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THE DUALITY CORRESPONDENCE OF INFINITESIMAL CHARACTERS

BY

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We determine the correspondence of infinitesimal characters of representations which occur in Howe's Duality Theorem. In the appendix we identify the lowest K-types, in the sense of Vogan, of the unitary highest weight representations of real reductive dual pairs with at least one member compact.

0. Introduction. Let $(W, \langle\ ,\ \rangle)$ be a finite-dimensional, real or complex, symplectic vector space. Let $\mathrm{Sp}(W, \langle\ ,\ \rangle) = \mathrm{Sp}$ denote the isometry group of the form $\langle\ ,\ \rangle$, and let $\mathfrak{sp}$ be its Lie algebra.

DEFINITION 0.1 [8, 10]. A pair of subgroups G, G' of Sp is called a *reductive dual pair* if

(0.2) G' is the centralizer of G in Sp and vice versa; and

(0.3) both G, G' act reductively on W.

These pairs have been classified [7, 9]. For a real reductive dual pair G, G' (contained in Sp) let its *complexification*

(0.4) $\mathbf{G}, \mathbf{G}'$ be the smallest complex reductive dual pair in the complexification of the algebraic group Sp such that $\mathbf{G}$ contains G, and $\mathbf{G}'$ contains G'.

We will use bold letters to denote complexifications.

Suppose $W = W_1 \oplus W_2$ is an orthogonal direct sum decomposition of W and each W_j is invariant by G and G'. Let G_j be the restriction of G to W_j. Define G'_j similarly. Then $G = G_1 \times G_2$ and $G' = G'_1 \times G'_2$ and G_j, G'_j is a reductive dual pair in $\mathrm{Sp}(W_j)$, $j = 1, 2$.

DEFINITION 0.5 [8, 10]. We say that the reductive dual pair G, G' is *irreducible* if it has no non-trivial direct sum decomposition like that described above.

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By the *metaplectic group* $\widetilde{\mathrm{Sp}}$ one understands the unique connected twofold covering group of the real symplectic group Sp. For any reductive Lie subgroup E of Sp let

(0.6) $\widetilde{E}$ be its preimage in the metaplectic group $\widetilde{\mathrm{Sp}}$.

Denote by $R(\widetilde{E})$ the set of infinitesimal equivalence classes ([19, 0.3.9]) of continuous irreducible admissible representations of $\widetilde{E}$ on locally convex topological vector spaces. The group $\widetilde{\mathrm{Sp}}$ has a unitary representation ω called the *oscillator representation* [1, 10, 15, ...]. Let ω^∞ be the smooth representation associated with ω. Denote by

(0.7) $\mathcal{R}(\widetilde{E},\omega)$ the set of elements of $\mathcal{R}(\widetilde{E})$ which can be realized as $\omega^\infty(\widetilde{E})$-invariant quotients by closed subspaces of the space ω^∞.

The following theorem of Roger Howe reveals a very special character of the oscillator representation.

THEOREM 0.8 [7]. *The set $\mathcal{R}(G\cdot G',\omega)$ is the graph of a bijection between* (*all of*) $\mathcal{R}(G,\omega)$ *and* (*all of*) $\mathcal{R}(G',\omega)$. *In other words*, *for each $\Pi\in\mathcal{R}(G,\omega)$ there is a unique $\Pi'\in\mathcal{R}(G',\omega)$ such that*

$$\Pi\otimes\Pi'\in\mathcal{R}(G\cdot G',\omega), \tag{0.9}$$

and vice versa.

Here $\otimes$ means the outer tensor product. The topology of $\otimes$ is not uniquely determined but the infinitesimal equivalence class is. Moreover,

$$\dim \mathrm{Hom}_{\tilde G\cdot\tilde G'}(\omega^\infty,\Pi\otimes\Pi')=1. \tag{0.10}$$

We will call the (bijective) function

$$\mathcal{R}(G,\omega)\ni\Pi\to\Pi'\in\mathcal{R}(G',\omega), \tag{0.11}$$

defined by (0.9), the *Duality Correspondence*.

It is not easy to describe this function in terms of any known parameters classifying $\mathcal{R}(\widetilde{G})$ and $\mathcal{R}(\widetilde{G}')$. In this paper we determine the correspondence of infinitesimal characters (see [19, 0.3.18]) of Π and Π' induced by (0.11) (Theorems 1.8, 1.13, and 1.19).

The point is that this correspondence does not depend on the real form G, G' of $\mathbf{G},\mathbf{G}'$ (0.4). Moreover, for any real reductive dual pair G, G' one can find another pair G_1, G_1' with the same complexification and at least one member compact. For such pairs the Duality Correspondence (0.11) is known explicitly (see [2, 4, 15] and the Appendix).

1. The Duality Correspondence. Let G, G' be a real reductive dual pair (Def. (0.1)) with Lie algebras $\mathfrak{g}$, $\mathfrak{g}'$. The group $\widetilde{\mathrm{Sp}}$ acts by conjugation on the space $\omega(U(\mathfrak{sp}))$, the image under ω of the universal enveloping algebra $U(\mathfrak{sp})$ of the Lie algebra $\mathfrak{sp}$. One of the fundamental properties of the oscillator representation is that this action factorizes to an action of the group Sp and even extends to an action of the complexification **Sp** (see [10, Section 3]). In this sense **Sp** acts by conjugation on $\omega(U(\mathfrak{sp}))$.

Since the group G acts reductively on the universal enveloping algebra $U(\mathfrak{g})$, we have

$$\omega(Z(\mathfrak{g})^{\mathbf{G}}) = \omega(Z(\mathfrak{g}))^{\mathbf{G}}, \tag{1.1}$$

where X^G is the space of G-invariants in X, $Z(\mathfrak{g})$ denotes the center of the universal enveloping algebra $U(\mathfrak{g})$ and the action of G on the right hand side of (1.1) is by conjugation (as explained above). Let us notice that some members of dual pairs are disconnected. It may indeed happen that $Z(\mathfrak{g})^{\mathbf{G}}$ is strictly contained in $Z(\mathfrak{g})$.

A statement similar to (1.1) holds for G' and for the product $G \cdot G'$. It follows from [6, Theorem 7] (see also [10, Theorem 4.1]) that

$$\omega(U(\mathfrak{sp}))^{\mathbf{G}'} = \omega(U(\mathfrak{g})) \tag{1.2}$$

and therefore that

$$\begin{aligned}\omega(Z(\mathfrak{g}')^{\mathbf{G}'}) &\subseteq \omega(U(\mathfrak{sp})^{\mathbf{G}\cdot\mathbf{G}'}) \\ &= \omega(U(\mathfrak{sp}))^{\mathbf{G}\cdot\mathbf{G}'} = \omega(U(\mathfrak{g}))^{\mathbf{G}} = \omega(Z(\mathfrak{g}))^{\mathbf{G}},\end{aligned} \tag{1.3}$$

where the inclusion is obvious, the first equality follows from (1.1), the second from (1.2) and the third from (1.1). By permuting G and G' in (1.3) we get a known

THEOREM 1.4. *If G, G' is a real reductive dual pair, then*

$$\begin{aligned}\omega(Z(\mathfrak{g})^{\mathbf{G}}) &= \omega(U(\mathfrak{sp})^{\mathbf{G}\cdot\mathbf{G}'}) = \omega(Z(\mathfrak{g}')^{\mathbf{G}'}) = \omega(Z(\mathfrak{g}))^{\mathbf{G}} \\ &= \omega(U(\mathfrak{sp}))^{\mathbf{G}\cdot\mathbf{G}'} = \omega(Z(\mathfrak{g}'))^{\mathbf{G}'}.\end{aligned}$$

Let Π, Π' be as in (0.9) and let $\chi_\Pi : Z(\mathfrak{g}) \to \mathbb{C}$ be the infinitesimal character of Π and let $\chi_{\Pi'} : Z(\mathfrak{g}') \to \mathbb{C}$ be the infinitesimal character of Π'. By (0.10) there is a non-zero operator

$$T \in \mathrm{Hom}_{\tilde{G}\cdot\tilde{G}'}(\omega^\infty, \Pi \otimes \Pi'). \tag{1.5}$$

It satisfies

$$T\omega(a) = \chi_\Pi(a)T, \qquad T\omega(a') = \chi_{\Pi'}(a')T$$

for $a \in Z(\mathfrak{g})$ and $a' \in Z(\mathfrak{g}')$.

We restrict χ_Π to $Z(\mathfrak{g})^{\mathbf{G}}$ and $\chi_{\Pi'}$ to $Z(\mathfrak{g}')^{\mathbf{G}'}$. It follows from (1.5) that

$$\mathrm{Ker}(\chi_\Pi) = \mathrm{Ker}(\omega|_{Z(\mathfrak{g})^{\mathbf{G}}}) \quad \text{and the same for } \chi_{\Pi'}, \tag{1.6}$$

where $\omega|_{Z(\mathfrak{g})^{\mathbf{G}}}$ is the restriction of ω to $Z(\mathfrak{g})^{\mathbf{G}}$. Therefore both χ_Π and $\chi_{\Pi'}$ define the same character

$$\chi : \omega(U(\mathfrak{sp})^{\mathbf{G}\cdot\mathbf{G}'}) \to \mathbb{C}$$

and we get the following commuting diagram of surjections:

$$\begin{tikzcd} Z(\mathfrak{g})^{\mathbf{G}} \arrow[r, "\omega"] \arrow[d, "\chi_\Pi"'] & \omega(U(\mathfrak{sp})^{\mathbf{G}\cdot\mathbf{G}'}) \arrow[d, "\chi"'] & Z(\mathfrak{g}')^{\mathbf{G}'} \arrow[l, "\omega"'] \arrow[d, "\chi'_\Pi"'] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \arrow[r, equal] & \mathbb{C} \end{tikzcd} \tag{1.7}$$

An immediate consequence of (1.7) is the following theorem:

THEOREM 1.8. *Let G, G' be a reductive dual pair and let $\Pi_j \otimes \Pi'_j \in \mathcal{R}(G\cdot G', \omega)$, $j = 1, 2$. Then $\chi_{\Pi_1} = \chi_{\Pi_2}$ implies $\chi_{\Pi'_1} = \chi_{\Pi'_2}$.*

Assume that $G, G' \subseteq \mathrm{Sp}$ and $G_1, G'_1 \subseteq \mathrm{Sp}$ are two real reductive dual pairs with isomorphic complexifications $\mathbf{G}, \mathbf{G}' \subseteq \mathbf{Sp}$ and $\mathbf{G}_1, \mathbf{G}'_1 \subseteq \mathbf{Sp}$. It follows from the classification of such pairs [7, 9] that there is an element $g \in \mathbf{Sp}$ such that

$$\operatorname{Int} g(\mathbf{G}) = \mathbf{G}_1 \quad \text{and} \quad \operatorname{Int} g(\mathbf{G}') = \mathbf{G}'_1. \tag{1.8}$$

Since all the complexified Lie algebras $\mathfrak{g}_\mathbb{C}$, $\mathfrak{g}'_\mathbb{C}$, $\mathfrak{g}_{1\mathbb{C}}$, $\mathfrak{g}'_{1\mathbb{C}}$ are contained in $\mathfrak{sp}_\mathbb{C}$, their universal enveloping algebras are contained in $U(\mathfrak{sp})$. Let $\operatorname{Ad}\omega(g)$ denote the action by conjugation of g on the algebra $\omega(U(\mathfrak{sp}))$. It is apparent that the following diagram is commutative:

$$\begin{tikzcd} Z(\mathfrak{g})^{\mathbf{G}} \arrow[r, "\omega"] \arrow[d, "\operatorname{Ad} g"'] & \omega(U(\mathfrak{sp})^{\mathbf{G}\cdot\mathbf{G}'}) \arrow[d, "\operatorname{Ad}\omega(g)"'] & Z(\mathfrak{g}')^{\mathbf{G}'} \arrow[l, "\omega"'] \arrow[d, "\operatorname{Ad} g"'] \\ Z(\mathfrak{g}_1)^{\mathbf{G}_1} \arrow[r, "\omega"] & \omega(U(\mathfrak{sp})^{\mathbf{G}_1\cdot\mathbf{G}'_1}) & Z(\mathfrak{g}'_1)^{\mathbf{G}'_1} \arrow[l, "\omega"'] \end{tikzcd} \tag{1.9}$$

and that the vertical arrows are isomorphisms.

THEOREM 1.10. *Let G, G' be a real irreducible dual pair. Assume that $\operatorname{rank} G \le \operatorname{rank} G'$. Then the oscillator representation ω maps $Z(\mathfrak{g})^{\mathbf{G}}$ injectively into $\omega(U(\mathfrak{sp}))$.*

Proof. By inspection of the list of all possible G, G' ([7], [10, 4.1, 4.2]) we see that there is a reductive dual pair G_1, G'_1 with at least one member compact and the same complexification as G, G'.

The diagram (1.9) reduces the verification of this theorem to the case of pairs like G_1, G'_1. They are either irreducible (Def. (0.5)) or double of irreducible pairs. We may therefore assume that G or G' is compact. In this situation this theorem is an immediate consequence of (1.6) and Lemma A.7 (in the Appendix). ∎

Under the assumptions of Theorem 1.10, the diagram (1.7) defines a surjective homomorphism

$$Z(\mathfrak{g}')^{\mathbf{G}'} \to Z(\mathfrak{g})^{\mathbf{G}}, \tag{1.11}$$

which, by dualization, defines an injection

$$D : \max\operatorname{spec} Z(\mathfrak{g})^{\mathbf{G}} \to \max\operatorname{spec} Z(\mathfrak{g}')^{\mathbf{G}'}. \tag{1.12}$$

THEOREM 1.13. *Under the assumptions of Theorem* 1.10,

(1.14) *the map D does not depend on the real form G, G' of $\mathbf{G}, \mathbf{G}'$; and*
(1.15) *if $\Pi \otimes \Pi' \in \mathcal{R}(G \cdot G', \omega)$, then $D(\chi_\Pi) = \chi_{\Pi'}$.*

Proof. The first statement follows immediately from the commutation of the diagram (1.9) and the second from (1.7). ■

The statement (1.14) reduces the problem of understanding the map D (1.12) to the case when $\mathbf{G}, \mathbf{G}'$ is an irreducible complex dual pair.

Choose a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}'_{\mathbb{C}}$ of $\mathfrak{g}'_{\mathbb{C}}$. Let $e_1, e_2, \ldots$ be the standard orthonormal coordinatization of $\mathfrak{h}^*_{\mathbb{C}}$ and let $e'_1, e'_2, \ldots$ be the standard orthonormal coordinatization of $\mathfrak{h}'^*_{\mathbb{C}}$ as in (A.4)–(A.6), or [3]. Define an embedding

$$E : \mathfrak{h}^* \to \mathfrak{h}'^* \quad \text{by} \quad E(e_j) = e'_j \quad (j = 1, 2, \ldots, \operatorname{rank} G). \tag{1.16}$$

Let

$$\tau = \begin{cases} \sum_{j=m+1}^{n}((m+1+n)/2 - j)e'_j \\ \qquad\qquad \text{if } \mathbf{G} = \mathrm{GL}(m,\mathbb{C}),\ \mathbf{G}' = \mathrm{GL}(n,\mathbb{C}), \\ \sum_{j=n+1}^{[m/2]}(m/2 - j)e'_j \quad \text{if } \mathbf{G} = \mathrm{Sp}(n,\mathbb{C}),\ \mathbf{G}' = O(m,\mathbb{C}), \\ \sum_{j=[m/2]+1}^{n}(n + 1 + [m/2] - m/2 - j)e'_j \\ \qquad\qquad \text{if } \mathbf{G} = O(m,\mathbb{C}),\ \mathbf{G}' = \mathrm{Sp}(n,\mathbb{C}). \end{cases} \tag{1.17}$$

Here we use the convention that $\sum_{j=p}^{q} = 0$ if $q < p$. Define a map F from $\mathfrak{h}^*_{\mathbb{C}}$ to $\mathfrak{h}'^*_{\mathbb{C}}$ by

$$F(\gamma) = E(\gamma) + \tau \quad (\gamma \in \mathfrak{h}^*_{\mathbb{C}}). \tag{1.18}$$

THEOREM 1.19. *Let G, G' be a real irreducible dual pair whose complexification $\mathbf{G}, \mathbf{G}'$ is an irreducible complex dual pair. Then the map D* (1.12) *coincides with the map F* (1.18) *via the Harish-Chandra isomorphism*

$$\max\operatorname{spec} Z(\mathfrak{g})^{\mathbf{G}} \to (\mathfrak{h}^*_{\mathbb{C}})^{\mathbf{W}} \quad (\textit{and the same for } G'). \tag{1.20}$$

Here $\mathbf{W}$ is the Weyl group of type A (permutations of the e_j's) if $G = \mathrm{GL}(m,\mathbb{C})$; and of type C (permutations and all sign changes of the e_j's)) if $\mathbf{G} = \mathrm{Sp}(n,\mathbb{C})$ or $\mathbf{G} = O(m,\mathbb{C})$ (and the same for $\mathbf{W}'$).

Proof. By Theorem 1.13 we may assume that G or G' is compact. For such pairs the representations Π and Π' (0.11) are highest weight modules (see the Appendix). One obtains the infinitesimal character of such a module by adding half the sum of positive roots to its highest weight. Therefore a straightforward calculation using (A.4)–(A.6) verifies this theorem. ■

Appendix. The Duality Correspondence for real irreducible dual pairs with at least one member compact. Let G, G' be such a pair. Assume that G' is compact. In this case the Duality Correspondence (0.11) is known explicitly [2, 4, 15]. The point is that both representations Π and Π' which occur in (0.11) are unitary highest weight modules. We will describe them here.

Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$. Our assumptions on G imply that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Fix a Borel subalgebra $\mathbf{b} \subseteq \mathfrak{g}_{\mathbb{C}}$ containing $\mathfrak{h}_{\mathbb{C}}$. Let

(A.1) Δ denote the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and Δ^+ the positive root system determined by $\mathbf{b}$.

Since (G, K) is a hermitian symmetric pair [5], we may assume that $\mathbf{b}$ is chosen so that $\mathfrak{k}_{\mathbb{C}} \oplus \mathbf{b}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let

(A.2) Δ^+ denote the set of positive compact roots and Δ_n^+ be the remaining roots of Δ^+.

Similarly we choose a Cartan subalgebra $\mathfrak{h}'_{\mathbb{C}}$ of $\mathfrak{g}'_{\mathbb{C}}$, a Borel subalgebra $\mathbf{b}' \subseteq \mathfrak{g}'_{\mathbb{C}}$ containing $\mathfrak{h}'_{\mathbb{C}}$,

(A.3) the root system Δ' of $(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ and the positive root system Δ'^+ determined by $\mathbf{b}'$.

Let H be the centralizer of $\mathfrak{h}$ in G and let H' be the centralizer of $\mathfrak{h}'$ in G'. Also let $\mathbf{n} \subseteq \mathbf{b}$ and $\mathbf{n}' \subseteq \mathbf{b}'$ be the nilradicals determined by Δ^+ and Δ'^+ respectively.

The representations Π, Π' (0.11) are uniquely determined by the irreducible representations Λ, Λ' of H, H' on the annihilators of $\mathbf{n} \subseteq \mathbf{b}$, $\mathbf{n}' \subseteq \mathbf{b}'$ in the Harish-Chandra modules of Π and Π' respectively. The representation Λ is always one-dimensional with derivative $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$, but Λ' is either one- or two-dimensional. In any case the derivative $d\Lambda'$ of Λ' has only one Δ'^+-dominant component $\lambda' \in \mathfrak{h}'^*_{\mathbb{C}}$. We are going to list all pairs λ, λ' defined above. We will use the standard coordinate expressions of the root systems as in [3].

(A.4) $G = \mathrm{Sp}(n,\mathbb{R})$**,** $G' = O(c)$ $(n, c \geq 1)$**.**

(A.4.1)
$$\begin{aligned} \Delta_c^+ &= \{e_i - e_j : 1 \leq i < j \leq n\}, \\ \Delta_n^+ &= \{e_i + e_j : 1 \leq i \leq j \leq n\}, \\ \Delta'^+ &= \begin{cases} \{e'_i \pm e'_j : 1 \leq i < j \leq l\} \cup \{e'_i : 1 \leq i \leq l\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } c = 2l+1 \geq 3, & \\ \{e'_i \pm e'_j : 1 \leq i < j \leq l\} \qquad \text{if } c = 2l \geq 4. & \end{cases} \end{aligned}$$

Here $\mathfrak{g}' = 0$, $\lambda' = 0$ if $c = 1$, and e'_1 is the standard basis element of $\mathfrak{h}'^*_{\mathbb{C}} = \mathfrak{g}'^*_{\mathbb{C}}$ if $c = 2$.

(A.4.2) The corresponding pairs of highest weights:

(A.4.2.1)
$$\lambda = -\sum_{a=1}^{n} \frac{c}{2} e_a - \sum_{a=1}^{k} \lambda_a e_{n+1-a}, \qquad \lambda' = \sum_{a=1}^{k} \lambda_a e'_a$$

for $0 \leq k \leq l, n$ and integers $\lambda_1 \geq \ldots \geq \lambda_k > 0$;

(A.4.2.2)
$$\lambda = -\sum_{a=1}^{n} \frac{c}{2} e_a - \sum_{a=1}^{c-k} \lambda_a e_{n+1-a}, \qquad \lambda' = \sum_{a=1}^{k} \lambda_a e'_a$$

for $c - n \leq k \leq l$ and integers $\lambda_1 \geq \ldots \geq \lambda_k > \lambda_{k+1} = \ldots = \lambda_{c-k} = 1$.

(A.5) $G = U(p,q)$**,** $G' = U(c)$ $(p, q \geq 0; p+q \geq 1, c \geq 1, p \leq q)$**.**

(A.5.1)
$$\begin{aligned} \Delta_c^+ &= \{e_i - e_j : 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq p+q\}, \\ \Delta_n^+ &= \{e_i - e_{p+j} : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\} \text{ for } p+q \geq 2, \\ \Delta'^+ &= \{e'_i - e'_j : 1 \leq i < j \leq c\} \text{ for } c \geq 2, \end{aligned}$$

and e_1 (resp. e'_1) is the standard basis element of $\mathfrak{h}^*_{\mathbb{C}} = \mathfrak{g}^*_{\mathbb{C}}$ if $p = 0, q = 1$ (resp. of $\mathfrak{h}'^*_{\mathbb{C}} = \mathfrak{g}'^*_{\mathbb{C}}$ if $c = 1$).

(A.5.2) The corresponding pairs of highest weights:

$$\begin{aligned} \lambda &= -\sum_{a=1}^{p} \frac{c}{2} e_a + \sum_{a=p+1}^{q} \frac{c}{2} e_a - \sum_{a=1}^{r} \nu_a e_{p+1-a} + \sum_{a=1}^{s} \mu_a e_{p+a}, \\ \lambda' &= \sum_{a=1}^{c} \frac{q-p}{2} e'_a - \sum_{a=1}^{r} \nu_a e'_{c+1-a} + \sum_{a=1}^{s} \mu_a e'_a \end{aligned}$$

for $0 \leq r \leq p$; $0 \leq s \leq q$; $r + s \leq c$; and integers $\nu_1 \geq \ldots \geq \nu_r > 0$, $\mu_1 \geq \ldots \geq \mu_s > 0$.

(A.6) $G = O^*_{2n}$**,** $G' = \mathrm{Sp}(c)$**,** $(n \geq 2, c \geq 1)$**.**

(A.6.1)
$$\begin{aligned} \Delta_c^+ &= \{e_i - e_j : 1 \leq i < j \leq n\}, \\ \Delta_n^+ &= \{e_i + e_j : 1 \leq i < j \leq n\}, \\ \Delta'^+ &= \{e'_i \pm e'_j : 1 \leq i < j \leq c\} \cup \{2e'_i : 1 \leq i \leq c\}. \end{aligned}$$

(A.6.2) The corresponding pairs of highest weights:

$$\lambda = -\sum_{a=1}^{n} ce_a - \sum_{a=1}^{k} \lambda_a e_{n+1-a}, \qquad \lambda' = \sum_{a=1}^{k} \lambda_a e'_a$$

for $k = \min\{n, c\}$ and integers $\lambda_1 \geq \ldots \geq \lambda_k \geq 0$.

Using this list we verify the following.

LEMMA A.7. *Let G, G' be a real irreducible dual pair with G' compact. Denote by S (resp. S') the set of all infinitesimal characters of representations $\Pi \in \mathcal{R}(G, \omega)$ (resp. $\Pi' \in \mathcal{R}(G', \omega)$)* (see (1.7)). *Then S (resp. S') is a Zariski dense subset of* $\max\operatorname{spec} Z(\mathfrak{g})^{\mathbf{G}}$ (*resp.* $\max\operatorname{spec} Z(\mathfrak{g}')^{\mathbf{G}'}$) *if* $\operatorname{rank} G \leq \operatorname{rank} G'$ (*resp.* $\operatorname{rank} G' \leq \operatorname{rank} G$).

Proof. Using Harish-Chandra's isomorphism (1.20), we obtain the set S (resp. S') from (A.4.2), (A.5.2), (A.6.2) via a translation by the half sum of the positive roots Δ^+ (resp. Δ'^+). By inspection of these formulas we see that there is no non-zero polynomial function on $\mathfrak{h}^*_{\mathbb{C}}$ (resp. $\mathfrak{h}'^*_{\mathbb{C}}$) which could vanish on S (resp. S') if $\dim \mathfrak{h} \leq \dim \mathfrak{h}'$ (resp. $\dim \mathfrak{h}' \leq \dim \mathfrak{h}$). ∎

We conclude this paper with an easy observation about the $\widetilde{K}$-types of Π (0.11). Let

(A.8) $A(\Pi)$ be the set of lowest $\widetilde{K}$-types of Π in the sense of Vogan

(see [20, Def. 3.2]).

THEOREM A.9. *Let G, G' be a real irreducible pair with G' compact. Assume that $\Pi \otimes \Pi' \in \mathcal{R}(G \cdot G', \omega)$. Then $A(\Pi) = \{\pi\}$, where π is the unique $\widetilde{K}$-type of Π with highest weight equal to the highest weight λ of Π.*

Proof. Let $\mathbf{p}^+$ (resp. $\mathbf{p}^-$) be the span of root spaces for roots from Δ_n^+ (resp. $-\Delta_n^+$). Let

(A.10) $\omega_{\Pi'}$ = the Π'-isotypic component of ω considered as a $\widetilde{G}'$-module,

and $H_{\Pi'} = \{v \in \omega_{\Pi'} : \omega(\mathbf{p}^+)v = 0\}$. Here $\omega_{\Pi'}$ is isomorphic to $\Pi \otimes \Pi'$ as a $\widetilde{G} \cdot \widetilde{G}'$-module. Howe ([7, (3.9) c) and d)]) has shown that

$$\omega_{\Pi'} = \omega(U(\mathbf{p}^-))H_{\Pi'} \tag{A.11}$$

and that

$$H_{\Pi'} = \pi \otimes \Pi' \quad \text{as a } \widetilde{K} \times \widetilde{G}' \text{ -module.}$$

Here $U(\mathbf{p}^-)$ denotes the subalgebra of $U(\mathfrak{g})$ generated by $\mathbf{p}^-$. It follows from [12, 2.4.4, exercise 12] and from (A.11) that

(A.13) if $\pi_\mu \in \widetilde{K}^\wedge$, with highest weight $\mu \in \mathfrak{h}^*_{\mathbb{C}}$, is a $\widetilde{K}$-type of Π,

then

(A.13.1) $\mu = \nu + \lambda$, where ν is a non-positive integral combination of roots from Δ_n^+.

Let 2ϱ (resp. $2\varrho_c$) be the sum of roots from Δ^+ (resp. Δ_c^+). Parthasarathy [18] has shown that

(A.14) $$\|\mu + \varrho\| > \|\lambda + \varrho\| \quad \text{for } \mu \neq \lambda \text{ as in (A.13.1).}$$

(For a short proof see [4, Proof 3.9].) We prove that

(A.15) $\|\mu + 2\varrho_c\|^2 - \|\lambda + 2\varrho_c\|^2$ is positive for $\mu \neq \lambda$ ([20, Def. 3.2]).

It follows from (A.14) that (A.15) is strictly greater than

(A.16) $$2(\nu, 2\varrho_c - \varrho) \quad \text{where } \nu \text{ is as in (A.13.1).}$$

Here (,) denotes the inner product on $\mathfrak{h}_{\mathbb{C}}^*$. Therefore it will suffice to verify

(A.17) $$(-\alpha, 2\varrho_c - \varrho) \geq 0 \quad \text{for } \alpha \in \Delta_n^+.$$

We check it case by case:

(A.4)$'$ $$2\varrho_c - \varrho = -\sum_{a=1}^{n} a e_a, \quad \alpha = e_i + e_j, \quad (-\alpha, 2\varrho_c - \varrho) = i + j > 0;$$

(A.5)$'$ $$2\varrho_c - \varrho = \sum_{a=1}^{p} \left(\frac{p+1-q}{2} - a \right) e_a + \sum_{a=1}^{p} \left(\frac{p+1+q}{2} - a \right) e_{p+a},$$
$$\alpha = e_i - e_{p+j}, \quad (-\alpha, 2\varrho_c - \varrho) = q + i - j > 0;$$

(A.6)$'$ $$2\varrho_c - \varrho = \sum_{a=1}^{n} (1-a) e_a, \quad \alpha = e_i + e_j,$$
$$(-\alpha, 2\varrho_c - \varrho) = i + j - 2 \geq 0. \ \blacksquare$$

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