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A NOTE ON A MULTI-VARIABLE POLYNOMIAL LINK INVARIANT

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In 1985 the Homfly polynomial was discovered independently by several groups of authors (see [PT], [FYHLMO]). Various possible generalizations were also discussed in [PT] and [HW]; see also [L]. In particular, one many-variable polynomial link invariant was defined in [PT], Example 3.9. In this note we show that this invariant is in fact equivalent to the Homfly polynomial.

First of all we recall the definition of the invariant. This is a polynomial w in variables $y_1^{\pm 1}$, ${x'_2}^{\pm 1}$, z'_2, z_i, $x_i^{\pm 1}$, $i \in \mathbb{N}$, satisfying the following three conditions.

(0) For the trivial link T_n of n components the following equality holds:

$$w(T_n) = \prod_{i=1}^{n-1}(x_i + y_i) + z_1 \prod_{i=2}^{n-1}(x_i + y_i) + \ldots + z_{n-2}(x_{n-1} + y_{n-1}) + z_{n-1}.$$

The next two conditions involve the notion of multiplicity pattern. We say that a triple $D_+,\ D_-,\ D_0$ of oriented diagrams *has multiplicity pattern* (n, k) if D_+ and D_- have n components each, and D_0 has k components. If the specified crossing of D_+ is a self-crossing of one component then $k = n + 1$. Otherwise $k = n - 1$. Thus, the only patterns that may appear are $(n, n + 1)$ and $(n, n - 1)$. Let $w_+ = w(D_+)$, $w_- = w(D_-)$, $w_0 = w(D_0)$. Then the next two conditions defining the polynomial w are:

(1) $x_n w_+ + y_n w_- = w_0 - z_n$ for multiplicity pattern $(n, n + 1)$ and

(2) $x'_n w_+ + y'_n w_- = w_0 - z'_n$ for multiplicity pattern $(n, n - 1)$.

Moreover, $x_i,\ y_i,\ z_i,\ x'_i,\ y'_i$, and z'_i are supposed to satisfy

$$y_i = x_i \frac{y_1}{x_1}, \quad x'_i = \frac{x'_2 x_1}{x_{i-1}}, \quad y'_i = \frac{x'_i y_1}{x_1}, \tag{3}$$
$$\frac{z'_{i+1} - z_{i-1}}{x_1 x'_2} = \left(1 + \frac{y_1}{x_1}\right)\left(\frac{z'_i}{x'_i} - \frac{z_i}{x_i}\right),$$

for $i = 1, 2, \ldots$

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In Problem 4.4 of [PT] it is asked whether the polynomial w is a better invariant of links than the Homfly polynomial. We will show that the answer is negative, namely:

THEOREM. *The polynomial w is equivalent to the Homfly polynomial.*

Here "equivalent" means that given the value of $h(K)$ we can calculate $w(K)$ and conversely.

Proof. We will use the Homfly polynomial, denoted by h, in variables x and y as defined by the equalities

$$xh_+ + yh_- = h_0, \qquad h(T_n) = (x+y)^{n-1}.$$

Let us observe that if the link K has n components, then $\deg h(K) = n-1$. Obviously, the substitutions $x_i = x_2' = x$, $y_1 = y$, $z_i = z_2' = 0$ for $i \in \mathbb{N}$ yield $w(K) = h(K)$. Therefore, it is enough to show that given the value of the Homfly polynomial for a link K we can determine the polynomial $w(K)$.

Let us begin with a simplification of the definition of w. Let $x := x_1$, $y := y_1$. In this notation, after multiplication of both sides of formulas (1) and (2) by x/x_n and x_{n-1}/x_2' respectively, we obtain the following identities:

$$(0') \qquad w(T_n) = \left(1+\frac{y}{x}\right)^{n-1} x x_2 \ldots x_{n-1} + \sum_{k=1}^{n-1} z_k \left(1+\frac{y}{x}\right)^{n-1-k} x_{k+1} \ldots x_{n-1},$$

$$(1') \qquad xw_+ + yw_- = \frac{x}{x_n}(w_0 - z_n) \qquad \text{for } (n, n+1),$$

$$(2') \qquad xw_+ + yw_- = \frac{x_{n-1}}{x_2'}(w_0 - z_n') \qquad \text{for } (n, n-1).$$

From (3) we have

$$z_{n+1}' = z_{n-1} + x x_2' \left(1+\frac{y}{x}\right)\left(\frac{z_n' x_{n-1}}{x_2' x} - \frac{z_n}{x_n}\right) \qquad \text{for } n \geq 2.$$

LEMMA 1. *The following equality holds*:

$$z_n' = \sum_{k=1}^{n-1} c_{n,k} z_k + c_{n,2'} z_2',$$

where the parameters $c_{n,k}, c_{n,2'}$ are defined in the following manner. Set

$$c_{n,1} = \begin{cases} (1+y/x)^{n-3} x_2 \ldots x_{n-2} & \text{for } n > 3, \\ 1 & \text{for } n = 3, \\ 0 & \text{for } n \leq 2. \end{cases}$$

Let $\alpha = 1 - xx_2'(1+y/x)^2$. *For* $k \ge 2$ *define* $c_{n,k}$ *as*

$$c_{n,k} = \begin{cases} \alpha(1+y/x)^{n-k-2}x_{k+1}\ldots x_{n-2} & \textit{for } n > k+2,\\ \alpha & \textit{for } n = k+2,\\ -(x_2'/x_k)(x+y) & \textit{for } n = k+1,\\ 0 & \textit{for } n \le k, \end{cases}$$

and

$$c_{n,2'} = \begin{cases} 1 & \textit{for } n = 2,\\ (1+y/x)^{n-2}x_1\ldots x_{n-2} & \textit{for } n \ge 3. \end{cases}$$

Proof. By induction on n. ∎

We will prove the next two lemmas using the method described in [K] and [PT], namely: Let D be an oriented diagram of n components, and let $\mathrm{cr}(D)$ denote the number of crossings in D. Let $b = (b_1, \ldots, b_n)$ be base points of D, one point for each component of D, none of them a crossing point. Now, one travels along D (according to the orientation of D) starting from b_1, then (when the walk along the first component is completed) from b_2 and so on. Any crossing that is passed by a tunnel when first encountered is called a *bad crossing*.

Let us consider all possible choices of $(b_1, \ldots, b_n)$. We denote the minimal number of bad crossings in D (over all possible choices of base points) by $\chi(D)$. For a given diagram D let $(b_1, \ldots, b_n)$ be base points of D such that the number of bad crossings in D is minimal possible. We may assume that the first bad crossing is positive. We denote D by D_+ with respect to this crossing. Then $\chi(D_-) < \chi(D_+)$, and D_0 has less crossings than D. Therefore, in order to prove some property of $w(D)$, it is convenient to use induction on $\mathrm{cr}(D)$ and $\chi(D)$.

Lemma 2. *For every diagram D of n components we can group the terms of $w(D)$ as follows*:

$$w(D) = w_0(D) + w_1(D)z_1 + \sum_{k=2}^{\infty} d_{n,k}z_k + w_{2'}(D)z_2',$$

where

$$d_{n,k} = \begin{cases} (x_{k+1}+y_{k+1})\ldots(x_{n-1}+y_{n-1}) & \textit{for } n \ge k+2,\\ 1 & \textit{for } n = k+1,\\ 0 & \textit{for } n \le k, \end{cases}$$

and w_0, w_1, $w_{2'}$ *are polynomials in the variables* $y^{\pm 1}$, ${x_2'}^{\pm 1}$, *and* $x_i^{\pm 1}$, $i \in \mathbb{N}$.

Proof (by induction on cr and χ). If $\mathrm{cr}(D) = 0$ then from $(0')$ the lemma is true. Let us assume that it holds for all D such that $\mathrm{cr}(D) \le c$. Let $\mathrm{cr}(D) \le c+1$. Now we apply induction on $\chi(D)$. If $\chi(D) = 0$ then D is a trivial link and the lemma is true. Let us assume that it holds for D such that $\chi(D) \le s$. If $\chi(D) = s+1$ then there is a crossing in D (we assume that it is positive) such that $D = D_+$, $\mathrm{cr}(D_0) \le c$ and $\chi(D_-) \le s$. Therefore D_0 and D_- satisfy the inductive hypothesis. If D_+, D_0 have respectively n and $n+1$ components then we have

$$w(D_0) = w_0(D_0) + w_1(D_0)z_1 + \sum_{k=2}^{\infty} d_{n+1,k} z_k + w_{2'}(D_0) z_{2'},$$

$$w(D_-) = w_0(D_-) + w_1(D_-)z_1 + \sum_{k=2}^{\infty} d_{n,k} z_k + w_{2'}(D_-) z_{2'}.$$

Then from $(1')$ for $n > 2$ we obtain

$$\begin{aligned}
w(D_+) &= \frac{1}{x_n} w(D_0) - \frac{z_n}{x_n} - \frac{y}{x} w(D_-) \\
&= \left[\frac{1}{x_n} w_0(D_0) - \frac{y}{x} w_0(D_-)\right] + \left[\frac{1}{x_n} w_1(D_0) - \frac{y}{x} w_1(D_-)\right] z_1 \\
&\quad + \sum_{k=2}^{n-1} \left(\frac{1}{x_n} d_{n+1,k} - \frac{y}{x} d_{n,k}\right) z_k + \left[\frac{1}{x_n} d_{n+1,n} - \frac{y}{x} d_{n,n} - \frac{1}{x_n}\right] z_n \\
&\quad + \left[\frac{1}{x_n} w_{2'}(D_0) - \frac{y}{x} w_{2'}(D_-)\right] z_2'.
\end{aligned}$$

One readily sees that

$$\frac{1}{x_n} d_{n+1,n} - \frac{y}{x} d_{n,n} - \frac{1}{x_n} = 0$$

and

$$\frac{1}{x_n} d_{n+1,k} - \frac{y}{x} d_{n,k} = d_{n,k} \qquad \text{for } k = 2, \ldots, n-1.$$

This completes the proof for this case.

For $n = 1$ and $n = 2$ the proof is similar and even simpler. What remains to prove is the case when D_+ and D_0 have respectively n and $n-1$ components. In this situation $(2')$ implies

$$\begin{aligned}
w(D_+) &= \frac{x_{n-1}}{x x_2'} w(D_0) - \frac{x_{n-1} z_n'}{x x_2'} - \frac{y}{x} w(D_-) \\
&= \left[\frac{x_{n-1}}{x x_2'} w_0(D_0) - \frac{y}{x} w_0(D_-)\right] + \left[\frac{x_{n-1}}{x x_2'} w_1(D_0) - \frac{y}{x} w_1(D_-) - \frac{x_{n-1}}{x x_2'} c_{n,1}\right] z_1 \\
&\quad + \sum_{k=2}^{\infty} \left[\frac{x_{n-1}}{x x_2'} (d_{n-1,k} - c_{n,k}) - \frac{y}{x} d_{n,k}\right] z_k \\
&\quad + \left[\frac{x_{n-1}}{x x_2'} w_{2'}(D_0) - \frac{x_{n-1}}{x x_2'} c_{n,2'} - \frac{y}{x} w_{2'}(D_-)\right] z_{2'}.
\end{aligned}$$

It is easy to prove that the equalities

$$\frac{x_{n-1}}{xx_2'}(d_{n-1,k}-c_{n,k})-\frac{y}{x}d_{n,k}=d_{n,k}$$

hold for $k\ge 2$ by checking directly the cases $n\ge k+3$, $n=k+2$, $n=k+1$, and $n\le k$. This completes the proof of Lemma 2. ∎

LEMMA 3. *Let D have n components. Then $w_0(D)$, $w_1(D)$, and $w_{2'}(D)$ are sums of monomials of the form*

$$\begin{cases} cx_2\ldots x_{n-1}x^\alpha y^\beta \dfrac{1}{{x_2'}^\gamma} & \textit{for } n\ge 3,\\ cx^\alpha y^\beta \dfrac{1}{{x_2'}^\gamma} & \textit{for } n\le 2,\end{cases}$$

and the sum of the exponents of $x, y, 1/x_2', x_2, \ldots, x_{n-1}$ in each of these monomials is equal to $n-2$ for w_1, and to $n-1$ for w_0 and $w_{2'}$.

Proof. Follows from $(1')$ and $(2')$ by induction on $\mathrm{cr}(D)$ and $\chi(D)$. ∎

Let v_0, v_1, $v_{2'}$ be polynomials (in the variables $x^{\pm1}$, $y^{\pm1}$, ${x_2'}^{\pm1}$) equal to w_0, w_1, $w_{2'}$ after substitution $x_i := x$, $i\in\mathbb{N}$, and let $\overline{v}_0$, $\overline{v}_1$, $\overline{v}_{2'}$ be the polynomials obtained respectively from v_0, v_1, $v_{2'}$ by putting x in place of x_2'. Now from $(0')$, $(1')$, and $(2')$ we have

$$x\overline{v}_{0_+}+y\overline{v}_{0_-}=\overline{v}_{0_0} \quad\text{and}\quad \overline{v}_0(T_n)=(x+y)^{n-1}.$$

Therefore $\overline{v}_0(D)=h(D)$ for each D.

For $\overline{v}_1$ the identities $(0')$, $(1')$, and $(2')$ take the form

$$\begin{cases} x\overline{v}_{1_+}+y\overline{v}_{1_-}=\overline{v}_{1_0}-1 & \text{for } (1,2) \text{ pattern},\\ x\overline{v}_{1_+}+y\overline{v}_{1_-}=\overline{v}_{1_0} & \text{for } (2,1) \text{ and } (n,n+1) \text{ patterns},\\ & \qquad n\ne 1,\\ x\overline{v}_{1_+}+y\overline{v}_{1_-}=\overline{v}_{1_0}-(x+y)^{(n-3)} & \text{for } (n,n-1) \text{ pattern, } n\ne 2,\\ \overline{v}_1(T_1)=0, & \\ \overline{v}_1(T_n)=(x+y)^{n-2} & \text{for } n\ge 2.\end{cases}$$

One readily sees that the polynomial

$$\frac{x+y}{(x+y)^2-1}(h-(x+y)^{n-1})+\begin{cases}(x+y)^{n-2} & \text{for } n\ge 2,\\ 0 & \text{for } n=1,\end{cases}$$

satisfies all the above conditions. Since these conditions uniquely define $\overline{v}_1$, therefore

$$\overline{v}_1=\frac{x+y}{(x+y)^2-1}(h-(x+y)^{n-1})+\begin{cases}(x+y)^{n-2} & \text{for } n\ge 2,\\ 0 & \text{for } n=1.\end{cases}$$

Finally, $\overline{v}_{2'}$ is defined by

$$\begin{cases} x\overline{v}_{2'_+} + y\overline{v}_{2'_-} = \overline{v}_{2'_0} & \text{for } (n, n+1) \text{ pattern,} \\ x\overline{v}_{2_+} + y\overline{v}_{2'_-} = \overline{v}_{2'_0} - (x+y)^{n-2} & \text{for } (n, n-1) \text{ pattern,} \\ \overline{v}_{2'}(T_n) = 0. \end{cases}$$

In the same way we prove that $\overline{v}_{2'} = (h - (x+y)^{n-1})/((x+y)^2 - 1)$. Since the number of components of D is equal to $\deg h(D) + 1$ we have the following:

Corollary. *From h we can calculate $\overline{v}_0$, $\overline{v}_1$ and $\overline{v}_{2'}$.*

If we know the form of $\overline{v}_0$ then we can reconstruct v_0: if a monomial $cx^a y^b$ appears in $\overline{v}_0$ then a monomial $cx^\alpha y^b \frac{1}{{x'_2}^\gamma}$ appears in v_0 with $\alpha, \gamma \in \mathbb{Z}$ satisfying

$$\alpha - \gamma = a, \qquad \alpha + \gamma + b = n - 1.$$

In a similar way we can deal with $\overline{v}_1$ and $\overline{v}_{2'}$. If we know $v_0(D)$ we can calculate $w_0(D)$: if $n \le 2$ then $w_0(D) = v_0(D)$, and for $n > 2$ the monomials of the form $cx^\alpha y^\beta \frac{1}{{x'_2}^\gamma}$ in $v_0(D)$ correspond to monomials

$$cx_2 \ldots x_{n-2} x^{\alpha-(n-3)} y^\beta \frac{1}{{x'_2}^\gamma}$$

in $w_0(D)$. The case of v_1 and $v_{2'}$ is similar. This completes the proof of the Theorem. ■

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