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# The Banach–Mazur game and $\sigma$-porosity

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**Abstract.** It is well known that the sets of the first category in a metric space can be described using the so-called Banach–Mazur game. We will show that if we change the rules of the Banach–Mazur game (by forcing the second player to choose large balls) then we can describe sets which can be covered by countably many closed uniformly porous sets. A characterization of $\sigma$-very porous sets and a sufficient condition for $\sigma$-porosity are also given in the terminology of games.

Let $(P,\varrho)$ be a metric space. The open ball with center $x\in P$ and radius $r>0$ is denoted by $B(x,r)$. Such a ball, considered as a set, does not uniquely determine its center and its radius, therefore a ball will be identified with the pair (center, radius). From this point of view, two distinct balls need not be geometrically different. Since inclusion will be used in the usual sense, the inclusions $B_1\subset B_2$, $B_2\subset B_1$ do not imply $B_1=B_2$ in general.

The center of an open ball $B$ is denoted by $c(B)$ and its radius by $r(B)$. The symbol $2\star B$ denotes the ball with twice the radius of $B$ and the same center. The symbol $\mathbb{N}_0$ denotes the set of non-negative integers.

Let $M\subset P$, $x\in P$ and $R>0$. We define

$$\gamma(x,R,M)=\sup\{r>0 : \text{for some } z\in P,\ B(z,r)\subset B(x,R)\setminus M\},$$
$$p(x,M)=\limsup_{R\to 0+}\gamma(x,R,M)/R,\qquad vp(x,M)=\liminf_{R\to 0+}\gamma(x,R,M)/R.$$

A set $M\subset P$ is said to be *porous* if $p(x,M)>0$ for every $x\in M$, *uniformly porous* if there exists $c>0$ such that $p(x,M)>c$ for every $x\in M$, *very porous* if $vp(x,M)>0$ for every $x\in M$. A countable union of porous (very porous, respectively) sets is called *$\sigma$-porous* (*$\sigma$-very porous*, respectively).

The notion of $\sigma$-porosity was introduced by E. P. Dolzhenko ([D]) to describe a certain class of exceptional sets which appears in the study of

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boundary behavior of complex functions. There are many other results describing sets of exceptional points in terms of $\sigma$-porous sets (cf. [Z$_2$]). The main goal of this paper is to develop methods to prove that certain sets are "small" in various senses, which are defined using porosity. A result (Corollary of Theorem 2) was already used to obtain results concerning differentiation of functions ([Z$_1$]).

Now we define an infinite game $G(M)$ of two players, where $M \subset P$, which is closely related to the well known Banach–Mazur game (cf. Remark below). The first player chooses a ball $B_1$, the second chooses a ball $B_2 \subset B_1$, the first chooses a ball $B_3 \subset B_2$ and so on. The first player wins if

$$\bigcap_{n=1}^{\infty} B_n \cap M \neq \emptyset,$$

otherwise the second player wins.

A *strategy for the second player* in the game $G(M)$ is a sequence of open-ball-valued mappings $(f_n)_{n=1}^{\infty}$ such that $f_n$ is defined on all $(2n-1)$-tuples of open balls and

$$f_n(B_1, \ldots, B_{2n-1}) \subset B_{2n-1} \quad \text{for every } n \in \mathbb{N}.$$

We say that a decreasing sequence $(B_n)_{n=1}^{k}$ of open balls, $k \in \mathbb{N} \cup \{\infty\}$, is *compatible with the strategy* $(f_n)_{n=1}^{\infty}$ if

$$f_n(B_1, \ldots, B_{2n-1}) = B_{2n} \quad \text{for every } n \in \mathbb{N} \text{ with } 2n \leq k.$$

A strategy $\mathcal{S}$ for the second player is a *winning strategy* in the game $G(M)$ if for every sequence $(B_n)_{n=1}^{\infty}$ compatible with $\mathcal{S}$ we have

$$\bigcap_{n=1}^{\infty} B_n \cap M = \emptyset.$$

The notions of a strategy for the first player and a winning strategy for the first player can be defined in the obvious way.

Remark. Our game $G(M)$ and the Banach–Mazur game for $M$ are closely related, but they are not the same. The Banach–Mazur game is played with open sets or equivalently with open balls (cf. [K], [O]). The players of $G(M)$ choose elements from $P \times (0, \infty)$. These games are in fact identical in those metric spaces where each open ball uniquely determines its center and radius. They are also equivalent in general metric spaces because, if some player has a winning strategy in the Banach–Mazur game for $M$, then the same player has a winning strategy in $G(M)$ and vice versa. If the winning strategy for the second player should fulfil some extra conditions concerning centers and radii of balls then differences may appear. It will be our case.

CONVENTION. The term *family* will be used for indexed sets and the term *collection* for (non-indexed) sets.

DEFINITION. We say that a family $(B_\alpha)_{\alpha\in A}$ of subsets of $P$ is *discrete* if there exists $\eta > 0$ such that $\operatorname{dist}(B_\alpha, B_{\alpha'}) > \eta$ for any distinct $\alpha, \alpha' \in A$. We then also say that $(B_\alpha)_{\alpha\in A}$ is discrete *with constant* $\eta$. We say that $(B_\alpha)_{\alpha\in A}$ is $\sigma$-*discrete* if there exist sets $A_n$ such that $A = \bigcup_{n=1}^\infty A_n$ and the families $(B_\alpha)_{\alpha\in A_n}$ are all discrete.

CONVENTION. The discreteness and $\sigma$-discreteness of a collection $\mathcal{B}$ of open balls are always understood in the sense of the family indexed by pairs of centers and radii of balls from $\mathcal{B}$.

NOTATION. Let $\mathcal{D}$ be a set of sequences $D = (D_n)_{n=1}^\infty$ of open balls. We put

$$\mathcal{D}|n = \{(D_i)_{i=1}^n : D \in \mathcal{D}\} \quad \text{for } n \in \mathbb{N}, \quad \text{and} \quad \mathcal{D}|\infty = \mathcal{D}.$$

LEMMA A. *Let $(B_\alpha)_{\alpha\in A}$ be a non-empty family of open balls in $P$. Then there exists $Q \subset A$ such that $(B_\alpha)_{\alpha\in Q}$ is $\sigma$-discrete and for every $\alpha \in A$ there exists $\alpha^\star \in Q$ with*

$$\tfrac{1}{2} r(B_{\alpha^\star}) \le r(B_\alpha) \le 2r(B_{\alpha^\star}) \quad \textit{and} \quad \operatorname{dist}(B_\alpha, B_{\alpha^\star}) \le r(B_\alpha).$$

Proof. For every $n \in \mathbb{Z}$ we define

$$L_n = \{\alpha \in A : r(B_\alpha) \in (2^{-n}, 2^{-n+1}]\}.$$

We let $C_n$ be the maximal subset (with respect to inclusion) of $L_n$ such that for any distinct elements $\alpha, \beta \in C_n$ we have

$$\operatorname{dist}(B_\alpha, B_\beta) > 2^{-n}.$$

Put $Q = \bigcup_{n\in\mathbb{Z}} C_n$. The family $(B_\alpha)_{\alpha\in Q}$ is clearly $\sigma$-discrete. Choose $\alpha \in A$. There exists $n \in \mathbb{Z}$ such that $\alpha \in L_n$. Since $C_n$ is maximal with respect to inclusion, there exists $\alpha^\star \in C_n$ such that $\operatorname{dist}(B_\alpha, B_{\alpha^\star}) \le 2^{-n}$. Hence

$$\operatorname{dist}(B_\alpha, B_{\alpha^\star}) \le r(B_\alpha).$$

We also have

$$r(B_\alpha) \ge 2^{-n} \ge \tfrac{1}{2} r(B_{\alpha^\star}) \quad \text{and} \quad r(B_\alpha) \le 2^{-n+1} \le 2r(B_{\alpha^\star}).$$

This proves our lemma. ∎

LEMMA B. *Let $(B_\alpha)_{\alpha\in A}$ be a $\sigma$-discrete family of open balls. For every $\alpha \in A$, let $(K_\beta)_{\beta\in C_\alpha}$ be a $\sigma$-discrete family of open balls contained in $B_\alpha$. Then the family $(K_\beta)_{\beta\in C}$, where $C = \bigcup_{\alpha\in A} C_\alpha$, is $\sigma$-discrete.*

Proof. We have $A = \bigcup_{n=1}^\infty A_n$, where $(B_\alpha)_{\alpha\in A_n}$ is a discrete family with a constant $\varepsilon_n > 0$. For every $\alpha \in A$ we also have $C_\alpha = \bigcup_{k=1}^\infty C_{k,\alpha}$, where $(K_\beta)_{\beta\in C_{k,\alpha}}$ is a discrete family with a constant $\eta_{k,\alpha}$. For $n,k,l\in\mathbb{N}$ we define

$$H_{n,k,l}=\bigcup\{C_{k,\alpha}:\alpha\in A_n,\ \eta_{k,\alpha}>1/l\}.$$

Clearly

$$C=\bigcup\{C_{k,\alpha}:k\in\mathbb{N},\ \alpha\in A\}=\bigcup\{H_{n,k,l}:n,k,l\in\mathbb{N}\}.$$

For two distinct elements $\xi,\zeta\in H_{n,k,l}$ we obtain

$$\operatorname{dist}(K_\xi,K_\zeta)>\min\{\varepsilon_n,1/l\}$$

and we are done. ∎

Lemma C. *Let $M\subset P$. Let $\mathcal{S}=(f_n)_{n=1}^\infty$ be a strategy for the second player in the game $G(M)$. Then there is a set $\mathcal{D}$ of infinite sequences $D=(D_n)_{n=1}^\infty$ of open balls such that*

(P1) *every $D\in\mathcal{D}$ is compatible with $\mathcal{S}$,*

(P2) *for every $k\in\mathbb{N}$ the family $(S_{2k})_{S\in\mathcal{D}|2k}$ of balls is $\sigma$-discrete,*

(P3) *if $k\in\mathbb{N}_0$, $(D_n)_{n=1}^\infty\in\mathcal{D}$, and $C_{2k+1}$, $C_{2k+2}$ are open balls such that*

$$(D_1,\ldots,D_{2k},C_{2k+1},C_{2k+2})$$

*is compatible with $\mathcal{S}$, then there exists $(D_i^\star)_{i=1}^\infty\in\mathcal{D}$ such that*

$$D_i^\star=D_i\quad\text{for } 1\le i\le 2k,$$

$$\tfrac12 r(D_{2k+2}^\star)\le r(C_{2k+2})\le 2r(D_{2k+2}^\star)\quad\text{and}\quad\operatorname{dist}(D_{2k+2}^\star,C_{2k+2})\le r(C_{2k+2}).$$

Proof. We will define sets $\mathcal{D}_n$, $n\in\mathbb{N}_0$, of sequences of open balls such that for every $j\in\mathbb{N}$ we have:

$(\mathrm{P1})_j$ the set $\mathcal{D}_j$ contains sequences $(D_i)_{i=1}^{2j}$ compatible with $\mathcal{S}$,

$(\mathrm{P2})_j$ the family $(S_{2j})_{S\in\mathcal{D}_j}$ is $\sigma$-discrete,

$(\mathrm{P3})_j$ if $(D_i)_{i=1}^{2j-2}\in\mathcal{D}_{j-1}$, $C_{2j-1}$, $C_{2j}$ are open balls such that

$$(D_1,\ldots,D_{2j-2},C_{2j-1},C_{2j})$$

is compatible with $\mathcal{S}$, then there exists $(D_i^\star)_{i=1}^{2j}\in\mathcal{D}_j$ such that

$$D_i^\star=D_i\quad\text{for } 1\le i\le 2j-2,$$

$$\tfrac12 r(D_{2j}^\star)\le r(C_{2j})\le 2r(D_{2j}^\star)\quad\text{and}\quad\operatorname{dist}(D_{2j}^\star,C_{2j})\le r(C_{2j}).$$

$(\mathrm{P4})_j$ for every $(D_i)_{i=1}^{2j}\in\mathcal{D}_j$ we have $(D_i)_{i=1}^{2j-2}\in\mathcal{D}_{j-1}$.

Put $\mathcal{D}_0=\{\emptyset\}$. Define

$$\mathcal{U}^1=\{B\subset P:\text{there exists an open ball } B_1\subset P \text{ such that } f_1(B_1)=B\}.$$

Lemma A gives us a $\sigma$-discrete collection $\mathcal{H}^1\subset\mathcal{U}^1$ such that for every $B\in\mathcal{U}^1$ there exists $B^\star\in\mathcal{H}^1$ with

$$\tfrac12 r(B^\star)\le r(B)\le 2r(B^\star)\quad\text{and}\quad\operatorname{dist}(B,B^\star)\le r(B).$$

For every $B \in \mathcal{H}^1$ let $B_1(B)$ be an open ball such that $f_1(B_1(B)) = B$. Define

$$\mathcal{D}_1 = \{(B_1(B), B) : B \in \mathcal{H}^1\}.$$

The set $\mathcal{D}_1$ clearly has the properties $(\mathrm{P1})_1$, $(\mathrm{P2})_1$, $(\mathrm{P3})_1$ and $(\mathrm{P4})_1$.

Now suppose that a set $\mathcal{D}_j$ satisfying $(\mathrm{P1})_j$, $(\mathrm{P2})_j$, $(\mathrm{P3})_j$ and $(\mathrm{P4})_j$ has been defined. We construct $\mathcal{D}_{j+1}$ as follows. Fix $S = (B_1, \ldots, B_{2j}) \in \mathcal{D}_j$. Put

$$\mathcal{U}^{j+1}(S) = \{B : \text{there exists an open ball } B_{2j+1} \subset B_{2j} \text{ such that } f_{j+1}(B_1, \ldots, B_{2j}, B_{2j+1}) = B\}.$$

There exists a $\sigma$-discrete collection $\mathcal{H}^{j+1}(S) \subset \mathcal{U}^{j+1}(S)$ such that for every $B \in \mathcal{U}^{j+1}(S)$ there exists $B^\star \in \mathcal{H}^{j+1}(S)$ with

$$\tfrac{1}{2} r(B^\star) \le r(B) \le 2r(B^\star) \quad \text{and} \quad \operatorname{dist}(B, B^\star) \le r(B).$$

For every $B \in \mathcal{H}^{j+1}(S)$ let $B^S_{2j+1}(B)$ be an open ball such that

$$B^S_{2j+1}(B) \subset B_{2j} \quad \text{and} \quad f_{j+1}(B_1, \ldots, B_{2j}, B^S_{2j+1}(B)) = B.$$

Define

$$\mathcal{D}_{j+1} = \{(B_1, \ldots, B_{2j}, B^{(B_1,\ldots,B_{2j})}_{2j+1}(B), B) : (B_1, \ldots, B_{2j}) \in \mathcal{D}_j, B \in \mathcal{H}^{j+1}(B_1, \ldots, B_{2j})\}.$$

The set $\mathcal{D}_{j+1}$ clearly satisfies the conditions $(\mathrm{P1})_{j+1}$, $(\mathrm{P3})_{j+1}$ and $(\mathrm{P4})_{j+1}$. We check that $(\mathrm{P2})_{j+1}$ is also satisfied. The family $(S_{2j})_{S\in\mathcal{D}_j}$ and the collections $\mathcal{H}^{j+1}(S)$, $S \in \mathcal{D}_j$, are $\sigma$-discrete. By Lemma B, the family $(S_{2j+2})_{S\in\mathcal{D}_{j+1}}$ is $\sigma$-discrete. Thus we have constructed $\mathcal{D}_j$'s. Now we define

$$\mathcal{D} = \{(D_i)_{i=1}^\infty : \text{for every } j \in \mathbb{N} \text{ we have } (D_i)_{i=1}^{2j} \in \mathcal{D}_j\}.$$

Then $\mathcal{D}$ clearly has the properties (P1)–(P3). ∎

LEMMA D. *Let $(A_\alpha)_{\alpha\in I}$ be a discrete family of subsets of $P$.*

(i) *If each $A_\alpha$ can be covered by countably many closed uniformly porous sets, then $\bigcup_{\alpha\in I} A_\alpha$ has the same property.*

(ii) *If each $A_\alpha$ is $\sigma$-very porous, then so is $\bigcup_{\alpha\in I} A_\alpha$.*

Proof. (i) Let $(A_\alpha)_{\alpha\in I}$ be a discrete family with a constant $\varepsilon > 0$. There clearly exist sets $B_{\alpha,n}$, $\alpha \in I$, $n \in \mathbb{N}$, such that

(a) $A_\alpha \subset \bigcup_{n=1}^\infty B_{\alpha,n}$ for every $\alpha \in I$,

(b) for every $\alpha \in I$ and $n \in \mathbb{N}$ there exists $c_{\alpha,n} > 0$ such that $p(x, B_{\alpha,n}) > c_{\alpha,n}$ whenever $x \in B_{\alpha,n}$,

(c) $B_{\alpha,n}$ is closed for every $\alpha \in I$, $n \in \mathbb{N}$,

(d) $\operatorname{dist}(B_{\alpha,n}, B_{\beta,m}) \ge \varepsilon$ for every $\alpha, \beta \in I$, $\alpha \ne \beta$, $n, m \in \mathbb{N}$.

Put

$$C_{n,k} = \bigcup\{B_{\alpha,k} : c_{\alpha,k} > 1/n\}.$$

The discreteness of the family $(B_{\alpha,k})_{\alpha\in I}$ implies that $C_{n,k}$ is closed and $p(x,C_{n,k})\ge 1/n$ whenever $x\in C_{n,k}$. We have

$$\bigcup_{\alpha\in I} A_\alpha \subset \bigcup_{n,k=1}^{\infty} C_{n,k}$$

and we are done.

(ii) The proof of this assertion is straightforward. ■

Definition. Let $\mathcal{D}$ be a set of sequences $D=(D_i)_{i=1}^\infty$ of open balls. Then for $S\in\mathcal{D}|k$, $k\in\mathbb{N}\cup\{\infty\}$, $n\in\mathbb{N}$, $n\le k$, we define the sets

$$H_n(S,\mathcal{D}) = S_n \setminus \bigcup\{T_{n+2} : T\in\mathcal{D},\ T_i = S_i,\ i\le n\}$$

and

$$H_0(\mathcal{D}) = P\setminus \bigcup\{T_2 : T\in\mathcal{D}\}.$$

Theorem 1. *A set $M\subset P$ can be covered by a countable union of closed uniformly porous sets if and only if there exist a sequence $(c_n)_{n=1}^\infty$ of positive numbers and a winning strategy $\mathcal{S}$ for the second player in the game $G(M)$ such that*

$$r(B_{2n}) > c_n\varrho(c(B_{2n-1}),c(B_{2n})) \quad \textit{whenever } (B_1,\ldots,B_{2n}) \textit{ is compatible with } \mathcal{S}. \tag{1}$$

Proof. First we suppose that there exist closed uniformly porous sets $(A_n)_{n\in\mathbb{N}}$ covering $M$. Let $c_n>0$ be such that $p(x,A_n)>c_n$ for every $x\in A_n$. As $A_n$'s are closed we see that

$$p(x,A_n)>c_n \quad \text{for every } x\in P. \tag{2}$$

A winning strategy for the second player is the following: In the $n$th step he avoids the set $A_n$. More precisely: the second player's answer to the $n$th move $B(x_{2n-1},r_{2n-1})$ of the first player is a ball $B(x_{2n},r_{2n})$ such that

$$B(x_{2n},r_{2n})\subset B(x_{2n-1},r_{2n-1})\setminus A_n,\qquad r_{2n}>c_n\varrho(x_{2n},x_{2n-1}).$$

This ball exists because (2) holds. This strategy is clearly winning.

Now suppose that there exist a sequence $(c_n)_{n=1}^\infty$ of positive numbers and a winning strategy $\mathcal{S}=(f_n)_{n=1}^\infty$ for the second player in the game $G(M)$ satisfying (1). Define

$$I(P)=\{x\in P : x \text{ is an isolated point of } P\}.$$

Observe that $M\cap I(P)=\emptyset$, otherwise the first player has a winning strategy in $G(M)$.

There exists a set $\mathcal{D}$ of sequences of open balls satisfying (P1), (P2) and (P3) of Lemma C for our strategy $\mathcal{S}$. We have

$$M \subset \Big(H_0(\mathcal{D}) \cup \bigcup_{n=1}^{\infty} \bigcup \{H_{2n}(D,\mathcal{D}) : D \in \mathcal{D}\}\Big) \setminus I(P).$$

Indeed, if $x \notin H_0(\mathcal{D}) \cup \bigcup_{n=1}^{\infty} \bigcup\{H_{2n}(D,\mathcal{D}) : D \in \mathcal{D}\}$, then there exists a sequence $D \in \mathcal{D}$ such that $x \in D_i$ for every $i \in \mathbb{N}$. Thus $x \notin M$.

Now we show that each set $H_{2j}(D,\mathcal{D}) \setminus I(P)$ $(j \in \mathbb{N},\ D \in \mathcal{D})$ can be covered by countably many closed uniformly porous sets. Let $B(y,r) = D_{2j}$. Put

$$Q_n = (\overline{B(y, r(1-2^{-n}))} \cap H_{2j}(D,\mathcal{D})) \setminus I(P).$$

Then $Q_n$ is clearly closed (recall that $I(P)$ is open in $P$). Let $x \in Q_n$. Fix $R > 0$ such that $B(x,R) \subset D_{2j}$ and there exists $w \in P$ such that $\varrho(x,w) = R$. Put

$$B = f_{j+1}(D_1, \ldots, D_{2j}, B(x,R)).$$

We have $r(B) > c_{j+1}\varrho(x, c(B))$. The property (P3) of Lemma C implies that there exists a sequence $D^\star \in \mathcal{D}$ such that

$$D^\star_i = D_i \quad \text{for } i \le 2j, \quad \operatorname{dist}(D^\star_{2j+2}, B) \le r(B)$$

and

$$\tfrac{1}{2} r(D^\star_{2j+2}) \le r(B) \le 2r(D^\star_{2j+2}).$$

Put

$$T = \varrho(x, c(B)) + r(B) + \operatorname{dist}(B, D^\star_{2j+2}) + 2r(D^\star_{2j+2}).$$

We have

$$\begin{gathered} D^\star_{2j+2} \subset B(x,T) \setminus H_{2j}(D,\mathcal{D}) \subset B(x,T) \setminus Q_n, \\ r(B) \le 2R \quad (\text{since } w \notin B), \quad r(D^\star_{2j+2}) \le 2r(B) \le 4R, \\ \operatorname{dist}(B, D^\star_{2j+2}) \le r(B) \le 2R \quad \text{and} \quad \varrho(x, c(B)) \le R. \end{gathered}$$

This gives $T \le 13R$. We obtain

$$\begin{aligned} \frac{\gamma(x,T,Q_n)}{T} &\ge \frac{r(D^\star_{2j+2})}{r(B)/c_{j+1} + r(B) + r(B) + 4r(B)} \\ &\ge \frac{r(B)/2}{(1/c_{j+1} + 6) r(B)} = \frac{1/2}{(1/c_{j+1} + 6)} = \kappa > 0. \end{aligned}$$

Now $x$ is not an isolated point of $P$. Therefore $R$ and $T$ can be arbitrarily small. Thus we have proved that $p(x, Q_n) > \frac{1}{2}\kappa$ for every $x \in Q_n$. Each $Q_n$ is closed, thus $H_{2j}(D,\mathcal{D}) \setminus I(P)$ can be covered by countably many closed uniformly porous sets.

Fix $j \in \mathbb{N}$. The property (P2) implies that there exist sets $\mathcal{S}_m$, $m \in \mathbb{N}$, such that

(i) $\bigcup_{m=1}^{\infty} \mathcal{S}_m = \mathcal{D}|2j$,
(ii) the family $(S_{2j})_{S \in \mathcal{S}_m}$ is discrete for every $m \in \mathbb{N}$.

Thus we have

$$\bigcup\{H_{2j}(D,\mathcal{D}) : D \in \mathcal{D}\} \setminus I(P) = \bigcup\{H_{2j}(S,\mathcal{D}) : S \in \mathcal{D}|2j\} \setminus I(P) = \bigcup_{m=1}^{\infty} \bigcup\{H_{2j}(S,\mathcal{D}) \setminus I(P) : S \in \mathcal{S}_m\}.$$

The sets $H_{2j}(S,\mathcal{D}) \setminus I(P)$, $S \in \mathcal{S}_m$, form a discrete family. Lemma D shows that their union can be covered by countably many closed uniformly porous sets.

The set $H_0(\mathcal{D})$ is clearly closed. If we replace $Q_n$ with $H_0(\mathcal{D}) \setminus I(P)$ and $D_{2j}$ with $P$ in the proof above we conclude that $H_0(\mathcal{D}) \setminus I(P)$ is a closed uniformly porous set. This completes the proof. ∎

THEOREM 2. *A set $M \subset P$ is $\sigma$-very porous if and only if there exist a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers and a winning strategy $\mathcal{S}$ for the second player in the game $G(M)$ such that*

(3) $r(B_{2n}) > c_n r(B_{2n-1})$ *whenever $(B_1,\ldots,B_{2n})$ is compatible with $\mathcal{S}$.*

Proof. Consider a $\sigma$-very porous set $M$. We have $M = \bigcup_{n=1}^{\infty} M_n$, where $M_n$ is very porous for every $n \in \mathbb{N}$. For $n,m,q \in \mathbb{N}$ we define

$$M_{n,m,q} = \{x \in M_n : \gamma(x,R,M_n)/R > 1/m \text{ for every } R \in (0,1/q)\}.$$

As $M_n$ is very porous we have

$$M_n = \bigcup_{m,q=1}^{\infty} M_{n,m,q}.$$

We order the set $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ into a sequence $((n_k,m_k,q_k))_{k=1}^{\infty}$. Now we are able to define a winning strategy for the second player. We define

$$c_n = \min\{1/(3m_k) : k \le n\}.$$

Clearly $c_n < 1/2$ for every $n \in \mathbb{N}$. Let $B(x_{2n-1},r_{2n-1})$ be the $n$th move of the first player. Let $k \in \mathbb{N}$ be the largest natural number such that

$$M_{n_j,m_j,q_j} \cap B(x_{2n-1},r_{2n-1}) = \emptyset \quad \text{for } j < k \text{ and } k \le n.$$

If $\frac{1}{2}r_{2n-1} \ge 1/q_k$, then we define $x_{2n} = x_{2n-1}$ and $r_{2n} = \frac{1}{2}r_{2n-1}$. Suppose that $\frac{1}{2}r_{2n-1} < 1/q_k$. If $B\big(x_{2n-1},\frac{1}{2}r_{2n-1}\big) \cap M_{n_k,m_k,q_k} = \emptyset$, then we put $x_{2n} = x_{2n-1}$ and $r_{2n} = \frac{1}{2}r_{2n-1}$. Otherwise there exist a point

$x \in B\big(x_{2n-1}, \frac{1}{2}r_{2n-1}\big) \cap M_{n_k,m_k,q_k}$ and a ball $B(z,p)$ such that

$$B(z,p) \subset B\bigg(x, \frac{1}{2}r_{2n-1}\bigg) \setminus M_{n_k,m_k,q_k} \quad \text{and} \quad p > \frac{1}{m_k} \cdot \frac{1}{2}r_{2n-1}.$$

Put $x_{2n} = z$ and $r_{2n} = \min\big\{\frac{1}{2}r_{2n-1}, p\big\}$. We have

$$r_{2n} > \frac{1}{2m_k} r_{2n-1} \geq c_k r_{2n-1} \geq c_n r_{2n-1}.$$

This strategy satisfies the condition (3). We claim that it is a winning strategy. Suppose that a sequence $(B(x_n,r_n))_{n=1}^{\infty}$ of open balls is compatible with our strategy and

$$\bigcap_{n=1}^{\infty} B(x_n,r_n) \cap M \neq \emptyset.$$

This implies that there exists a smallest $k_0 \in \mathbb{N}$ such that $B(x_n,r_n) \cap M_{n_{k_0},m_{k_0},q_{k_0}} \neq \emptyset$ for every $n \in \mathbb{N}$. Since $r_{2n} \leq \frac{1}{2}r_{2n-1}$ there exists $m \in \mathbb{N}$ such that $\frac{1}{2}r_{2m-1} < 1/q_{k_0}$, $k_0 \leq m$ and $M_{n_j,m_j,q_j} \cap B(x_{2m-1}, r_{2m-1}) = \emptyset$ for every $j < k_0$. Both possibilities: $B\big(x_{2m-1}, \frac{1}{2}r_{2m-1}\big) \cap M_{n_{k_0},m_{k_0},q_{k_0}}$ empty or non-empty, imply $B(x_{2m},r_{2m}) \cap M_{n_{k_0},m_{k_0},q_{k_0}} = \emptyset$. This contradiction proves our claim.

Now suppose that there exist a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers and a winning strategy $\mathcal{S} = (f_n)_{n=1}^{\infty}$ for the second player in the game $G(M)$ such that (3) holds.

Define

$$N(P) = \{x \in P : vp(x, \{x\}) \leq 0\}.$$

Suppose $x \in N(P) \cap M$. Choose $r_1 > 0$ such that

$$\gamma(x, r_1, \{x\})/r_1 < c_1$$

and let $B(x,r_1)$ be the first move of the first player in the game $G(M)$. Let

$$B(x_2,r_2) = f_1(B(x,r_1)).$$

We have $r_2 > c_1 r_1$ and therefore $x \in B(x_2,r_2)$. Suppose that $(B_1,\ldots,B_{2n})$ is compatible with $\mathcal{S}$ and $x \in B_{2n}$. Choose $r_{2n+1} > 0$ such that

$$\gamma(x, r_{2n+1}, \{x\})/r_{2n+1} < c_{n+1}$$

and $B(x,r_{2n+1}) \subset B_{2n}$. Let $B(x,r_{2n+1})$ be the $(n+1)$th move of the first player. Let

$$B(x_{2n+2}, r_{2n+2}) = f_{n+1}(B_1,\ldots,B_{2n}, B(x_{2n+1}, r_{2n+1})).$$

We have $r_{2n+2} > c_{n+1} r_{2n+1}$. This implies $x \in B(x_{2n+2}, r_{2n+2})$. Thus $x \in \bigcap_{n=1}^{\infty} B_n \cap M$ and $(B_i)_{i=1}^{\infty}$ is compatible with $\mathcal{S}$. This contradiction implies

$$N(P) \cap M = \emptyset.$$

There exists a set $\mathcal{D}$ of sequences of open balls having the properties (P1)–(P3) of Lemma C for our strategy $\mathcal{S}$. Similarly to the proof of Theorem 1 we have

$$M \subset \Big(H_0(\mathcal{D}) \cup \bigcup_{n=1}^{\infty} \bigcup \{H_{2n}(D,\mathcal{D}) : D \in \mathcal{D}\}\Big) \setminus N(P).$$

Since the discreteness argument from the proof of Theorem 1 and Lemma D work as well, it is sufficient to prove that the sets $H_{2j}(D,\mathcal{D}) \setminus N(P)$ ($j \geq 1$, $D \in \mathcal{D}$) and $H_0(\mathcal{D}) \setminus N(P)$ are very porous. Let $x \in H_{2j}(D,\mathcal{D}) \setminus N(P)$. We have $vp(x,\{x\}) = \eta$, $\eta \in (0,1]$. There exists $R_0 > 0$ such that

$$\gamma(x,R,\{x\})/R > \tfrac{1}{2}\eta \quad \text{for every } R \in (0,R_0). \tag{4}$$

Let $R \in (0,R_0)$ and $B(x,R) \subset D_{2j}$. Define

$$B = f_{j+1}\big(D_1,\ldots,D_{2j},B\big(x,\tfrac{1}{32}\eta R\big)\big).$$

We have $r(B) > c_{j+1}\frac{1}{32}\eta R$ and $c(B) \in B\big(x,\frac{1}{32}\eta R\big)$. The property (P3) gives us a sequence $D^\star \in \mathcal{D}$ such that

$$D^\star_i = D_i \quad \text{for } 1 \leq i \leq 2j, \quad \tfrac{1}{2}r(D^\star_{2j+2}) \leq r(B) \leq 2r(D^\star_{2j+2})$$

and

$$\operatorname{dist}(D^\star_{2j+2},B) \leq r(B).$$

We have $r(B) < \frac{1}{8}R$, since otherwise

$$B\big(x,\tfrac{1}{16}R\big) \subset B\big(c(B),\tfrac{1}{8}R\big) \subset B \subset B\big(x,\tfrac{1}{32}\eta R\big)$$

and therefore

$$\frac{\gamma\big(x,\frac{1}{16}R,\{x\}\big)}{\frac{1}{16}R} \leq \frac{\frac{1}{32}\eta R}{\frac{1}{16}R} = \frac{1}{2}\eta,$$

contrary to (4). For every $y \in D^\star_{2j+2}$ we have

$$\begin{aligned}\varrho(x,y) &\leq \varrho(x,c(B)) + r(B) + \operatorname{dist}(B,D^\star_{2j+2}) + 2r(D^\star_{2j+2})\\ &\leq \tfrac{1}{32}R + \tfrac{1}{8}R + \tfrac{1}{8}R + 4\cdot\tfrac{1}{8}R < R.\end{aligned}$$

We conclude that

$$D^\star_{2j+2} \subset B(x,R) \setminus H_{2j}(D,\mathcal{D}).$$

We obtain

$$\begin{aligned}\frac{\gamma(x,R,H_{2j}(D,\mathcal{D}) \setminus N(P))}{R} &\geq \frac{r(D^\star_{2j+2})}{R} \geq \frac{\frac{1}{2}r(B)}{R}\\ &\geq \frac{\frac{1}{2}c_{j+1}\frac{1}{32}\eta R}{R} = \frac{1}{64}c_{j+1}\eta > 0.\end{aligned}$$

We proved that $H_{2j}(D,\mathcal{D}) \setminus N(P)$ is very porous. It is easy to check that this method also works for $H_0(\mathcal{D}) \setminus N(P)$. ∎

DEFINITION. Let $M \subset P$. Then we say that $M$ is *globally very porous* if there exists $c > 0$ such that for each ball $B(x,r)$ there exists a ball $B(y,r')$ such that $B(y,r') \subset B(x,r)$, $B(y,r') \cap M = \emptyset$ and $r' > cr$. We say that $M$ is *σ-globally very porous set* if it is a countable union of globally very porous sets.

LEMMA E. *Let $A \subset P$, where $P$ is a normed linear space. Then $A$ is σ-very porous if and only if it is σ-globally very porous.*

Proof. Each σ-globally very porous set is clearly σ-very porous.

If a set $A$ is σ-very porous, then there exist very porous sets $A_n$, $n \in \mathbb{N}$, such that $A = \bigcup_{n=1}^{\infty} A_n$. Put

$$A_{n,k} = \{x \in A_n : vp(x, A_n) > 1/k\}.$$

We have $A = \bigcup_{n,k=1}^{\infty} A_{n,k}$ and $vp(x, A_{n,k}) > 1/k$ for every $x \in A_{n,k}$. Therefore it is sufficient to prove that each set $A$ satisfying $vp(x, A) > c > 0$ for every $x \in A$ is σ-globally very porous set.

Consider a continuous linear functional $\varphi$ on $P$ and a point $x_0 \in P$ such that $\|\varphi\| = 1$, $\|x_0\| = 1$ and $\varphi(x_0) = 1$. Define

$$\begin{aligned} C_{p,q} &= \varphi^{-1}([p/q, (p+1)/q]), \qquad p \in \mathbb{Z},\ q \in \mathbb{N}, \\ B_q &= \{x \in A : \gamma(x, r, A)/r > c \text{ for } r \in (0, 1/q]\}, \qquad q \in \mathbb{N}. \end{aligned}$$

Clearly $\bigcup_{q=1}^{\infty} B_q = A$ and $\bigcup_{p=-\infty}^{\infty} B_q \cap C_{p,q} = B_q$.

We will show that $B_q \cap C_{p,q}$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}$) is globally very porous. Consider a ball $B(x,r)$. Then we have the following possibilities:

1) Suppose $r > 2/q$. If $\varphi(x) \le p/q$, then we put $y = x - (r/2)x_0$ and $r' = r/4$. If $\varphi(x) > p/q$, then we put $y = x + (3r/4)x_0$ and $r' = r/8$. In these cases we have $B(y,r') \subset B(x,r)$ and $B(y,r') \cap C_{p,q} = \emptyset$.

2) Suppose $r \le 2/q$. If $B(x, r/2) \cap B_q \cap C_{p,q} = \emptyset$, then we put $y = x$ and $r' = r/2$. If there exists $z \in B(x, r/2) \cap B_q \cap C_{p,q}$, then there exists a ball $B(z_0, cr/2)$ such that $B(z_0, cr/2) \subset B(z, r/2)$ and $B(z_0, cr/2) \cap B_q = \emptyset$. We put $y = z_0$ and $r' = cr/2$.

We have $r' \ge \min\{1/4, 1/8, 1/2, c/2\}r$. This fact proves our lemma. ∎

COROLLARY. *A subset $M$ of a normed linear space $P$ is σ-globally very porous if and only if there exist a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers and a winning strategy $\mathcal{S}$ for the second player in the game $G(M)$ such that*

$$r(B_{2n}) > c_n r(B_{2n-1}) \quad \text{whenever } (B_1, \ldots, B_{2n}) \text{ is compatible with } \mathcal{S}.$$

Now we derive from Theorem 1 a sufficient condition for σ-porosity using the following modification of the game $G(M)$. Let $M \subset P$. Two players play *the game* $\widetilde{G}(M)$ with the set $M$ in the same way as in the game $G(M)$ and the first player wins if

(i) $c(B_{2n-1}) \in M$ for every $n \in \mathbb{N}$,
(ii) $M \cap \bigcap_{n=1}^{\infty} B_n \neq \emptyset$,

else the second player wins.

THEOREM 3. *Let $M \subset P$, $(c_n)_{n=1}^{\infty}$ be a sequence of positive numbers and $\mathcal{S} = (f_n)_{n=1}^{\infty}$ be a winning strategy for the second player in the game $\widetilde{G}(M)$ such that*

$$r(B_{2n}) > c_n \varrho(c(B_{2n-1}), c(B_{2n})) \quad \textit{whenever } (B_1, \ldots, B_{2n}) \textit{ is compatible with } \mathcal{S}.$$

*Then $M$ is $\sigma$-porous.*

Proof. This proof was suggested by the referee.

Put $P^\star = P \setminus (\overline{M} \setminus M)$. It is easy to see that $P^\star$ is dense in $P$ and the set $M$ is closed in $P^\star$. We will use the following notation. The symbol $B^\star(z, p)$ stands for the open ball in $P^\star$ with center $z \in P^\star$ and radius $p > 0$. We will show that the second player has a winning strategy $\mathcal{S}^\star$ in the game $G(M)$ in the space $P^\star$ such that

$$r(B^\star_{2n}) > c_n \varrho(c(B^\star_{2n-1}), c(B^\star_{2n})) \quad \text{whenever } (B^\star_1, \ldots, B^\star_{2n}) \text{ is compatible with } \mathcal{S}^\star.$$

We define inductively mappings $f^\star_n$ describing the strategy $\mathcal{S}^\star$.

Let $B^\star_1$ be an open ball in $P^\star$. If $c(B^\star_1) \notin M$, then there exists an open ball $B^\star_2$ such that $c(B^\star_2) = c(B^\star_1)$ and $B^\star_2 \cap M = \emptyset$, since $M$ is closed in $P^\star$. We define $f^\star_1(B^\star_1) = B^\star_2$. If $c(B^\star_1) \in M$, then we consider the ball $B_1 = B(c(B^\star_1), r(B^\star_1))$ in $P$. Define

$$B_2 = f_1(B_1).$$

We have $r(B_2) > c_1 \varrho(c(B_1), c(B_2))$. The density of $P^\star$ in $P$ implies that there exists an open ball $C$ such that

$$C \subset B_2, \quad r(C) \in P^\star \quad \text{and} \quad r(C) > c_1 \varrho(c(B^\star_1), c(C)).$$

We put $f^\star_1(B^\star_1) = B^\star(c(C), r(C))$. Suppose that the mappings $f^\star_1, \ldots, f^\star_n$ are defined. Let $(B^\star_1, \ldots, B^\star_{2n+1})$ be a sequence of open balls in $P^\star$ such that

$$B^\star_{k+1} \subset B^\star_k \quad \text{for } k \leq 2n \quad \text{and} \quad B^\star_{2k} = f^\star_k(B^\star_1, \ldots, B^\star_{2k-1}) \quad \text{for } k \leq n.$$

We distinguish the two cases $c(B^\star_{2n+1}) \notin M$ and $c(B^\star_{2n+1}) \in M$ again and define the mapping $f^\star_{n+1}$ in a similar way to $f^\star_1$ above. Let $(B^\star_n)_{n=1}^{\infty}$ be a sequence of open balls compatible with our strategy $\mathcal{S}^\star$. If there exists $n_0 \in \mathbb{N}$ such that $c(B^\star_{2n_0-1}) \notin M$, then the second player wins. If $c(B^\star_{2n-1}) \in M$

for every $n \in \mathbb{N}$, then there exists a sequence $(B_n)_{n=1}^{\infty}$ of open balls in $P$ compatible with $\mathcal{S}$ such that

$$B^\star(c(B_{2n-1}), r(B_{2n-1})) = B^\star_{2n-1}, \qquad B(c(B^\star_{2n}), r(B^\star_{2n})) \subset B_{2n}.$$

Therefore we have

$$\bigcap_{n=1}^{\infty} B^\star_n \subset \bigcap_{n=1}^{\infty} B_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} B_n \cap M = \emptyset.$$

Thus the second player wins also in this case. This and Theorem 1 imply that the set $M$ can be covered by countably many closed uniformly porous subsets of $P^\star$. These sets are clearly porous in $P$. Thus the theorem is proved. ∎

R e m a r k. The condition of Theorem 3 is not a necessary condition. To show this, consider a closed $\sigma$-porous set $M \subset \mathbb{R}$ such that the set

$$\{x \in M : p(x, M) = 0\} \tag{5}$$

is dense in $M$. Such a set can be constructed in this way. If $I = [a, b] \subset \mathbb{R}$ is a closed bounded interval, then we define the set $\{I_k\}_{k \in \mathbb{Z} \setminus \{0\}}$ of open intervals as follows:

If $k > 0$, then $I_k = (x_k, x_{k-1})$, where

$$x_n = \frac{a+b}{2} + \frac{b-a}{n+2}, \qquad n \in \mathbb{N}_0.$$

If $k < 0$, then $I_k = (y_{|k|-1}, y_{|k|})$, where

$$y_n = \frac{a+b}{2} - \frac{b-a}{n+2}, \qquad n \in \mathbb{N}_0.$$

The symbol $I_{k_1,\ldots,k_n}$ means $J_{k_n}$, where $J = I_{k_1,\ldots,k_{n-1}}$.

Now we are able to define the set $M$: Put

$$I = [-1, 1], \quad G = \bigcup_{n=1}^{\infty} \bigcup \{I_{k_1,\ldots,k_n} : k_n \text{ is even}\} \quad \text{and} \quad M = I \setminus G.$$

It is not difficult to prove that $M$ is $\sigma$-porous and that (5) is dense in $M$.

We shall show that the first player has a winning strategy in the game $\widetilde{G}(M)$ for any sequence $(c_n)_{n=1}^{\infty}$. Fix some sequence $(c_n)_{n=1}^{\infty}$. Let $B(0, r_1)$ be the first move of the first player such that

$$\gamma(0, R, M)/R < \tfrac{1}{2} c_1 \qquad \text{whenever } R \in (0, r_1).$$

Such an $r_1$ exists since $0 \in \{x \in M : p(x, M) = 0\}$. The other moves of the first player are chosen to fulfil $\overline{B}_{2n+1} \subset B_{2n}$, $c(B_{2n+1}) \in \{x \in M : p(x, M) = 0\}$ and

$$\gamma(c(B_{2n+1}), R, M)/R < \tfrac{1}{2} c_{n+1} \qquad \text{whenever } R \in (0, r(B_{2n+1})).$$

This gives $B_n \cap M \neq \emptyset$ for every $n \in \mathbb{N}$. Since $M$ is compact we have $\bigcap_{n=1}^{\infty} B_n \cap M \neq \emptyset$.

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