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NEGATIVELY REDUCED IDEALS IN ORDERS OF REAL QUADRATIC FIELDS: EVEN DISCRIMINANTS

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1. Introduction. Let Δ be a positive discriminant, that is, a nonsquare positive integer congruent to 0 or 1 modulo 4. Let O_Δ be the order of discriminant Δ in the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. The primitive ideals of O_Δ are the $\mathbb{Z}$-modules

$$(*)_\Delta \qquad I = [a, (b+\sqrt{\Delta})/2], \text{ where } a, b \in \mathbb{Z},$$
$$a > 0, \ c = (b^2 - \Delta)/(4a) \in \mathbb{Z} \text{ and } (a, b, c) = 1.$$

Our main reference is [4]; however, we depart from notation there in requiring $a > 0$. In $(*)_\Delta$, $a = \mathcal{N}(I) = (O_\Delta : I)$ is the norm of I, and b is uniquely determined modulo $2a$.

For a real number λ, we denote by $[\lambda]$ the greatest integer not exceeding λ. For $\varphi = x + y\sqrt{\Delta} \in \mathbb{Q}(\sqrt{\Delta})$ $(x, y \in \mathbb{Q})$, we denote by $\overline{\varphi} = x - y\sqrt{\Delta}$ its conjugate.

If I is given by $(*)_\Delta$, the number $\varphi = (b+\sqrt{\Delta})/2a$ is determined modulo 1 by I, while $I = a[1, \varphi]$ is uniquely determined by φ. The quantity $\varphi + [-\overline{\varphi}]$ depends only on I. Following P. Kaplan [2], we call the ideal I *k-reduced* if $\varphi + [-\overline{\varphi}] > k$, and *strictly k-reduced* if $k < \varphi + [-\overline{\varphi}] < k + 1$. With this terminology, 1-reduced ideals are just the reduced ideals in the classical sense, 0-reduced ideals are the negatively reduced ideals considered in [6] and [3] (see also [7]), and strictly 0-reduced ideals are negatively reduced ideals which are not reduced. For each $k \ge 0$, the number of k-reduced ideals of O_Δ is finite.

These notions have been used by P. Kaplan [2] in the case of odd discriminants D, to relate the 0-reduced and 1-reduced primitive ideals of O_{4D} to the primitive ideals of O_D in a precise way. When $D \equiv 5 \pmod 8$ these results have application to Eisenstein's problem concerning the existence

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of odd solutions of the equation $x^2 - Dy^2 = 4$. This connection was first observed by Mimura [6] and investigated in a systematic way by P. Kaplan and P. A. Leonard [3].

In the present note we study the relationship between primitive 0-reduced ideals of O_{4D} and primitive ideals of O_D in the case when D is even, and we give an application to the Pell equation.

2. Notations and results. Let $D = 4d$ be a discriminant. We start with a description of the primitive ideals of O_D and O_{4D}.

LEMMA 1. (i) *Each primitive ideal J of O_D is of the form*

$$(*)_D \qquad J = [A, B + \sqrt{d}], \quad \text{where } A, B \in \mathbb{Z},\ A > 0,\ C = (B^2 - d)/A \in \mathbb{Z} \text{ and } (A, 2B, C) = 1.$$

(ii) *Each primitive ideal I of O_{4D} is of the form*

$$(*)_{4D} \qquad I = [a, 2b + 2\sqrt{d}], \quad \text{where } a, b \in \mathbb{Z},\ a > 0,\ c = 4(b^2 - d)/a \in \mathbb{Z} \text{ and } (a, 2b, c) = 1.$$

In particular, *we have either $a \equiv 1 \pmod 2$ or $a \equiv 0 \pmod 4$.*

Proof. (i) If J is a primitive ideal of O_D, then $J = [A, (b_1 + \sqrt{4d})/2]$, where $A, b_1 \in \mathbb{Z}$, $A > 0$, $C = (b_1^2 - 4d)/(4A) \in \mathbb{Z}$ and $(A, b_1, C) = 1$. This implies $b_1 \equiv 0 \pmod 2$, and with $b_1 = 2B$ we obtain the asserted form.

(ii) If I is a primitive ideal of O_{4D}, then $I = [a, (b_1 + \sqrt{16d})/2]$, where $a, b_1 \in \mathbb{Z}$, $a > 0$, $c = (b_1^2 - 16d)/(4a) \in \mathbb{Z}$ and $(a, b_1, c) = 1$. This implies $b_1 \equiv 0 \pmod 2$, and with $b_1 = 2b_2$ we obtain $I = [a, b_2 + 2\sqrt{d}] = [a, a + b_2 + 2\sqrt{d}]$, $c = (b_2^2 - 4d)/a \in \mathbb{Z}$ and $(a, 2b_2, c) = 1$. If b_2 is odd, then so is a, and we replace b_2 by the even number $a + b_2$. Therefore we may assume that b_2 is even, $b_2 = 2b, c = 4(b^2 - d)/a$ and $(a, 2b, c) = 1$. ■

For a primitive ideal I of O_{4D} in the form $(*)_{4D}$, we define a primitive ideal $\theta(I)$ of O_D by the formula

$$\theta(I) = \begin{cases} [a, b + \sqrt{d}] & \text{if } a \equiv 1 \pmod 2, \\ [a/4, b + \sqrt{d}] & \text{if } a \equiv 0 \pmod 4. \end{cases}$$

This map, already studied by Gauss, was investigated in detail in [4], §3, and in [3], §3. Let $\mathcal{C}_D^+$ resp. $\mathcal{C}_{4D}^+$ be the group of strict equivalence classes of primitive ideals of O_D resp. O_{4D}. Then θ induces a surjective group homomorphism (also denoted by θ)

$$\theta : \mathcal{C}_{4D}^+ \to \mathcal{C}_D^+$$

such that, for any class $\mathfrak{c} \in \mathcal{C}_{4D}^+$ and each primitive ideal $I \in \mathfrak{c}$, we have $\theta(I) \in \theta(\mathfrak{c})$. Concerning the kernel of θ, we have the following result.

LEMMA 2. *Consider Pell's equation*

$$x^2 - dy^2 = 1. \tag{P}$$

If (P) *has a solution* (x, y) *with* $y \equiv 1 \pmod 2$, *then* θ *is an isomorphism*; *otherwise the kernel of* θ *has order* 2.

Proof. For a discriminant Δ, let h_Δ be the number of classes of properly equivalent primitive binary quadratic forms with discriminant Δ. Then we have $h_\Delta = \#\mathcal{C}_\Delta^+$ ([5], Theorem 1.20), and therefore

$$r = \#\ker(\theta) = \frac{\#\mathcal{C}_{4D}^+}{\#\mathcal{C}_D^+} = \frac{h_{4D}}{h_D},$$

and the latter quotient is calculated in [1], §151 as follows. Let (x_0, y_0) resp. (x_1, y_1) be the least positive solution of $x^2 - dy^2 = 1$ resp. $x^2 - 4dy^2 = 1$; then

$$r = \frac{2\log(x_0 + y_0\sqrt{d})}{\log(x_1 + 2y_1\sqrt{d})}.$$

From the theory of Pell's equation (cf. [1], §85) it follows that (P) has a solution (x, y) with $y \equiv 1 \pmod 2$ if and only if $y_0 \equiv 1 \pmod 2$, and in this case $x_1 + 2y_1\sqrt{d} = (x_0 + y_0\sqrt{d})^2$, whence $r = 1$. If $y_0 \equiv 0 \pmod 2$, then $y_1 = y_0/2$ and $r = 2$. ∎

In what follows let E (respectively E^*) denote the set of primitive 0-reduced ideals of O_{4D} (respectively O_D). Our next lemma provides a useful normalization of ideals in E^*.

LEMMA 3. *For each* $J \in E^*$, *there exists a unique* $C = C_J \in \mathbb{Z}$ *such that* $J = [A, B + \sqrt{d}]$, *where* $A, B \in \mathbb{Z}$, $A > 0$, $C = (B^2 - d)/A$, $(A, 2B, C) = 1$, *and* $\omega = (B + \sqrt{d})/A$ *satisfies*

$$1 < \overline{\omega} < 2 < \omega.$$

The number $\omega = \omega_J$ *is also uniquely determined by* J, *and* J *is* 1*-reduced if and only if* $\omega > 3$.

Proof. By Lemma 1, $J = [A, B + \sqrt{d}]$ where $A, B \in \mathbb{Z}$, $A > 0$, $C = (B^2 - d)/A \in \mathbb{Z}$ and $(A, 2B, C) = 1$. In this representation, $A = \mathcal{N}(J)$ is uniquely determined, B is uniquely determined modulo A, and each normalization of B also fixes C. There is a unique choice of B modulo A such that the number $\omega = (B + \sqrt{d})/A$ satisfies $1 < \overline{\omega} < 2$, and since $\omega + [-\overline{\omega}] > 0$, we infer $\omega > 2$. J is 1-reduced if and only if $\omega + [-\overline{\omega}] > 1$, i.e., $\omega > 3$. ∎

DEFINITION. (a) Let $J \in E^*$ be an ideal, $A = \mathcal{N}(J)$ and $C = C_J \in \mathbb{Z}$ the number introduced in Lemma 3. The ideal J is called

- *of type* 1 if $C \equiv 0 \pmod 2$ and J is strictly 0-reduced;
- *of type* 2 if either $C \equiv 0 \pmod 2$ and J is 1-reduced
  or $A \equiv 0 \pmod 2$ and J is strictly 0-reduced;
- *of type* 3 if $A \equiv C \equiv 1 \pmod 2$
  or $A \equiv 0 \pmod 2$ and J is 1-reduced.

(b) For a class $\mathfrak{c} \in \mathcal{C}_{4D}^+$ and $j \in \{1,2,3\}$, denote by $N_j^*(\mathfrak{c})$ the number of ideals of type j in $\theta(\mathfrak{c}) \cap E^*$, and set $N(\mathfrak{c}) = \#(E \cap \mathfrak{c})$, $N = \#E$.

(c) For $j \in \{1,2,3\}$, let N_j^* denote the number of ideals of type j in E^*.

THEOREM. *For any class* $\mathfrak{c} \in \mathcal{C}_{4D}^+$, *we have*

$$\sum_{j=1}^{3} j N_j^*(\mathfrak{c}) = \sum_{\mathfrak{c}' \in \theta^{-1}(\theta(\mathfrak{c}))} N(\mathfrak{c}').$$

The proof of Theorem 1 will be given in §3. Here we draw two corollaries.

COROLLARY 1. $N = N_1^* + 2N_2^* + 3N_3^*$.

Proof. With $r = \#\ker(\theta)$, we obtain

$$r\sum_{j=1}^{3} jN_j^* = \sum_{\mathfrak{c}\in\mathcal{C}_{4D}^+}\sum_{j=1}^{3} jN_j^*(\mathfrak{c}) = \sum_{\mathfrak{c}\in\mathcal{C}_{4D}^+}\sum_{\mathfrak{c}'\in\theta^{-1}(\theta(\mathfrak{c}))} N(\mathfrak{c}') = r\sum_{\mathfrak{c}\in\mathcal{C}_{4D}^+} N(\mathfrak{c}) = rN,$$

whence the assertion. ∎

COROLLARY 2. *The following assertions are equivalent*:

(a) *Pell's equation* $x^2 - dy^2 = 1$ *has a solution* (x, y) *with* $y \equiv 1 \pmod 2$.
(b) *For any class* $\mathfrak{c} \in \mathcal{C}_{4D}^+$, *we have* $N(\mathfrak{c}) = N_1^*(\mathfrak{c}) + 2N_2^*(\mathfrak{c}) + 3N_3^*(\mathfrak{c})$.

Proof. Since $N(\mathfrak{c}) > 0$ for every $\mathfrak{c} \in \mathcal{C}_{4D}^+$, Theorem 1 implies that (b) holds if and only if θ is an isomorphism. Now the assertion follows from Lemma 2. ∎

3. Proof of the theorem.

Throughout, we fix an ideal class $\mathfrak{c} \in \mathcal{C}_{4D}^+$, and we set

$$\overline{\mathfrak{c}} = \bigcup_{\mathfrak{c}' \in \theta^{-1}(\theta(\mathfrak{c}))} \mathfrak{c}'.$$

Clearly, $\overline{\mathfrak{c}} = \mathfrak{c}$ if θ is injective; otherwise $\overline{\mathfrak{c}} = \mathfrak{c} \cup \mathfrak{c}_1$ where $\mathfrak{c} \neq \mathfrak{c}_1 \in \mathcal{C}_{4D}^+$ and $\theta(\mathfrak{c}_1) = \theta(\mathfrak{c})$. We will study the effect of θ on the ideals $I \in E \cap \overline{\mathfrak{c}}$, given by

$(*)_{4D}$. To this end, we partition $E \cap \bar{\mathfrak{c}}$, defining E_i $(i = 1, 2, 3)$ by

$$E_1 = \{I \in E \cap \bar{\mathfrak{c}} \mid a \equiv 1 \pmod 2 \text{ and } \theta(I) \text{ is 0-reduced}\},$$
$$E_2 = \{I \in E \cap \bar{\mathfrak{c}} \mid a \equiv 1 \pmod 2 \text{ and } \theta(I) \text{ is not 0-reduced}\},$$
$$E_3 = \{I \in E \cap \bar{\mathfrak{c}} \mid a \equiv 0 \pmod 4\}.$$

For an ideal $J \in E^*$ with associated number ω, we denote by J' the ideal associated with $\omega' = ([\omega + 1] - \omega)^{-1}$; see [3]. Moreover, we set $A_J = \mathcal{N}(J)$ and we denote by $C_J \in \mathbb{Z}$ the number introduced in Lemma 3.

For $I \in E \cap \bar{\mathfrak{c}}$, we define

$$\psi(I) = \begin{cases} \theta(I) & \text{if } I \in E_1 \cup E_3, \\ \theta(I)' & \text{if } I \in E_2. \end{cases}$$

The theorem follows from Propositions 1, 2 and 3 below, giving the effect of ψ on E_1, E_2 and E_3, respectively.

PROPOSITION 1. *ψ maps E_1 bijectively to $E_1^* = \{J \in E^* \cap \theta(\mathfrak{c}) \mid A_J \equiv 1 \pmod 2\}$.*

Proof. By definition, $\psi(E_1) \subset E_1^*$. We must prove that, given $J \in E_1^*$, there is exactly one $I \in E_1$ with $\psi(I) = J$. Let $J \in E_1^*$ be given, $J = [A, B + \sqrt{d}]$, A odd, $B^2 - AC = d$ and $(A, 2B, C) = 1$.

For $I = [a, 2b + 2\sqrt{d}] \in E_1$, $\theta(I) = J$ if and only if $[a, b + \sqrt{d}] = [A, B + \sqrt{d}]$, that is, if and only if $a = A$ and $b = B + kA$ for some $k \in \mathbb{Z}$. Since $[A, 2B + 2kA + 2\sqrt{d}] = [A, 2B + 2\sqrt{d}]$ for all k, there is at most one ideal $I \in E_1$ such that $\theta(I) = J$, namely, $I = [A, 2B + 2\sqrt{d}]$. Now $c = (4b^2 - 4d)/a = 4(B^2 - d)/A = 4C$, so that $(a, 2b, c) = (A, 2B, 4C) = (A, 2B, C) = 1$, and therefore $[A, 2B + 2\sqrt{d}]$ is primitive.

If $\omega = (B + \sqrt{d})/A$, then as $J = A[1, \omega]$ is 0-reduced, $\omega + [-\overline{\omega}] > 0$. Now $[A, 2B + 2\sqrt{d}] = A[1, 2\omega]$ and $2\omega + [-2\overline{\omega}] \geq 2\omega + 2[-\overline{\omega}] > 0$ so that I is 0-reduced. This completes the proof. ∎

PROPOSITION 2. *ψ maps E_2 bijectively to $E_2^* = \{J \in E^* \cap \theta(\mathfrak{c}) \mid C_J \equiv 1 \pmod 2\}$.*

Proof. We first prove that $I \in E_2$ implies $\psi(I) = \theta(I)' \in E_2^*$. Let $I \in E_2$ be given, $I = a[1, \varphi] = [a, 2b + 2\sqrt{d}]$, where $a, b \in \mathbb{Z}$, $a > 0$, $a \equiv 1 \pmod 2$, $4b^2 - ac = 4d$, where $c \in \mathbb{Z}$ and $(a, 2b, c) = 1$. Since a is odd, we have $c_1 = c/4 \in \mathbb{Z}$. Furthermore,

$$(*) \qquad \varphi + [-\overline{\varphi}] > 0 \quad \text{and} \quad \frac{\varphi}{2} + \left[\frac{-\varphi}{2}\right] < 0,$$

since I is 0-reduced and $\theta(I)$ is not. Now $J = \psi(I) = [A, B + \sqrt{d}] = A[1, \omega]$, where $A = A_J$, $B \in \mathbb{Z}$, $C = (B^2 - d)/A \in \mathbb{Z}$ and $\omega = (B + \sqrt{d})/A = ([\varphi/2 + 1] - \varphi/2)^{-1}$. If $k = [-\overline{\varphi}/2]$, then $(*)$ implies $[-\overline{\varphi}] > -\varphi > 2k$; consequently, $[-\overline{\varphi}] = 2k + 1, -2k - 1 < \varphi < -2k$ and $[\varphi/2 + 1] = -k$.

Since $\omega = -2/(2k+\varphi)$, we obtain $\omega > 2$, and $\overline{\omega} = -2/(2k+\overline{\varphi})$ implies $1 < \overline{\omega} < 2$, since $-2 < 2k+\overline{\varphi} < -1$. In particular, we obtain $\omega + [-\overline{\omega}] > 0$, whence $J \in E^*$, and $C = C_J$.

Since $\varphi = (2b+2\sqrt{d})/a$, we infer $\omega = -2/(2k+\varphi) = -a/(ka+b+\sqrt{d}) = (B+\sqrt{d})/A$, which implies

$$B = -(ka+b), \qquad A = k^2a + 2bk + c_1$$

and therefore $C = (B^2-d)/A = a \equiv 1 \pmod 2$; thus $J \in E_2^*$.

Next, suppose $J \in E_2^*$, $J = [A, B+\sqrt{d}]$, where $A = A_J$, $B \in \mathbb{Z}$, $\omega = (B+\sqrt{d})/A$ satisfies $1 < \overline{\omega} < 2 < \omega$ and $C = C_J = (B^2-d)/A$ is odd. If $I \in E_2$ and $\psi(I) = J$, we normalize I in the form $I = a[1,\varphi]$, where $-2 < \overline{\varphi} < -1 < \varphi$. Since $\varphi/2 + [-\overline{\varphi}/2] < 0$ and $[-\overline{\varphi}/2] = 0$, we have $\varphi < 0$ and $J = A[1,\omega']$, where $\omega' = 1/([\varphi/2+1] - \varphi/2) = -2/\varphi$. Since $1 < \overline{\omega}' < 2 < \omega'$ we have $\omega = \omega'$, and so $\varphi = -2/\omega = (-2B+2\sqrt{d})/C$. Therefore, the only candidate for $I \in E_2$ satisfying $\psi(I) = J$ is

$$I_0 = [C, -2B+2\sqrt{d}] = C\cdot[1,\varphi], \qquad \text{where } \varphi = -2/\omega.$$

It remains to show that $I_0 \in E_2$. As $a = C$ and $b = -B$, we have $c = (4b^2-4d)/a = 4(B^2-d)/C = 4A$, and therefore $(a,2b,c) = (C,-2B,4A) = 1$ since C is odd. Thus I_0 is primitive. As $1 < \overline{\omega} < 2 < \omega$ we have $-2 < \overline{\varphi} < -1 < \varphi < 0$. This implies $\varphi + [-\overline{\varphi}] > \varphi + 1 > 0$ (so that I_0 is 0-reduced) and $\varphi/2 + [-\overline{\varphi}/2] = \varphi/2 < 0$ (so that $\theta(I_0)$ is not 0-reduced). Therefore, $I_0 \in E_2$ and the proof of Proposition 2 is complete. ∎

PROPOSITION 3. *Let $J = [A, B+\sqrt{d}] \in E^* \cap \theta(\mathfrak{c})$, where $A = A_J, B \in \mathbb{Z}$ and $C = C_J = (B^2-d)/A \in \mathbb{Z}$, be given. Then*

$$\#\{I \in E_3 \mid \psi(I) = J\} = \begin{cases} 2 & \text{if } J \text{ is 1-reduced and } A \equiv 0 \pmod 2,\\ 1 & \text{if } J \text{ is 1-reduced and } C \equiv 0 \pmod 2\\ & \text{or } J \text{ is 1-reduced and } A \equiv C \equiv 1 \pmod 2\\ & \text{or } J \text{ is not 1-reduced and } C \equiv 1 \pmod 2,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. For J given as above, $\omega = (B+\sqrt{d})/A$ satisfies $1 < \overline{\omega} < 2 < \omega$. If $I \in E_3$ is such that $\psi(I) = J$ then $I = [4a_1, 2b+2\sqrt{d}]$, where $a_1, b, c \in \mathbb{Z}$, $a_1 > 0$, $(4a_1, 2b, c) = 1$, $b^2 - a_1c = d$ and $\theta(I) = [a_1, b+\sqrt{d}] = [A, B+\sqrt{d}]$. Thus $a_1 = A$ and $b = B + kA$ for some $k \in \mathbb{Z}$, so that $I = I_k = [4A, 2B+2kA+2\sqrt{d}]$ for some $k \in \mathbb{Z}$. Since $I_k = I_{k+2}$ for each k, we have $I \in \{I_0, I_1\}$ and $\{I \in E_3 \mid \psi(I) = J\} = \{I_0, I_1\} \cap E$. It remains to determine the conditions under which each of the two candidates, I_0 and I_1, is a member of E.

First, $I_0 = [4A, 2B+2\sqrt{d}] = [a, 2b+2\sqrt{d}]$ has $c = (4b^2-4d)/a = 4(B^2-d)/(4a) = C$, and so $(a,2b,c) = (4A,2B,C)$. Hence I_0 is primitive if and only if C is odd. Now $\varphi_0 = (2B+2\sqrt{d})/(4A) = \omega/2$ satisfies $\varphi_0 + [-\overline{\varphi}_0] =$

$\omega/2 + [-\overline{\omega}/2] = \omega/2 - 1 > 0$ so that I_0 is always 0-reduced. Thus, $I_0 \in E$ precisely when it is primitive, that is, when $C_J \equiv 1 \pmod 2$.

Next, $I_1 = [4A, 2B + 2A + 2\sqrt{d}] = [a, 2b + 2\sqrt{d}]$ has $c = (4b^2 - 4d)/a = 4((A + B)^2 - d)/(4A) = A + 2B + C$ and so $(a, 2b, c) = (4A, 2B + 2A, A + 2B + C)$. Hence I is primitive if and only if $A + C$ is odd.

Now $\varphi_1 = (2B + 2A + 2\sqrt{d})/(4A) = (\omega + 1)/2$ satisfies $\varphi_1 + [-\overline{\varphi}_1] = (\omega + 1)/2 + [-(\omega + 1)/2] = (\omega + 1)/2 + (-2) > 0$ if and only if $\omega > 3$, that is, if and only if J is 1-reduced. Thus, $I_1 \in E$ precisely when $A + C$ is odd and J is 1-reduced. Proposition 3 follows easily from the preceding two observations. ■

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