COLLOQUIUM MATHEMATICUM

VOL. LXIX 1995 FASC. 1

A NOTE ON JEŚMANOWICZ' CONJECTURE

BY

MAOHUA LE (ZHANJIANG)

1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of integers and positive integers respectively. Let (a,b,c) be a primitive Pythagorean triple such that

$$a^2+b^2=c^2,\quad a,b,c\in\mathbb{N},\ \gcd(a,b,c)=1,\ 2\mid b. \tag{1}$$

Then we have

$$a=r^2-s^2,\quad b=2rs,\quad c=r^2+s^2, \tag{2}$$

where $r,s\in\mathbb{N}$ satisfy $\gcd(r,s)=1$, $r>s$ and $2\mid rs$. In this respect, Jeśmanowicz [6] conjectured that the equation

$$a^x+b^y=c^z,\quad x,y,z\in\mathbb{N}, \tag{3}$$

has only the solution $(x,y,z)=(2,2,2)$. This problem was solved for some special cases by Sierpiński [16], Ke [8, 9, 10], Ke and Sun [11], Lu [12], Rao [15], Chen [2], Józefiak [7], Podsypanin [14], Dem'yanenko [3], Grytczuk and Grelak [4]. In general, the problem is not solved yet. In this note we prove the following result.

THEOREM. *If $2\parallel rs$ and $c=p^n$, where p is an odd prime and $n\in\mathbb{N}$, then* (3) *has only the solution $(x,y,z)=(2,2,2)$.*

2. Preliminaries. For any $k\in\mathbb{N}$ with $k>1$ and $4\nmid k$, let

$$V(k)=\prod_{q\mid k}(1+\chi(q)),$$

where q runs over distinct prime factors of k, and

$$\chi(q)=\begin{cases}0 & \text{if } q=2,\\ (-1)^{(q-1)/2} & \text{if } q\neq 2.\end{cases}$$

1991 *Mathematics Subject Classification*: Primary 11D61.

Supported by the National Natural Science Foundation of China.

Lemma 1 ([5, Theorems 6·7·1 and 6·7·4]). *The equation*

$$X_1^2 + Y_1^2 = k, \qquad X_1, Y_1 \in \mathbb{Z},\ \gcd(X_1, Y_1) = 1, \tag{4}$$

has exactly $4V(k)$ *solutions* (X_1, Y_1).

Lemma 2 ([13, Chapter 15]). *If* $2 \nmid k$, *then all solutions* (X, Y, Z) *of the equation*

$$X^2 + Y^2 = k^Z, \qquad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0,$$

are given by

$$Z \in \mathbb{N}, \quad X + Y\sqrt{-1} = (X_1 + Y_1\sqrt{-1})^Z \quad \text{or} \quad Y + X\sqrt{-1} = (X_1 + Y_1\sqrt{-1})^Z,$$

where (X_1, Y_1) *runs over all solutions of* (4).

Lemma 3 ([1]). *Let* $D \in \mathbb{N}$ *with* $D > 1$, *and let* p *be an odd prime with* $p \nmid D$. *If the equation*

$$X^2 + DY^2 = p^Z, \qquad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0, \tag{5}$$

has solutions (X, Y, Z), *then it has a unique solution* (X_1, Y_1, Z_1) *such that* $X_1 > 0$, $Y_1 > 0$ *and* $Z_1 \le Z$, *where* Z *runs over all solutions of* (5). *The solution* (X_1, Y_1, Z_1) *is called the least solution of* (5). *Moreover, all solutions of* (5) *are given by*

$$Z = Z_1 t, \quad X + Y\sqrt{-D} = \lambda_1 (X_1 + \lambda_2 Y_1 \sqrt{-D})^t, \quad t \in \mathbb{N},\ \lambda_1, \lambda_2 \in \{-1, 1\}.$$

Lemma 4. *If* $c = p^n$, *then* $(X_1, Y_1, Z_1) = (r - s, 1, n)$ *is the least solution of the equation*

$$X^2 + bY^2 = p^Z, \qquad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0. \tag{6}$$

P r o o f. Clearly, $(X, Y, Z) = (r - s, 1, n)$ is a solution of (6). By Lemma 3, if $(X_1, Y_1, Z_1) \ne (r - s, 1, n)$, then there exists $t \in \mathbb{N}$ such that $t > 1$ and $n = Z_1 t$. Since $X_1^2 + bY_1^2 = p^{Z_1}$, we get $r^2 + s^2 = c = p^n \ge p^{2Z_1} \ge (1 + b)^2 = (1 + 2rs)^2 > 4(r^2 + s^2)$, a contradiction. The lemma is proved.

3. Proof of Theorem. Let (x, y, z) be a solution of (3). If $2 \nmid x$ and $2 \mid y$, then we have $(-a/c) = 1$, where $(\cdot/\cdot)$ denotes Jacobi's symbol. Since $2 \parallel rs$, we see from (2) that $c \equiv 5 \pmod 8$. Hence, by (2),

$$1 = \left(\frac{-a}{c}\right) = \left(\frac{s^2 - r^2}{r^2 + s^2}\right) = \left(\frac{2s^2}{r^2 + s^2}\right) = \left(\frac{2}{r^2 + s^2}\right) = \left(\frac{2}{c}\right) = -1,$$

a contradiction. Similarly, if $2 \nmid xy$, then we have

$$1 = \left(\frac{-ab}{c}\right) = \left(\frac{2rs(s^2 - r^2)}{r^2 + s^2}\right) = \left(\frac{4rs^3}{r^2 + s^2}\right)$$

$$= \left(\frac{4rs}{r^2+s^2}\right) = \left(\frac{2(r+s)^2}{r^2+s^2}\right) = \left(\frac{2}{r^2+s^2}\right) = \left(\frac{2}{c}\right) = -1,$$

a contradiction.

If $2 \mid x$ and $2 \mid y$, then $a^x + b^y \equiv 1 \pmod 8$. Since $c \equiv 5 \pmod 8$, we see from (3) that $2 \mid z$. Then $(X, Y, Z) = (a^{x/2}, b^{y/2}, z/2)$ is a solution of the equation

$$X^2 + Y^2 = c^{2Z}, \qquad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0.$$

Notice that c is an odd prime power. By Lemmas 1 and 2, we get the following four cases ($\lambda_1, \lambda_2 \in \{-1, 1\}$):

$$\begin{aligned} a^{x/2} + b^{y/2}\sqrt{-1} &= \lambda_1(a + \lambda_2 b\sqrt{-1})^{z/2}, \\ a^{x/2} + b^{y/2}\sqrt{-1} &= \lambda_1(b + \lambda_2 a\sqrt{-1})^{z/2}, \\ b^{y/2} + a^{x/2}\sqrt{-1} &= \lambda_1(a + \lambda_2 b\sqrt{-1})^{z/2}, \\ b^{y/2} + a^{x/2}\sqrt{-1} &= \lambda_1(b + \lambda_2 a\sqrt{-1})^{z/2}. \end{aligned} \tag{7}$$

When $z = 2$, we find from (7) that $x = y = 2$.

When $z > 2$ and $2 \mid z/2$, (7) is impossible, since $a > 1$, $b > 1$ and $\gcd(a, b) = 1$.

When $z > 2$ and $2 \nmid z/2$, we see from (7) that

$$a^{x/2} + b^{y/2}\sqrt{-1} = \lambda_1(a + \lambda_2 b\sqrt{-1})^{z/2}, \qquad \lambda_1, \lambda_2 \in \{-1, 1\},$$

whence we get

$$a^{x/2} = \lambda_1 a \sum_{i=0}^{(z-2)/4} \binom{z/2}{2i+1} a^{2i}(-b^2)^{(z-2)/4-i} \tag{8}$$

and

$$b^{y/2} = \lambda_1\lambda_2 b \sum_{i=0}^{(z-2)/4} \binom{z/2}{2i+1} a^{z/2-2i-1}(-b^2)^{i}. \tag{9}$$

Since $2 \nmid a$, $2 \mid b$ and $2 \nmid z/2$, we see from (9) that $y = 2$. Further, since $z > 2$, we get $x > 2$ by (1), and $z/2 \equiv 0 \pmod a$ by (8). Let q be a prime factor of a, and let $q^\alpha \parallel a$, $q^\beta \parallel z/2$ and $q^{\gamma_i} \parallel 2i+1$ for any $i \in \mathbb{N}$. Notice that $q \ge 3$ and

$$\gamma_i \le \frac{\log(2i+1)}{\log q} < 2i, \qquad i \in \mathbb{N}.$$

We have

$$\begin{aligned} \binom{z/2}{2i+1} a^{2i} &= \frac{z}{2}\binom{z/2-1}{2i}\frac{a^{2i}}{2i+1} \\ &\equiv 0 \pmod{q^{\beta+1}}, \qquad i = 1, \ldots, (z-2)/4. \end{aligned} \tag{10}$$

On applying (10) together with (8), we obtain $\beta = \alpha(x/2-1)$ for any prime factor q of a. This implies that

$$z/2 \equiv 0 \pmod{a^{x/2-1}}. \tag{11}$$

Since $y = 2$, we see from (3) and (11) that

$$c^{x+2} > a^x + b^2 = a^x + b^y = c^z \geq c^{2a^{x/2-1}},$$

whence we get

$$x + 2 > 2a^{x/2-1}. \tag{12}$$

But since $a \geq 3$ and $x > 2$, (12) is impossible.

If $2 \mid x$ and $2 \nmid y$, then $(X, Y, Z) = (a^{x/2}, b^{(y-1)/2}, nz)$ is a solution of (6). By Lemmas 3 and 4, we have

$$a^{x/2} + b^{(y-1)/2}\sqrt{-b} = \lambda_1((r-s) + \lambda_2\sqrt{-b})^z, \qquad \lambda_1, \lambda_2 \in \{-1, 1\}. \tag{13}$$

When $2 \mid z$, we see from (13) that $b^{(y-1)/2} \equiv 0 \pmod{r-s}$. By (1) and (2), this implies that $r - s = 1$. By [3], the theorem holds in this case.

When $2 \nmid z$, since $c \equiv 5 \pmod 8$, we have $a^x \equiv 1 \pmod 8$, $c^z \equiv 5 \pmod 8$ and $y = 1$ by (3). On the other hand, we deduce from (13) that

$$\begin{aligned} \frac{a^{x/2}}{r-s} &= (r-s)^{x/2-1}(r+s)^{x/2} \\ &= \lambda_1 \sum_{i=0}^{(z-1)/2} \binom{z}{2i+1}(r-s)^{2i}(-b)^{(z-1)/2-i}. \end{aligned} \tag{14}$$

If $x = 2$, then we have $c = r^2 + s^2 < (r^2 - s^2)^2 + 2rs = a^2 + b = c^z < c^2$, a contradiction. If $x > 2$, then $z \equiv 0 \pmod{r-s}$. Let q be a prime factor of $r - s$, and let $q^\alpha \parallel r - s$, $q^\beta \parallel z$ and $q^{\gamma_i} \parallel 2i + 1$ for any $i \in \mathbb{N}$. Since $2 \nmid r - s$, $q \geq 3$ and $\gamma_i \leq (\log(2i+1))/\log q < 2i$ for any $i \in \mathbb{N}$, we find from (14) and

$$\begin{aligned} \binom{z}{2i+1}(r-s)^{2i} &= z\binom{z-1}{2i}\frac{(r-s)^{2i}}{2i+1} \\ &\equiv 0 \pmod{q^{\beta+1}}, \qquad i = 1, \ldots, (z-1)/2, \end{aligned}$$

that $\beta = \alpha(x/2-1)$. This implies that

$$z \equiv 0 \pmod{(r-s)^{x/2-1}}. \tag{15}$$

Recalling that $y = 1$, we see from (3) and (15) that

$$c^x > a^x + b = c^z = c^{(r-s)^{x/2-1}z_1}, \qquad z_1 \in \mathbb{N},\ 2 \nmid z_1. \tag{16}$$

Since $r - s \geq 3$ and $x \geq 4$, we find from (16) that $r - s = 3$, $x = 4$, $z_1 = 1$ and $z = 3$. In this case, (14) can be written as $(r+s)^2 = b - 3 = 2rs - 3$, a contradiction. Thus, the proof is complete.

Acknowledgments. The author would like to thank the referee for his valuable suggestions.

REFERENCES

[1] R. Apéry, *Sur une équation diophantienne*, C. R. Acad. Sci. Paris Sér. I Math. 251 (1960), 1451–1452.
[2] J.-R. Chen, *On Jeśmanowicz' conjecture*, J. Sichuan Univ. Nat. Sci. 1962 (2), 19–25 (in Chinese).
[3] V. A. Dem'yanenko, *On Jeśmanowicz' problem for Pythagorean numbers*, Izv. Vyssh. Uchebn. Zaved. Mat. 1965 (5), 52–56 (in Russian).
[4] A. Grytczuk and A. Grelak, *On the equation* $a^x + b^y = c^z$, Comment. Math. Prace Mat. 24 (1984), 269–275.
[5] L.-K. Hua, *Introduction to Number Theory*, Springer, Berlin, 1982.
[6] L. Jeśmanowicz, *Several remarks on Pythagorean numbers*, Wiadom. Mat. (2) 1 (1955/1956), 196–202 (in Polish).
[7] T. Józefiak, *On a conjecture of L. Jeśmanowicz concerning Pythagorean numbers*, Comment. Math. Prace Mat. 5 (1961), 119–123 (in Polish).
[8] Z. Ke, *On Pythagorean numbers*, J. Sichuan Univ. Nat. Sci. 1958 (1), 73–80 (in Chinese).
[9] —, *On Jeśmanowicz' conjecture*, ibid. 1958 (2), 81–90 (in Chinese).
[10] —, *On the diophantine equation* $(2n+1)^x + (2n(n+1))^y = (2n(n+1)+1)^z$, ibid. 1959 (3), 25–34 (in Chinese).
[11] Z. Ke and Q. Sun, *On Pythagorean numbers* $2n+1, 2n(n+1)$ *and* $2n(n+1)+1$, ibid. 1964 (2), 1–12 (in Chinese).
[12] W.-D. Lu, *On the Pythagorean numbers* $4n^2-1, 4n$ *and* $4n^2+1$, ibid. 1959 (2), 39–42 (in Chinese).
[13] L. J. Mordell, *Diophantine Equations*, Academic Press, London, 1969.
[14] V. D. Podsypanin, *On a property of Pythagorean numbers*, Izv. Vyssh. Uchebn. Zaved. Mat. 1962 (4), 130–133 (in Russian).
[15] D.-M. Rao, *A note on the diophantine equation* $(2n+1)^x + (2n(n+1))^y = (2n(n+1)+1)^z$, J. Sichuan Univ. Nat. Sci. 1960 (1), 79–80 (in Chinese).
[16] W. Sierpiński, *On the equation* $3^x + 4^y = 5^z$, Wiadom. Mat. (2) 1 (1955/1956), 194–195 (in Polish).

DEPARTMENT OF MATHEMATICS
ZHANJIANG TEACHERS' COLLEGE
P.O. BOX 524048
ZHANJIANG, GUANGDONG
P.R. CHINA

Reçu par la Rédaction le 20.6.1994