_2, x^i_3)$, $i = 1, \ldots, m$, be local real coordinates at the point p. We can introduce the quaternionic coordinates $(q_1, \ldots, q_m)$ as follows:

$$q_k := x^k_0 + ix^k_1 + jx^k_2 + kx^k_3, \quad k = 1, \ldots, m.$$

If the almost quaternionic structure (I, J, K) is integrable then we can assume that I, J and K are given by

$$
\begin{aligned}
&I\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big)=-\frac{\partial}{\partial x^k_{1\,|p}}, && J\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big)=-\frac{\partial}{\partial x^k_{2\,|p}}, && K\Big(\frac{\partial}{\partial x^k_{0\,|p}}\Big)=-\frac{\partial}{\partial x^k_{3\,|p}},\\
&I\Big(\frac{\partial}{\partial x^k_{1\,|p}}\Big)=\frac{\partial}{\partial x^k_{0\,|p}}, && J\Big(\frac{\partial}{\partial x^k_{1\,|p}}\Big)=\frac{\partial}{\partial x^k_{3\,|p}}, && K\Big(\frac{\partial}{\partial x^k_{1\,|p}}\Big)=-\frac{\partial}{\partial x^k_{2\,|p}},\\
&I\Big(\frac{\partial}{\partial x^k_{2\,|p}}\Big)=-\frac{\partial}{\partial x^k_{3\,|p}}, && J\Big(\frac{\partial}{\partial x^k_{2\,|p}}\Big)=\frac{\partial}{\partial x^k_{0\,|p}}, && K\Big(\frac{\partial}{\partial x^k_{2\,|p}}\Big)=\frac{\partial}{\partial x^k_{1\,|p}},\\
&I\Big(\frac{\partial}{\partial x^k_{3\,|p}}\Big)=\frac{\partial}{\partial x^k_{2\,|p}}, && J\Big(\frac{\partial}{\partial x^k_{3\,|p}}\Big)=-\frac{\partial}{\partial x^k_{1\,|p}}, && K\Big(\frac{\partial}{\partial x^k_{3\,|p}}\Big)=\frac{\partial}{\partial x^k_{0\,|p}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{\partial}{\partial q_{k\,|p}}&=\frac14\Big(\frac{\partial}{\partial x^k_{0\,|p}}-i\frac{\partial}{\partial x^k_{1\,|p}}-j\frac{\partial}{\partial x^k_{2\,|p}}-k\frac{\partial}{\partial x^k_{3\,|p}}\Big),\\
\frac{\partial}{\partial q^1_{k\,|p}}&=\frac14\Big(\frac{\partial}{\partial x^k_{0\,|p}}-i\frac{\partial}{\partial x^k_{1\,|p}}+j\frac{\partial}{\partial x^k_{2\,|p}}+k\frac{\partial}{\partial x^k_{3\,|p}}\Big),\\
\frac{\partial}{\partial q^2_{k\,|p}}&=\frac14\Big(\frac{\partial}{\partial x^k_{0\,|p}}+i\frac{\partial}{\partial x^k_{1\,|p}}-j\frac{\partial}{\partial x^k_{2\,|p}}+k\frac{\partial}{\partial x^k_{3\,|p}}\Big),\\
\frac{\partial}{\partial q^3_{k\,|p}}&=\frac14\Big(\frac{\partial}{\partial x^k_{0\,|p}}+i\frac{\partial}{\partial x^k_{1\,|p}}+j\frac{\partial}{\partial x^k_{2\,|p}}-k\frac{\partial}{\partial x^k_{3\,|p}}\Big).
\end{aligned}
\tag{2.1}
$$

Then the system $\{\partial/\partial q_{1\,|p},\ldots,\partial/\partial q_{m\,|p}\}$ forms a basis for $U^{\mathbb{H}}_p$ and $\{\partial/\partial q^a_{1\,|p},\ldots,\partial/\partial q^a_{m\,|p}\}$ are bases for $\tau_a U^{\mathbb{H}}_p$, $a=1,2,3$. It is also clear that $\{dq_{1\,|p},\ldots,dq_{m\,|p}\}$ and $\{dq^a_{1\,|p},\ldots,dq^a_{m\,|p}\}$ are bases for $(U^{\mathbb{H}}_p)^*$ and $(\tau_a U^{\mathbb{H}}_p)^*$, $a=1,2,3$, respectively.

3. Main theorems. Let V be a real vector space of dimension $4n$. A quaternionic structure in V corresponds to the following decomposition (see Sect. 2):

$$V^{\mathbb{H}}=U^{\mathbb{H}}\oplus\tau_1U^{\mathbb{H}}\oplus\tau_2U^{\mathbb{H}}\oplus\tau_3U^{\mathbb{H}}.$$

Consider the tensor products

$$V^{\mathbb{H}}\otimes_{\mathbb{H}}(V^{\mathbb{H}})^*,\quad V^{\mathbb{H}}\otimes_{\mathbb{H}}\overline{V^{\mathbb{H}}},\quad \overline{(V^{\mathbb{H}})^*}\otimes_{\mathbb{H}}(V^{\mathbb{H}})^*,\ldots$$

Each of them can be decomposed into the direct sum of 16 parts with respect to the given quaternionic structure in V. In particular, to every real covariant 2-tensor S on V, $S\in V^*\otimes_{\mathbb{R}}V^*$, there corresponds a 2-tensor $S^{\mathbb{H}}\in\overline{(V^{\mathbb{H}})^*}\otimes_{\mathbb{H}}(V^{\mathbb{H}})^*$ which can be decomposed into 4 parts, namely real tensors $S^{\mathbb{H}}_0$, $S^{\mathbb{H}}_1$, $S^{\mathbb{H}}_2$ and $S^{\mathbb{H}}_3$ (of genus $0,1,2,3$, respectively) in the following way:

$$
\begin{aligned}
S^{\mathbb{H}}_0\in\ &\overline{(U^{\mathbb{H}})^*}\otimes_{\mathbb{H}}(U^{\mathbb{H}})^*+(\tau_1\overline{U^{\mathbb{H}}})^*\otimes_{\mathbb{H}}(\tau_1U^{\mathbb{H}})^*+(\tau_2\overline{U^{\mathbb{H}}})^*\otimes_{\mathbb{H}}(\tau_2U^{\mathbb{H}})^*\\
&+(\tau_3\overline{U^{\mathbb{H}}})^*\otimes_{\mathbb{H}}(\tau_3U^{\mathbb{H}})^*,
\end{aligned}
$$

$$
\begin{aligned}
S_1^{\mathbb{H}} \in \overline{(U^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (\tau_1 U^{\mathbb{H}})^* + (\tau_1 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (U^{\mathbb{H}})^* + (\tau_2 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_3 U^{\mathbb{H}})^* \\
+ (\tau_3 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_2 U^{\mathbb{H}})^*, \\
S_2^{\mathbb{H}} \in \overline{(U^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (\tau_2 U^{\mathbb{H}})^* + (\tau_2 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (U^{\mathbb{H}})^* + (\tau_3 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_1 U^{\mathbb{H}})^* \\
+ (\tau_1 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_3 U^{\mathbb{H}})^*, \\
S_3^{\mathbb{H}} \in \overline{(U^{\mathbb{H}})^*} \otimes_{\mathbb{H}} (\tau_3 U^{\mathbb{H}})^* + (\tau_3 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (U^{\mathbb{H}})^* + (\tau_1 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_2 U^{\mathbb{H}})^* \\
+ (\tau_2 \overline{U^{\mathbb{H}}})^* \otimes_{\mathbb{H}} (\tau_1 U^{\mathbb{H}})^*.
\end{aligned}
$$

In each line the tensors are mutally biconjugate.

Assume that

$$\dim_{\mathbb{R}} V = 4.$$

Let $x \in V$ and $(\partial/\partial x^0{}_{|x}, \partial/\partial x^1{}_{|x}, \partial/\partial x^2{}_{|x}, \partial/\partial x^3{}_{|x})$ be a base of the tangent space $T_x V$ (this base is good for every $x \in V$ and because $T_x V \cong V$, we can treat it as a base for V) and let (dx^0, dx^1, dx^2, dx^3) be the dual base. Consider a real symmetric 2-tensor on V:

$$S = dx^i S_{ij} dx^j.$$

The quaternionic decomposition of $S^{\mathbb{H}}$ in pure components looks as follows:

$$
\begin{aligned}
S^{\mathbb{H}} = d\overline{q} s_{00} dq + d\overline{q^1} s_{11} dq^1 + d\overline{q^2} s_{22} dq^2 + d\overline{q^3} s_{33} dq^3 \\
+ d\overline{q} s_{01} dq^1 + d\overline{q^1} s_{10} dq + d\overline{q^2} s_{23} dq^3 + d\overline{q^3} s_{32} dq^2 \\
+ d\overline{q} s_{02} dq^2 + d\overline{q^2} s_{20} dq + d\overline{q^1} s_{13} dq^3 + d\overline{q^3} s_{31} dq^1 \\
+ d\overline{q} s_{03} dq^3 + d\overline{q^3} s_{30} dq + d\overline{q^1} s_{12} dq^2 + d\overline{q^2} s_{21} dq^1,
\end{aligned}
$$

where

$$s_{00} = s_{11} = s_{22} = s_{33} \in \mathbb{R},$$

$$
\begin{aligned}
&s_{10} = (s_{01})^1, \quad s_{23} = \tau_2 s_{01}, \quad s_{32} = \tau_2 \overline{s}_{01}, \quad s_{23} = (s_{32})^1, \\
&s_{20} = (s_{02})^2, \quad s_{13} = \tau_1 s_{02}, \quad s_{31} = \tau_1 \overline{s}_{02}, \quad s_{21} = (s_{12})^3, \\
&s_{30} = (s_{03})^3, \quad s_{12} = \tau_3 s_{03}, \quad s_{21} = \tau_3 \overline{s}_{03}, \quad s_{31} = (s_{13})^2.
\end{aligned}
$$

The relationship between the quaternionic components s_{ij} and the real components S_{mn} is

$$
\begin{aligned}
&\tfrac{1}{4} S_{00} = s_{00} + \operatorname{Re}(s_{01} + s_{02} + s_{03}), \\
&\tfrac{1}{4} S_{11} = s_{00} + \operatorname{Re}(s_{01} - s_{02} - s_{03}), \\
&\tfrac{1}{4} S_{22} = s_{00} + \operatorname{Re}(-s_{01} + s_{02} - s_{03}), \\
&\tfrac{1}{4} S_{33} = s_{00} + \operatorname{Re}(-s_{01} - s_{02} + s_{03}), \\
&\tfrac{1}{4} S_{01} = -\operatorname{Re}[(s_{02} + s_{03}) i] = -\operatorname{Re}[(\pm s_{01} + s_{02} + s_{03}) i],
\end{aligned}
$$

$$\begin{aligned}
\tfrac{1}{4}S_{02} &= -\operatorname{Re}[(s_{01}+s_{03})j] = -\operatorname{Re}[(s_{01}\pm s_{02}+s_{03})j],\\
\tfrac{1}{4}S_{03} &= -\operatorname{Re}[(s_{01}+s_{02})k] = -\operatorname{Re}[(s_{01}+s_{02}\pm s_{03})k],\\
\tfrac{1}{4}S_{12} &= -\operatorname{Re}[(s_{01}-s_{02})k] = -\operatorname{Re}[(s_{01}-s_{02}\pm s_{03})k],\\
\tfrac{1}{4}S_{13} &= -\operatorname{Re}[-(s_{01}-s_{03})j] = -\operatorname{Re}[(-s_{01}\pm s_{02}+s_{03})j],\\
\tfrac{1}{4}S_{23} &= -\operatorname{Re}[-(s_{02}-s_{03})i] = -\operatorname{Re}[(\pm s_{01}-s_{02}+s_{03})i].
\end{aligned}$$

Writing $\pm s_{0k}$ we indicate that we are free to choose the sign due to the fact that $\operatorname{Re}(s_{01}i)=\operatorname{Re}(s_{02}j)=\operatorname{Re}(s_{03}k)=0$.

On the other hand, we have

$$\begin{aligned}
s_{00} &= \tfrac{1}{16}(S_{00}+S_{11}+S_{22}+S_{33}),\\
s_{01} &= \tfrac{1}{16}(S_{00}+S_{11}-S_{22}-S_{33})+\tfrac{1}{8}(S_{02}-S_{13})j+\tfrac{1}{8}(S_{03}+S_{12})k,\\
s_{02} &= \tfrac{1}{16}(S_{00}-S_{11}+S_{22}-S_{33})+\tfrac{1}{8}(S_{01}+S_{23})i+\tfrac{1}{8}(S_{03}-S_{12})k,\\
s_{03} &= \tfrac{1}{16}(S_{00}-S_{11}-S_{22}+S_{33})+\tfrac{1}{8}(S_{01}-S_{23})i+\tfrac{1}{8}(S_{02}+S_{13})j.
\end{aligned}$$

Hereafter we will consider a real Riemannian manifold M which is locally conformally flat with $\dim_{\mathbb{R}} M=4$ (e.g. the sphere $S^4\cong\mathbb{HP}^1$ or the torus $T^4\cong\mathbb{H}/\mathbb{Z}^4$, see e.g. [2]). Then we can assume that in a neighbourhood of every point p of M there exists a system of local coordinates (x^0,x^1,x^2,x^3) such that the metric g is expressed by

$$g=g^0_{\mathbb{R}}[(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2],$$

where $g^0_{\mathbb{R}}$ is a real positive C^∞-function defined about p. Consider the quaternionic coordinate $q:=x^0+ix^1+jx^2+kx^3$ associated with the given system of real coordinates. Then the quaternionic expression of g is

$$g=g^0_{\mathbb{H}}[d\overline{q}\otimes dq+d\overline{q^1}\otimes dq^1+d\overline{q^2}\otimes dq^2+d\overline{q^3}\otimes dq^3]. \tag{3.1}$$

Comparing the expressions for g in real and quaternionic coordinates we get:

PROPOSITION 3.1. $g^0_{\mathbb{R}}=4g^0_{\mathbb{H}}$.

Proof. This follows by straightforward calculations. ∎

DEFINITION 3.1 [5]. Let $\Phi:(M,g)\to(N,h)$ be a smooth map between Riemannian manifolds. The *stress-energy tensor* of Φ is the symmetric 2-tensor on M given by

$$S(\Phi):=e\cdot g-\Phi^*h,$$

where $e=e(\Phi)$ denotes the *energy density* of Φ:

$$e(\Phi):=\tfrac{1}{2}|d\Phi|^2=\tfrac{1}{2}g^{ij}h_{\alpha\beta}\Phi^\alpha_i\Phi^\beta_j,$$

and $(\Phi^\alpha_i)=(\partial\Phi^\alpha/\partial x^i)$ is a local representation of $d\Phi$.

We will compute the quaternionic components of $S(\Phi)$.

I. Computation of Φ^*h. The real expression of the metric h on N is

$$h=\sum_{\alpha,\beta} h_{\alpha\beta}dy^\alpha\otimes dy^\beta \qquad (h_{\alpha\beta}=h_{\beta\alpha}),$$

where (y^α) is a local system of real coordinates defined in an open neighbourhood of $\Phi(p)$, with p a fixed point in M. If (x^j) is a system of local coordinates about p then

$$\Phi^*h=dx^i\Phi_i^\alpha h_{\alpha\beta}\Phi_j^\beta dx^j.$$

In order to pass to quaternionic coordinates we have to consider the extension of the metric h to the quaternionified tangent bundle of N, $T^{\mathbb{H}}N:=TN\otimes_{\mathbb{R}}\mathbb{H}$. If $X^{\mathbb{H}},Y^{\mathbb{H}}\in T_p^{\mathbb{H}}N$ then

$$X^{\mathbb{H}}=(X^1,\dots,X^n),\qquad Y^{\mathbb{H}}=(Y^1,\dots,Y^n),$$

where $4n=\dim_{\mathbb{R}}N$ and

$$X^\alpha=X_0^\alpha+iX_1^\alpha+jX_2^\alpha+kX_3^\alpha,\qquad Y^\beta=Y_0^\beta+iY_1^\beta+jY_2^\beta+kY_3^\beta.$$

Then we set

$$\begin{aligned}\langle X^{\mathbb{H}},Y^{\mathbb{H}}\rangle&=\langle X^{\mathbb{H}},Y^{\mathbb{H}}\rangle_h\\ &:=\sum_{\alpha,\beta=1}^n (X_0^\alpha+iX_1^\alpha+jX_2^\alpha+kX_3^\alpha)h_{\alpha\beta}(Y_0^\beta+iY_1^\beta+jY_2^\beta+kY_3^\beta).\end{aligned}$$

Taking into account (2.1) we can write

$$\begin{aligned}\Phi^*h=&\left(d\overline{q}\frac{\partial\Phi^\alpha}{\partial\overline{q}}+d\overline{q^1}\frac{\partial\Phi^\alpha}{\partial\overline{q^1}}+d\overline{q^2}\frac{\partial\Phi^\alpha}{\partial\overline{q^2}}+d\overline{q^3}\frac{\partial\Phi^\alpha}{\partial\overline{q^3}}\right)h_{\alpha\beta}\\ &\times\left(\frac{\partial\Phi^\beta}{\partial q}dq+\frac{\partial\Phi^\beta}{\partial q^1}dq^1+\frac{\partial\Phi^\beta}{\partial q^2}dq^2+\frac{\partial\Phi^\beta}{\partial q^3}dq^3\right).\end{aligned}$$

Finally, we get

$$\begin{aligned}\Phi^*h=&\langle\Phi_{\bar q},\Phi_q\rangle(d\overline{q}dq+d\overline{q^1}dq^1+d\overline{q^2}dq^2+d\overline{q^3}dq^3)\\ &+[d\overline{q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle dq^1+d\overline{q^1}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^1dq+d\overline{q^2}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^2dq^3\\ &+d\overline{q^3}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^3dq^2]\\ &+[d\overline{q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle dq^2+d\overline{q^2}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^2dq+d\overline{q^1}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^1dq^3\\ &+d\overline{q^3}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^3dq^1]\\ &+[d\overline{q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle dq^3+d\overline{q^3}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^3dq+d\overline{q^1}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^1dq^2\\ &+d\overline{q^2}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^2dq^1].\end{aligned}$$

II. Computation of $e(\Phi)\cdot g$. By the definition we have

$$e(\Phi)=\tfrac{1}{2}|d\Phi|^2=\tfrac{1}{2}g^{ij}h_{\alpha\beta}\Phi_i^\alpha\Phi_j^\beta=\tfrac{1}{2}g^0_{\mathbb{H}}(|\Phi_0|^2+|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2).$$

Then

$$\begin{aligned}e(\Phi)\cdot g&=\tfrac{1}{8}(|\Phi_0|^2+|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2)(d\overline{q}dq+d\overline{q^1}dq^1+d\overline{q^2}dq^2+d\overline{q^3}dq^3)\\&=2\langle\Phi_{\bar q},\Phi_q\rangle(d\overline{q}dq+d\overline{q^1}dq^1+d\overline{q^2}dq^2+d\overline{q^3}dq^3).\end{aligned}$$

Now, we can write an explicit expression for the stress-energy tensor of Φ:

$$\begin{aligned}(3.2)\quad S(\Phi)=\langle\Phi_{\overline{q}},\Phi_q\rangle&(d\overline{q}dq+d\overline{q^1}dq^1+d\overline{q^2}dq^2+d\overline{q^3}dq^3)\\&-[d\overline{q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle dq^1+d\overline{q^1}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^1dq+d\overline{q^2}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^2dq^3\\&+d\overline{q^3}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^3dq^2]\\&-[d\overline{q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle dq^2+d\overline{q^2}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^2dq+d\overline{q^1}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^1dq^3\\&+d\overline{q^3}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^3dq^1]\\&-[d\overline{q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle dq^3+d\overline{q^3}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^3dq+d\overline{q^1}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^1dq^2\\&+d\overline{q^2}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^2dq^1].\end{aligned}$$

According to the above decomposition we can define four tensors S_0, S_1, S_2, S_3 so that every square bracket $[\cdot]$ corresponds to one component of the decomposition of $S(\Phi)$ in these tensors: $S(\Phi)=S_0+S_1+S_2+S_3$.

PROPOSITION 3.2. (Φ *is conformal*) $\Leftrightarrow$ ($S_1=S_2=S_3=0$).

P r o o f. Notice that the equations

$$\langle\Phi_{\bar q},\Phi_{q^1}\rangle=\langle\Phi_{\bar q},\Phi_{q^2}\rangle=\langle\Phi_{\bar q},\Phi_{q^3}\rangle=0,$$

which express the vanishing of the components in S_1,S_2 and S_3, are equivalent to the conditions

$$|\Phi_0|^2=|\Phi_1|^2=|\Phi_2|^2=|\Phi_3|^2,$$
$$\langle\Phi_0,\Phi_i\rangle=0,\quad i=1,2,3;\quad\langle\Phi_i,\Phi_j\rangle=0,\quad i\neq j,i,j\neq0,$$

which just express the conformality of Φ. ∎

R e m a r k 3.1. (Φ is conformal) $\Leftrightarrow$ ($S(\Phi)$ is pure of genus 0).

COROLLARY 3.1. *If* Φ *is locally regular*, *i.e.* $\Phi_{\overline{q}}=0$, *then* $S(\Phi)=0$.

COROLLARY 3.2. $S(\Phi)=0$ *if and only if* $\Phi=\text{const}$.

P r o o f. By (3.2), $S(\Phi)=0$ if and only if

$$\langle\Phi_{\bar q},\Phi_q\rangle=\langle\Phi_{\bar q},\Phi_{q^1}\rangle=\langle\Phi_{\bar q},\Phi_{q^2}\rangle=\langle\Phi_{\bar q},\Phi_{q^3}\rangle=0.$$

By straightforward calculations the above equalities are equivalent to

$$|\Phi_0|^2 = |\Phi_1|^2 = |\Phi_2|^2 = |\Phi_3|^2 = 0 \quad \text{and} \quad \langle \Phi_i, \Phi_k \rangle = 0, \quad i, k = 0, 1, 2, 3.$$

But this is possible if and only if Φ is a constant. ■

DEFINITION 3.2. A 2-tensor S defined on an almost quaternionic manifold (M^4, g) with $\dim_{\mathbb{R}} M^4 = 4$ and standard enhanced quaternionic structure Q_0 is *Hermitian* if for any $p \in M^4$ we have

$$S(I_\alpha X, I_\alpha Y) = S(X, Y) \quad \text{for } \alpha = 1, 2, 3,$$

where $(I_1, I_2, I_3) \in Q_{0|p}$ and $X, Y \in T_p M^4$.

Remark 3.2. (S is pure of genus 0) $\Leftrightarrow$ (S is Hermitian).

Remark 3.3. (Φ is conformal) $\Leftrightarrow$ ($S(\Phi)$ is Hermitian).

PROPOSITION 3.3. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth mapping between two almost-quaternionic-Hermitian manifolds. Assume that (M^4, g) is locally conformally flat. If Φ is Q-holomorphic then $S(\Phi)$ is Hermitian.*

Proof. By the definition of $S(\Phi)$ it is enough to show that $\Phi^* h$ is Hermitian on (M^4, g). Indeed,

$$\begin{aligned} \Phi^* h(I_\alpha X, I_\alpha Y) &= h(d\Phi(I_\alpha X), d\Phi(I_\alpha Y)) = h(I'_\alpha(d\Phi(X), I'_\alpha(d\Phi(Y))) \\ &= h(d\Phi(X), d\Phi(Y)) = \Phi^* h(X, Y). \ ■ \end{aligned}$$

PROPOSITION 3.4. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth mapping between almost quaternionic Hermitian manifolds. Assume that (M^4, g) is locally conformally flat. If Φ is Q-holomorphic then Φ is harmonic if and only if it is homothetic.*

Proof. Note that

$$e(\Phi) = 2 g^0_{\mathbb{H}} \langle \Phi_{\bar{q}}, \Phi_q \rangle. \tag{3.3}$$

If Φ is Q-holomorphic then Φ is conformal. By Proposition 3.2 and (3.2) we get

$$S(\Phi) = g^0_{\mathbb{H}} \langle \Phi_{\bar{q}}, \Phi_q \rangle g. \tag{3.4}$$

On the other hand, conformality of Φ means that $\Phi^* h = \mu g$ for some continuous and non-negative function μ defined on M. By (3.3) and (3.4) we obtain $\mu = g^0_{\mathbb{H}} \langle \Phi_{\bar{q}}, \Phi_q \rangle$. Thus

$$\begin{aligned} (\operatorname{div} S(\Phi) = 0) &\Leftrightarrow [\langle d(g^0_{\mathbb{H}} \langle \Phi_{\bar{q}}, \Phi_q \rangle), g \rangle = 0] \Leftrightarrow [d(g^0_{\mathbb{H}} \langle \Phi_{\bar{q}}, \Phi_q \rangle) = 0] \\ &\Leftrightarrow (d\mu = 0) \Leftrightarrow (\mu = \text{const}) \Leftrightarrow (\Phi \text{ is homothetic}). \end{aligned}$$

If Φ is homothetic then $\Phi^*h = \mu_0 g$, where $\mu_0 = \text{const}$. So, we have $S = (e - \mu_0)g$ and $\operatorname{div} S = \langle de, g\rangle$. But on the other hand, $e = 0$, and so Φ is harmonic. ∎

Recall that if S is a real 2-tensor on a (real) Riemannian manifold (M, g), then one defines the *divergence* of S (see e.g. [5]) in the local coordinates (x^i) by

$$(\operatorname{div}_{\mathbb{R}} S)_i = (\operatorname{div} S)_i := g^{jk}\nabla_{\partial_j} S_{ki}.$$

DEFINITION 3.3. We define a *quaternionic divergence* of the quaternionic 2-tensor $s_{\mathbb{H}}$ by

$$(\operatorname{div}_{\mathbb{H}} s_{\mathbb{H}})_\gamma := g_{\mathbb{H}}^{\alpha\beta}\nabla_{\partial_\alpha} s_{\beta\gamma},$$

where α, β, γ stand for q, q^1, q^2, q^3.

PROPOSITION 3.5. *If the metric g is locally conformally flat then*

$$\begin{aligned} 64\operatorname{Re}[(\operatorname{div}_{\mathbb{H}} s_{\mathbb{H}})_q] &= (\operatorname{div}_{\mathbb{R}} S)_0,\\ 64\operatorname{Re}[i(\operatorname{div}_{\mathbb{H}} s_{\mathbb{H}})_{\overline{1}}] &= (\operatorname{div}_{\mathbb{R}} S)_1,\\ 64\operatorname{Re}[j(\operatorname{div}_{\mathbb{H}} s_{\mathbb{H}})_{\overline{2}}] &= (\operatorname{div}_{\mathbb{R}} S)_2,\\ 64\operatorname{Re}[k(\operatorname{div}_{\mathbb{H}} s_{\mathbb{H}})_{\overline{3}}] &= (\operatorname{div}_{\mathbb{R}} S)_3, \end{aligned} \tag{3.5}$$

Proof. This follows by straightforward computations. ∎

Let us quote a very important result of Eells [5]:

THEOREM 3.1. *Suppose that $\Phi : (M, g) \to (N, h)$ is a smooth mapping between smooth Riemannian manifolds. If Φ is harmonic then $S(\Phi)$ is conservative* (*i.e.* $\operatorname{div}_{\mathbb{R}} S(\Phi) = 0$). *If Φ is a differentiable submersion almost everywhere and* $\operatorname{div}_{\mathbb{R}} S(\Phi) = 0$, *then Φ is harmonic.*

Now, we can state

THEOREM 3.2. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth map between smooth Riemannian manifolds. Assume that M is locally conformally flat. If Φ is harmonic then Φ satisfies the following system of real equations*:

$$\begin{aligned} &\tfrac{1}{32}\nabla_{\partial_0}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\ &\qquad + \nabla_{\partial_1}\langle\Phi_0, \Phi_1\rangle + \nabla_{\partial_2}\langle\Phi_0, \Phi_2\rangle + \nabla_{\partial_3}\langle\Phi_0, \Phi_3\rangle = 0,\\ &\tfrac{1}{32}\nabla_{\partial_1}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\ &\qquad - \nabla_{\partial_0}\langle\Phi_0, \Phi_1\rangle + \nabla_{\partial_2}\langle\Phi_0, \Phi_2\rangle - \nabla_{\partial_3}\langle\Phi_0, \Phi_3\rangle = 0,\\ &\tfrac{1}{32}\nabla_{\partial_2}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\ &\qquad - \nabla_{\partial_0}\langle\Phi_0, \Phi_1\rangle - \nabla_{\partial_1}\langle\Phi_0, \Phi_2\rangle + \nabla_{\partial_3}\langle\Phi_0, \Phi_3\rangle = 0,\\ &\tfrac{1}{32}\nabla_{\partial_3}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2)\\ &\qquad - \nabla_{\partial_0}\langle\Phi_0, \Phi_1\rangle + \nabla_{\partial_1}\langle\Phi_0, \Phi_2\rangle - \nabla_{\partial_2}\langle\Phi_0, \Phi_3\rangle = 0, \end{aligned} \tag{3.6}$$

where $\Phi_i := \partial\Phi/\partial x^i$ *and* $\langle\cdot,\cdot\rangle$ *means a real scalar product. Moreover*, *if* Φ *is a differentiable submersion almost everywhere and the system* (3.6) *is satisfied then* Φ *is harmonic.*

Proof. By the result of Eells, Theorem 3.1, and by Proposition 3.5 the condition $\mathrm{div}_{\mathbb{R}} S(\Phi) = 0$ is equivalent to the following system of quaternionic equations:

$$\begin{gathered}\mathrm{Re}[(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}}(\Phi))_0] = 0, \quad \mathrm{Re}[i(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}}(\Phi))_{\overline{1}}] = 0,\\ \mathrm{Re}[j(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}}(\Phi))_{\overline{2}}] = 0, \quad \mathrm{Re}[k(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}}(\Phi))_{\overline{3}}] = 0,\end{gathered} \tag{3.7}$$

where $s_{\mathbb{H}}(\Phi)$ denotes the quaternionic stress-energy tensor of Φ.

By the assumption, the metric g has the form (3.1). Hence, the only non-zero quaternionic components of g are $g^{00}, g^{\overline{11}}, g^{\overline{22}}, g^{\overline{33}}$ and they equal $g^0_{\mathbb{H}} \neq 0$, which is real. Then we have

$$\begin{aligned}(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}})_0 &= \sum_\alpha g^{\alpha\alpha}\nabla_{\partial_\alpha} s_{\alpha 0},\\ (\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}})_{\overline{\gamma}} &= \sum_\alpha g^{\alpha\alpha}\nabla_{\partial_\alpha} s_{\alpha\overline{\gamma}}, \qquad \gamma = 1,2,3.\end{aligned} \tag{3.8}$$

Substituting the quaternionic expression $s_{\mathbb{H}}$ for the stress-energy tensor into (3.8) we get

$$\begin{aligned}\frac{(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}})_0}{g^0_{\mathbb{H}}} &= \nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_q\rangle - \nabla_{\partial/\partial q^1}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^1\\ &\quad - \nabla_{\partial/\partial q^2}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^2 - \nabla_{\partial/\partial q^3}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^3,\\ \frac{(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}})_{\overline{1}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle + \nabla_{\partial/\partial q^1}\langle\Phi_{\bar q},\Phi_q\rangle\\ &\quad - \nabla_{\partial/\partial q^2}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^2 - \nabla_{\partial/\partial q^3}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^3,\\ \frac{(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}})_{\overline{2}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle - \nabla_{\partial/\partial q^1}\langle\Phi_{\bar q},\Phi_{q^3}\rangle^1\\ &\quad + \nabla_{\partial/\partial q^2}\langle\Phi_{\bar q},\Phi_q\rangle - \nabla_{\partial/\partial q^3}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^3,\\ \frac{(\mathrm{div}_{\mathbb{H}}\, s_{\mathbb{H}})_{\overline{3}}}{g^0_{\mathbb{H}}} &= -\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle - \nabla_{\partial/\partial q^1}\langle\Phi_{\bar q},\Phi_{q^2}\rangle^1\\ &\quad - \nabla_{\partial/\partial q^2}\langle\Phi_{\bar q},\Phi_{q^1}\rangle^2 + \nabla_{\partial/\partial q^3}\langle\Phi_{\bar q},\Phi_q\rangle.\end{aligned}$$

Since

$$\nabla_{\partial/\partial q^i}\langle\Phi_{\bar q},\Phi_{q^m}\rangle^i = [\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^m}\rangle]^i$$

for $i,m = 0,1,2,3$ and

$$\mathrm{Re}\,[\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^n}\rangle]^n = \mathrm{Re}\,\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^n}\rangle, \qquad n = 0,1,2,3,$$

we see that the system (3.7) is equivalent to

$$
(3.9)\quad
\begin{aligned}
&\operatorname{Re}[\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_q\rangle-\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle\\
&\qquad\qquad-\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle-\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle]=0,\\
&\operatorname{Re} i[\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle-\nabla_{\partial/\partial q^1}\langle\Phi_{\bar q},\Phi_q\rangle\\
&\qquad\qquad+\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle+\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle]=0,\\
&\operatorname{Re} j[\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle+\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle\\
&\qquad\qquad-\nabla_{\partial/\partial q^2}[\Phi_{\bar q},\Phi_q\rangle+\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle]=0,\\
&\operatorname{Re} k[\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle+\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle\\
&\qquad\qquad+\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle-\nabla_{\partial/\partial q^3}\langle\Phi_{\bar q},\Phi_q\rangle]=0.
\end{aligned}
$$

Now, note that

$$
\begin{aligned}
&\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_q\rangle=\tfrac{1}{64}\nabla_{\partial_0}(|\Phi_0|^2+|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2),\\
&\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle\\
&\qquad=\tfrac{1}{64}\nabla_{\partial_0}(|\Phi_0|^2+|\Phi_1|^2-|\Phi_2|^2-|\Phi_3|^2)\\
&\qquad\quad+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_2\rangle-\langle\Phi_1,\Phi_3\rangle]+2\nabla_{\partial_3}[\langle\Phi_0,\Phi_3\rangle+\langle\Phi_1,\Phi_2\rangle],\\
&\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle\\
&\qquad=\tfrac{1}{64}\nabla_{\partial_0}(|\Phi_0|^2-|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2)\\
&\qquad\quad+2\nabla_{\partial_1}[\langle\Phi_0,\Phi_1\rangle+\langle\Phi_2,\Phi_3\rangle]+2\nabla_{\partial_3}[\langle\Phi_0,\Phi_3\rangle-\langle\Phi_1,\Phi_2\rangle],\\
&\operatorname{Re}\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle\\
&\qquad=\tfrac{1}{64}\nabla_{\partial_0}(|\Phi_0|^2-|\Phi_1|^2-|\Phi_2|^2+|\Phi_3|^2)\\
&\qquad\quad+2\nabla_{\partial_1}[\langle\Phi_0,\Phi_1\rangle-\langle\Phi_2,\Phi_3\rangle]+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_2\rangle+\langle\Phi_1,\Phi_3\rangle],\\
&\operatorname{Re} i\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle\\
&\qquad=\tfrac{1}{64}\nabla_{\partial_1}(|\Phi_0|^2+|\Phi_1|^2-|\Phi_2|^2-|\Phi_3|^2)\\
&\qquad\quad-2\nabla_{\partial_3}[\langle\Phi_0,\Phi_2\rangle-\langle\Phi_1,\Phi_3\rangle]+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_3\rangle+\langle\Phi_1,\Phi_2\rangle],\\
&\operatorname{Re} i\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^2}\rangle\\
&\qquad=\tfrac{1}{64}\nabla_{\partial_1}(|\Phi_0|^2-|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2)\\
&\qquad\quad-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_1\rangle+\langle\Phi_2,\Phi_3\rangle]+2\nabla_{\partial_2}[\langle\Phi_0,\Phi_3\rangle-\langle\Phi_1,\Phi_2\rangle],\\
&\operatorname{Re} i\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^3}\rangle\\
&\qquad=\tfrac{1}{64}\nabla_{\partial_1}(|\Phi_0|^2-|\Phi_1|^2-|\Phi_2|^2+|\Phi_3|^2)\\
&\qquad\quad-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_1\rangle-\langle\Phi_2,\Phi_3\rangle]-2\nabla_{\partial_3}[\langle\Phi_0,\Phi_2\rangle+\langle\Phi_1,\Phi_3\rangle],\\
&\operatorname{Re} j\nabla_{\partial/\partial q}\langle\Phi_{\bar q},\Phi_{q^1}\rangle\\
&\qquad=\tfrac{1}{64}\nabla_{\partial_2}(|\Phi_0|^2+|\Phi_1|^2-|\Phi_2|^2-|\Phi_3|^2)\\
&\qquad\quad-2\nabla_{\partial_0}[\langle\Phi_0,\Phi_2\rangle-\langle\Phi_1,\Phi_3\rangle]-2\nabla_{\partial_1}[\langle\Phi_0,\Phi_3\rangle+\langle\Phi_1,\Phi_2\rangle],
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Re} j\nabla_{\partial/\partial q}\langle \Phi_{\bar q}, \Phi_{q^2}\rangle \\
&\qquad = \tfrac{1}{64}\nabla_{\partial_2}(|\Phi_0|^2 - |\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2) \\
&\qquad\quad + 2\nabla_{\partial_3}[\langle \Phi_0, \Phi_1\rangle + \langle \Phi_2, \Phi_3\rangle] - 2\nabla_{\partial_1}[\langle \Phi_0, \Phi_3\rangle - \langle \Phi_1, \Phi_2\rangle], \\
&\operatorname{Re} j\nabla_{\partial/\partial q}\langle \Phi_{\bar q}, \Phi_{q^3}\rangle \\
&\qquad = \tfrac{1}{64}\nabla_{\partial_2}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 + |\Phi_3|^2) \\
&\qquad\quad + 2\nabla_{\partial_3}[\langle \Phi_0, \Phi_1\rangle - \langle \Phi_2, \Phi_3\rangle] - 2\nabla_{\partial_0}[\langle \Phi_0, \Phi_2\rangle + \langle \Phi_1, \Phi_3\rangle], \\
&\operatorname{Re} k\nabla_{\partial/\partial q}\langle \Phi_{\bar q}, \Phi_{q^1}\rangle \\
&\qquad = \tfrac{1}{64}\nabla_{\partial_3}(|\Phi_0|^2 + |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) \\
&\qquad\quad + 2\nabla_{\partial_1}[\langle \Phi_0, \Phi_2\rangle - \langle \Phi_1, \Phi_3\rangle] - 2\nabla_{\partial_0}[\langle \Phi_0, \Phi_3\rangle + \langle \Phi_1, \Phi_2\rangle], \\
&\operatorname{Re} k\nabla_{\partial/\partial q}\langle \Phi_{\bar q}, \Phi_{q^2}\rangle \\
&\qquad = \tfrac{1}{64}\nabla_{\partial_3}(|\Phi_0|^2 - |\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2) \\
&\qquad\quad - 2\nabla_{\partial_2}[\langle \Phi_0, \Phi_1\rangle - \langle \Phi_2, \Phi_3\rangle] - 2\nabla_{\partial_0}[\langle \Phi_0, \Phi_3\rangle - \langle \Phi_1, \Phi_2\rangle], \\
&\operatorname{Re} k\nabla_{\partial/\partial q}\langle \Phi_{\bar q}, \Phi_{q^3}\rangle \\
&\qquad = \tfrac{1}{64}\nabla_{\partial_3}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 + |\Phi_3|^2) \\
&\qquad\quad - 2\nabla_{\partial_2}[\langle \Phi_0, \Phi_1\rangle - \langle \Phi_2, \Phi_3\rangle] + 2\nabla_{\partial_1}[\langle \Phi_0, \Phi_2\rangle + \langle \Phi_1, \Phi_3\rangle], \\
&\operatorname{Re} i\nabla_{\partial/\partial q^1}\langle \Phi_{\bar q}, \Phi_q\rangle = \tfrac{1}{64}\nabla_{\partial_1}(|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2), \\
&\operatorname{Re} j\nabla_{\partial/\partial q^2}\langle \Phi_{\bar q}, \Phi_q\rangle = \tfrac{1}{64}\nabla_{\partial_2}(|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2), \\
&\operatorname{Re} k\nabla_{\partial/\partial q^3}\langle \Phi_{\bar q}, \Phi_q\rangle = \tfrac{1}{64}\nabla_{\partial_3}(|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2).
\end{aligned}
$$

Now substituting the above expressions into (3.9) we get immediately (3.6), as required. ∎

DEFINITION 3.4. A function $f : \Omega \to \mathbb{H}$ is said to be *antiregular in the sense of Fueter* in a domain $\Omega \subset \mathbb{H}$ if f is differentiable (in the usual sense) as a mapping of Ω, regarded as an open set in $\mathbb{R}^4$, with values in $\mathbb{R}^4$ and

$$Df := \tfrac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3)(f_0 + if_1 + jf_2 + kf_3) = 0$$

in Ω; here $\partial_k := \partial/\partial x_k$, $k = 0, 1, 2, 3$.

Remark 3.4. Let $F = F_0 + iF_1 + jF_2 + kF_3 : \Omega \to \mathbb{H}$, where Ω is an open set in $\mathbb{H}$, be a differentiable mapping (i.e. each component is differentiable as a mapping $\mathbb{R}^4 \to \mathbb{R}$). Then $DF = 0$ if and only if

$$
\begin{aligned}
\partial_0 F_0 + \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 &= 0, \\
\partial_1 F_0 - \partial_0 F_1 + \partial_2 F_3 - \partial_3 F_2 &= 0, \\
\partial_2 F_0 - \partial_0 F_2 - \partial_1 F_3 + \partial_3 F_1 &= 0, \\
\partial_3 F_0 - \partial_0 F_3 + \partial_1 F_2 - \partial_2 F_1 &= 0.
\end{aligned}
$$

Proof. This follows by straightforward computations. ∎

Let us recall that the properties of antiregular functions are analogous to those of regular functions in the sense of Fueter (see e.g. [6, 7, 13]).

THEOREM 3.3. *Let $\Phi : (M^4, g) \to (N^{4n}, h)$ be a smooth map between smooth Riemannian manifolds. Assume that M is locally conformally flat. If Φ is harmonic then the function*

$$\begin{aligned} F_\Phi = F_0 + iF_1 + jF_2 + kF_3 := \tfrac{1}{32}(|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2) \\ + i\langle\Phi_0, \Phi_1\rangle + j\langle\Phi_0, \Phi_2\rangle + k\langle\Phi_0, \Phi_3\rangle \end{aligned} \tag{3.10}$$

is antiregular in the sense of Fueter. Moreover, if Φ is a differentiable submersion almost everywhere and F_Φ is antiregular in the sense of Fueter, then Φ is harmonic.

Proof. Note that the system (3.7) can be written in the following very suggestive and condensed form:

$$\nabla_D F_\Phi = 0, \tag{3.11}$$

where $D = \frac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3)$ and F_Φ is the quaternionic-valued function defined by (3.10). Since F_Φ is scalar, (3.11) is equivalent to $D\cdot F_\Phi = 0$, which proves the theorem. ∎

The above result is rather unexpected and it emphasizes the importance of the class of regular functions in the sense of Fueter.

THEOREM 3.4. *On the torus $T^4 := \mathbb{R}^4/\Lambda$, where Λ is a lattice, consider a global linear system of coordinates $q = x_0 + ix_1 + jx_2 + kx_3$. Then any harmonic map $\Phi : T^4 \to (N^{4n}, h)$ satisfies*

$$\begin{aligned} &|\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2 = a, \\ &\langle\Phi_0, \Phi_1\rangle = b_1, \quad \langle\Phi_0, \Phi_2\rangle = b_2, \quad \langle\Phi_0, \Phi_3\rangle = b_3, \end{aligned} \tag{3.12}$$

for suitable constants $a, b_1, b_2, b_3 \in \mathbb{R}$.

Proof. If Φ is harmonic then F_Φ is antiregular in the sense of Fueter. Any antiregular function satisfies the maximum principle. Since the torus T^4 is compact, F_Φ has to be constant. Then, by the definition of F_Φ, we get (3.12), as required. ∎

COROLLARY 3.3. *If $\Phi : (T^4, g) \to (N^{4n}, h)$ is harmonic and non-constant then either Φ_0 or (Φ_1, Φ_2, Φ_3) has no zero on T^4.*

Proof. Indeed, otherwise at a point $p_1 \in T^4$ where $\Phi_0(p_1) = 0$, by Theorem 3.4, we would have

$$-|\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2 = a \le 0 \quad \text{and} \quad b_1 = b_2 = b_3 = 0,$$

and at a point $p_2 \in T^4$ where $(\Phi_1(p_2), \Phi_2(p_2), \Phi_3(p_2)) = 0$ we would get

$$|\Phi_0|^2 = a \geq 0, \qquad b_1 = b_2 = b_3 = 0.$$

Then we would have

$$a = b_1 = b_2 = b_3 = 0.$$

This means that

$$\begin{cases} |\Phi_0|^2 = |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2, \\ \langle \Phi_0, \Phi_1 \rangle = \langle \Phi_0, \Phi_2 \rangle = \langle \Phi_0, \Phi_3 \rangle = 0 \end{cases} \tag{3.13}$$

at all points of T^4, with $\Phi_i := \partial\Phi/\partial x_i$, $i = 0, 1, 2, 3$.

But any smooth map $\Phi : (T^4, g) \to (N^{4n}, h)$ satisfying (3.13) has to be constant. Indeed, (3.13) must hold with x_0 replaced by any one of the variables x_1, x_2, x_3. Thus

$$\langle \Phi_i, \Phi_k \rangle = 0, \quad i \neq k, \quad i, k = 0, 1, 2, 3,$$

and

$$\begin{cases} |\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 - |\Phi_3|^2 = 0, \\ |\Phi_1|^2 - |\Phi_0|^2 - |\Phi_2|^2 - |\Phi_3|^2 = 0, \\ |\Phi_2|^2 - |\Phi_0|^2 - |\Phi_1|^2 - |\Phi_3|^2 = 0, \\ |\Phi_3|^2 - |\Phi_0|^2 - |\Phi_1|^2 - |\Phi_2|^2 = 0. \end{cases} \tag{3.14}$$

Note that the determinant of the system (3.14) is

$$\det \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \neq 0.$$

Thus, Φ would be constant, which contradicts our assumption. ■

Let S^4 denote the 4-dimensional sphere.

COROLLARY 3.4. *If* $\Phi : (T^4, g) \to (S^4, h)$ *is a* C^∞*-map such that* $(\Phi_0, \Phi_1, \Phi_2, \Phi_3) \neq 0$ *on* T^4 *then* $\deg(\Phi) = 0$.

Proof. The tangent bundle TS^4 is of rank 4, so we can complexify it. Denote by $TS^4_{\mathbb{C}}$ the complexification of TS^4. By the definition the bundle $\Phi^{-1}(TS^4_{\mathbb{C}})$ is also a complex vector bundle (now of rank 2).

Denote by $c_2(\Phi^{-1}(TS^4_{\mathbb{C}}))$ the second Chern class of $\Phi^{-1}(TS^4_{\mathbb{C}})$. Since the bundle $\Phi^{-1}(TS^4_{\mathbb{C}})$ is trivial we have

$$c_2(\Phi^{-1}(TS^4_{\mathbb{C}})) = 0.$$

On the other hand (see [4, 11]),

$$c_2(TS^4_{\mathbb{C}}) = \frac{3}{4\pi^2} v_g^{S^4},$$

where v_g is the volume form of S^4. Then we get

$$\begin{aligned} 0 &= \int_{T^4} c_2(\Phi^{-1}TS^4_{\mathbb{C}}) = \int_{T^4} \Phi^*[c_2(TS^4_{\mathbb{C}})] = \frac{3}{4\pi^2}\int_{T^4} \Phi^*(v_g^{S^4}) \\ &= \frac{3}{4\pi^2}\deg(\Phi)\operatorname{Vol}(S^4) = 2\deg(\Phi) \end{aligned}$$

because by the definition of the topological degree we have

$$\int_{T^4} \Phi^*(v_g^{S^4}) = \deg(\Phi)\operatorname{Vol}(S^4) \quad \text{and} \quad \operatorname{Vol}(S^4) = \tfrac{8}{3}\pi^2.$$

Thus, any C^∞-mapping $\Phi : T^4 \to S^4$ with $(\Phi_0, \Phi_1, \Phi_2, \Phi_3) \neq 0$ has degree 0, as required. ■

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Reçu par la Rédaction le 18.7.1994
Révisé le 15.2.1996