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Algebras of holomorphic functions with Hadamard multiplication

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Abstract. A systematic investigation of algebras of holomorphic functions endowed with the Hadamard product is given. For example we show that the set of all non-invertible elements is dense and that each multiplicative functional is continuous, answering some questions in the literature.

Introduction. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be power series with radii of convergence r_f and r_g respectively. Then the *Hadamard product* of f and g is defined by

$$f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1}$$

Note that the radius of convergence of $f * g$ is at least $r_f \cdot r_g$. Let now G be an open domain of $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ containing 0 and let $H(G)$ be the set of all holomorphic functions on G. We call G *admissible* if for all $f, g \in H(G)$ the Hadamard product $f * g$ extends to a (unique) function of $H(G)$, i.e. $H(G)$ is a commutative algebra. Examples of admissible domains are the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$, or more generally $D_r := \{z \in \mathbb{C} : |z| < r\}$ for $r \geq 1$, and also $\widehat{\mathbb{C}} \setminus \{1\}$ and so-called α-starlike regions like $\mathbb{C}_- := \{z \in \mathbb{C} : z \notin [1, \infty)\}$; see [9] for details. A necessary condition for admissibility is that G^c, the complement of G, is a multiplicative semigroup: if $a, b \in G^c$ then $f(z) = 1/(a - z)$ and $g(z) = 1/(b - z)$ are functions in $H(G)$ and $f * g(z) = 1/(ab - z)$ has a pole at ab and therefore ab must be in G^c. As a consequence G always contains the open unit disk D. Conversely, if G is a domain and G^c is a multiplicative semigroup then the famous Hadamard multiplication theorem states that $f * g$ is holomorphic on G (cf. [15]). Clearly $H(G)$ is a completely metrizable locally convex vector space (i.e. a Fréchet space) where the norms are given by $|f|_K := \sup_{z \in K} |f(z)|$ for an arbitrary compact subset K of G.

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It can be shown that the Hadamard multiplication is continuous, henc $H(G)$ is a so-called B_0-algebra. Nonetheless, it is important to notice tha the norms $|\cdot|_K$ are not submultiplicative: indeed, using a result in [14] we ca show that $H(G)$ is a Fréchet algebra if and only if $1 \in G$, or equivalently, i and only if G contains the closed unit disk (cf. Theorem 2.8 and Proposition 3.11 and 6.4).

The investigation of the algebra $H(G)$ usually depends on the propertie of the admissible domain G. But some results are generally true: In the firs section it will be shown that the set of all non-invertible elements is dense i $H(G)$, answering a question in [9] positively. In the following sections we wil show that each multiplicative functional is continuous, answering a questio in [10], and an explicit description is given. Let us now consider the (three different types of admissible domains. As already mentioned G contains th open unit disk. First assume that the number 1 is in the domain; then G must contain the closed unit disk (otherwise $G^c \cap \{z \in \mathbb{C} : |z| = 1\}$ i either a finite subgroup or a dense subset of the unit circle, contradictin the assumption $1 \in G$). This case will be discussed in Section 2 and it wil be shown that $H(G)$ is a so-called Q-algebra with respect to the norm give by $\|f\|_{\mathbb{N}} := \sup_{n\in\mathbb{N}_0} |a_n|$. As a consequence one deduces that

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{1+a_n} z^n$$

is in $H(G)$ provided that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H(G)$ and $a_n \neq -1$ for al $n \in \mathbb{N}_0$ (cf. Theorem 2.7).

Let us now assume that 1 is in G^c. Then $H(G)$ has a unit element give by

$$\gamma(z) := \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for all } |z| < 1. \tag{2}$$

We have to consider two completely different cases. First suppose that 1 i not isolated in G^c. By Lemma 1 in [10], G is of a rather special form, namel α-starlike, and this case will be discussed in Section 3. For our purposes i suffices to know that α-starlike domains are simply connected. This propert is the key to very simple proofs for characterizing the closed maximal ideal of $H(G)$. Moreover, the multiplicative functionals are given by considerin the nth Taylor coefficient, i.e., they are of the form δ_n defined by $\delta_n(f) := a_n$ It remains to consider the case where 1 is an isolated point in G^c. This cas is more involved and completely different from the previous one. First, it i clear that $A := G^c \cap \{z \in \mathbb{C} : |z| = 1\}$ is a finite subgroup of the unit circl and therefore A is the set of all kth roots of unity for a suitable $k \in \mathbb{N}$ Then $\widetilde{G} := G \cup A$ is an admissible domain containing the closed unit disk

Identifying $f \in H(\widetilde{G})$ with $f|G$ we can see $H(\widetilde{G})$ as a subalgebra of $H(G)$. By separating the singularities one obtains a topological linear isomorphism

$$T : H(G) \to H_k \oplus H(\widetilde{G}), \quad Tf = f_1 + f_2 \tag{3}$$

(cf. [10] for details), where H_k denotes the holomorphic functions $f : \widehat{\mathbb{C}} \setminus A \to \mathbb{C}$ with $f(\infty) = 0$ and $\widetilde{G} \supset G$ contains the closed unit disk. Hence the study of $H(G)$ can be reduced to the algebra H_k and the already discussed case where the domain contains the closed unit disk. Moreover, it is easy to see that H_k and $\bigoplus_{j=1}^{k} H_1$ are isomorphic topological vector spaces (see [10]). Thus investigating H_1 is the key to the general case, which will be done in the fourth section. Clearly f is in the algebra H_1 if and only if there exists an entire function g with $g(1/(1-z)) = f(z)$ and $g(0) = 0$. It is known that the algebra H_1 is topologically and algebraically isomorphic to the algebra E_0 of all entire functions of zero exponential type with pointwise multiplication and a suitable topology. The isomorphism is given by the theorem of Wigert: for $f \in H_1$ there exists a unique function $\widehat{f} \in E_0$ interpolating the Taylor coefficients of f in the sense that $\widehat{f}(n) = a_n$ for all $n \in \mathbb{N}_0$. As worked out in [10] the multiplicative functionals of H_1 are given by point evaluation, i.e., $f \mapsto \widehat{f}(\alpha)$ for $\alpha \in \mathbb{C}$. We give a quite elementary method for determining the multiplicative functionals. As a nice consequence it follows that the interpolating function $\widehat{f} \in E_0$ is just the Gelfand transform of $f \in H_1$.

The fifth section is devoted to the study of the algebra H_k, which was already started in [10]. One of our results states that every closed maximal ideal of H_k is the kernel of a multiplicative functional, answering a question in [10] positively. Moreover, an element in H_k is invertible iff it is not in the kernel of some multiplicative functional. This yields an elegant proof of an invertibility criterion given in [9]. In the sixth section an investigation of $H(G)$ is given for the third case, namely where 1 is isolated in G^c. As a matter of fact, most results which are valid for H_k carry over to this case.

Finally, let us fix some notations. The topological closure of a set M will be denoted by $\overline{M}$. Further, the kernel of a linear functional δ is denoted by $\ker(\delta)$.

1. Density of non-invertible elements in $H(G)$. In this section we assume that G is an admissible domain with $1 \notin G$. If 1 is not isolated in G^c then by Lemma 1 in [10], G is simply connected. Since polynomials are obviously non-invertible we infer that the set of non-invertible elements is dense in $H(G)$ for simply connected domains as polynomials are dense in $H(G)$ (this was already remarked in [9]). This is also true in the isolated case:

1.1. THEOREM. *Let G be an admissible domain with $1 \notin G$. Then the set of non-invertible elements is dense in $H(G)$.*

Proof. By the above remarks we can assume that 1 is isolated in G^c. Let $U \subset \mathbb{C}$ be an open set with $1 \in U$ and $(U \setminus \{1\}) \subset G$. Consider the sequence

$$\gamma_n(z) = \frac{1}{1-z} - \frac{1}{n} \cdot \frac{1}{(1-z)^2} = \sum_{k=0}^{\infty} \frac{n-k-1}{n} z^k. \tag{4}$$

Then $\gamma_n \in H(G)$ and $\gamma_n \to \gamma$. Obviously γ_n is not invertible since the coefficient of z^{n-1} in (4) is zero. We now show that every $f \in H(G)$ can be approximated by a sequence of non-invertible elements. Clearly we can assume that f is invertible. The sequence $f_n = f * \gamma_n$ converges to $f * \gamma = f$ by the continuity of multiplication. The functions f_n are not invertible since $\gamma_n = f^{-1} * f_n$. ■

2. The first case: G contains the closed unit disk. First we will prove two easy lemmas which are true for all admissible domains and which will be also used in other sections. We define the coefficient functionals $\delta_n : H(G) \to \mathbb{C}$ by $\delta_n(f) := a_n$ (where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < 1$). The functionals δ_n will play an important role throughout the paper. Note the formula $\delta_n(f * z^n) = \delta_n(f)$.

2.1. LEMMA. *Let G be an admissible domain. Then δ_n is a continuous multiplicative functional and $H(G)$ is semisimple.*

Proof. For the continuity it suffices to show that $f_k \to 0$ implies $\delta_n(f_k) \to 0$. But this is clear since $f_k \to 0$ implies $f_k * z^n = \delta_n(f_k) z^n \to 0$ by continuity of multiplication. Let now f be in the radical of $H(G)$. Then $\delta_n(f) = 0$ for all $n \in \mathbb{N}_0$. The identity theorem yields $f = 0$. ■

2.2. LEMMA. *Let G be an admissible domain and let M be an ideal. Then, for each $n \in \mathbb{N}_0$, either $M \subset \ker(\delta_n)$ or $z^n \in M$. A prime ideal either contains all polynomials or it is equal to some $\ker(\delta_n)$.*

Proof. If M is not contained in $\ker(\delta_n)$ then there exists $f \in M$ with $\delta_n(f) \neq 0$. Since $f * z^n \in M$ and $f * z^n = \delta_n(f) z^n$ it follows that $z^n \in M$. Now let M be a prime ideal. Suppose that $M \subset \ker(\delta_n)$ for some $n \in \mathbb{N}_0$. Let $f \in \ker(\delta_n)$. Then $f * z^n = \delta_n(f) z^n = 0 \in M$. Since M is prime we infer that $f \in M$ (as $z^n \notin M$). Hence $M = \ker(\delta_n)$. Finally, if $M \neq \ker(\delta_n)$ for all $n \in \mathbb{N}_0$ then $z^n \in M$ for all $n \in \mathbb{N}_0$. ■

Let now G be an admissible domain which contains the *closed* unit disk. Hence each $f \in H(G)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, has a convergence radius $r > 1$ and therefore $(a_n)_n$ converges to zero. It follows that $H(G)$ is a non-unital normed algebra with respect to the submultiplicative norm $\|\cdot\|_{\mathbb{N}}$ defined by

$$\|f\|_{\mathbb{N}} := \sup\{|\delta_n(f)| : n \in \mathbb{N}_0\} = \sup_{n \in \mathbb{N}_0} |a_n|. \tag{5}$$

Let τ_k be the compact-open topology on $H(G)$. Since $|a_n| \le \max\{|f(z)| : |z| = 1\}$ it is clear that $\mathrm{id} : (H(G), \tau_k) \to (H(G), \|\cdot\|_{\mathbb{N}})$ is continuous.

It is well known that each algebra A can be embedded in a unital algebra A_+ in quite a formal way; see [18]. For our purposes it is convenient to adjoin the function γ with $\gamma(z) = (1-z)^{-1}$ as a formal unit element. The unitization $H_+(G)$ is the vector space $\gamma\mathbb{C} \times H(G)$ endowed with the product $(\lambda\gamma + f) * (\mu\gamma + g) := \lambda\mu\gamma + \lambda g + \mu f + f * g$ for $\lambda, \mu \in \mathbb{C}$ and $f, g \in H(G)$. First we consider the simple case where G is a ball $D_r := \{z \in \mathbb{C} : |z| < r\}$ for $r \in (1, \infty]$.

2.3. THEOREM. *Let $r > 1$ and let $f \in H(D_r)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < r$. Then $\gamma - f$ is invertible in $H_+(D_r)$ iff $a_n \neq 1$ for all $n \in \mathbb{N}_0$. The spectrum of f in $H_+(D_r)$ is the set $\{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.*

Proof. The neccessity is clear. For the converse note that $(a_n)_n$ converges to zero. Hence $|1 - a_n| \ge 1/2$ for all $n \ge n_0$ for suitable n_0. Then $|a_n/(1 - a_n)| \le 2|a_n|$ for $n \ge n_0$. It follows that

$$h(z) := \sum_{n=0}^{\infty} \frac{a_n}{1 - a_n} z^n$$

defines a holomorphic function on D_r. But $(\gamma + h)(z) = (\gamma - f)^{-1}$. ■

In the sequel we want to show that

(i) Theorem 2.3 can be generalized to admissible domains containing the closed unit disk, and that

(ii) in this case each multiplicative functional on $H(G)$ is equal to some δ_n.

Recall that an ideal I of a commutative algebra is *modular* if there exists $u \in A$ such that $ux - x \in I$ for all $x \in A$. It is easy to see that I is modular iff A/I has a unit element. Let A_+ be the unitization of A. Then for each (maximal) modular ideal I of A there exists a unique (maximal) ideal I_+ of A_+ such that $I = I_+ \cap A$. Suppose now that A_+ is a *Q-algebra*, i.e. that A_+ is a topological algebra such that the set of all invertible elements is open, or equivalently, A_+ has a neighborhood of the unit element consisting of invertible elements. Clearly a maximal ideal in a Q-algebra is closed and it follows that each modular maximal ideal of A is closed. Suppose now that A is in addition a normed algebra. If I is a maximal modular ideal (hence closed) then A/I is a normed division algebra and is isomorphic to $\mathbb{C}$ by the theorem of Gelfand–Mazur. Hence each maximal modular ideal is the kernel of some multiplicative functional. Moreover, each multiplicative functional is continuous with respect to the norm. The following result implies that $H_+(G)$ is a Q-algebra with respect to $\|\cdot\|_{\mathbb{N}}$:

2.4. THEOREM. *Let G be an admissible domain and $f \in H(G)$ be a function with convergence radius $r > 1$. If $\|f\|_{\mathbb{N}} < 1$ then $\gamma - f$ is invertible in $H_+(G)$.*

Proof. Define $f^0 := \gamma$ and $f^k := f * \ldots * f$ (k times), which has convergence radius r^k for $k \in \mathbb{N}$ by the Cauchy–Hadamard convergence radius formula. Let $s_n := \sum_{k=0}^{n} f^k$ be the nth partial sum of the geometric series. It suffices to show that $(s_n)_n$ is a Cauchy sequence: if s is the limit then $(\gamma - f) * s_n = \gamma - f^{n+1} \to \gamma$ and therefore $(\gamma - f) * s = \gamma$. Let now K be an arbitrary compact subset of G and let $R > 0$ be so big that the ball D_R with center 0 and radius R contains K. Choose $k_0 \in \mathbb{N}$ such that $R < r^{k_0}$. Obviously f^k is equal to $\sum_{n=0}^{\infty} a_n^k z^n$ for a neighborhood of 0, and for $k \geq k_0$ this identity even holds for all $|z| < r^{k_0}$. In particular, we have $C := \sum_{n=0}^{\infty} |a_n^{k_0}| R^n < \infty$. This yields the estimate

$$(6)\qquad |f^k(z)| \leq \|f\|_{\mathbb{N}}^{k-k_0} \sum_{n=0}^{\infty} |a_n^{k_0}||z|^n \leq C\|f\|_{\mathbb{N}}^{k-k_0}$$

for all $z \in K \subset D_R$, $k \geq k_0$. Hence $|f^k|_K \leq C_1 \|f\|_{\mathbb{N}}^k$ for all $k \geq k_0$ and a suitable constant C_1. ∎

2.5. COROLLARY. *Let G be an admissible domain containing the closed unit disk. Then the algebra $H_+(G)$ is a Q-algebra with respect to the norm $\|\cdot\|_{\mathbb{N}}$. In particular, each maximal modular ideal of $H(G)$ and $H_+(G)$ is closed and is the kernel of a multiplicative functional. Each multiplicative functional on $H(G)$ and $H_+(G)$ is continuous with respect to $\|\cdot\|_{\mathbb{N}}$ and the topology of compact convergence.*

Proof. Since each $f \in H(G)$ has convergence radius $r > 1$, Theorem 2.4 yields the first statement. For the last statement note that $\mathrm{id} : (H(G), \tau_k) \to (H(G), \|\cdot\|_{\mathbb{N}})$ is continuous. ∎

Although $(H_+(G), \|\cdot\|_{\mathbb{N}})$ is a Q-algebra the norm is not complete: otherwise the continuous map $\mathrm{id} : (H(G), \tau_k) \to (H(G), \|\cdot\|_{\mathbb{N}})$ would be a topological isomorphism by the open mapping theorem, which is impossible (consider for example $f_n(z) = z^n/n$, which converges to zero with respect to $\|\cdot\|_{\mathbb{N}}$ but not with respect to τ_k).

We are now able to characterize the spectrum of $H(G)$.

2.6. THEOREM. *Let G be an admissible domain containing the closed unit disk. Then each non-trivial multiplicative functional on $H(G)$ and $H_+(G)$ is equal to some δ_n with $n \in \mathbb{N}_0$.*

Proof. Let h be multiplicative. By Corollary 2.5, h is continuous. Suppose that the kernel of h is not contained in some $\ker(\delta_n)$. Then by Lemma 2.2 we have $h(z^n) = 0$ for all $n \in \mathbb{N}_0$. Now let $f \in H(G)$ with $h(f) = 1$ and define f^n as the nth power of f. Let $r > 1$ be the convergence radius of f. For each $n \in \mathbb{N}$ there exists a polynomial p_n such that $|f^n(z) - p_n(z)| < 1/n^2$ for all $|z| < r^n/2$. Then $\sum_{n=1}^{\infty}(f^n - p_n) \in H(G)$. Since $h(f^n) = 1$ and $h(p_n) = 0$ we obtain $h(\sum_{n=1}^{\infty}(f^n - p_n)) = \infty$ by continuity, a contradiction. ∎

Since $H_+(G)$ is a normed Q-algebra the spectrum of an element f coincides with the set $\{h(f) : \text{the functional } h : H_+(G) \to \mathbb{C} \text{ is multiplicative}\}$. This proves

2.7. THEOREM. *Let G be an admissible domain containing the closed unit disk and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(G)$. Then the spectrum of f in $H_+(G)$ is the set $\{a_n : n \in \mathbb{N}_0\} \cup \{0\}$. Hence $\gamma - f$ is invertible if and only if $a_n \neq 1$ for all $n \in \mathbb{N}_0$.* ∎

Simple examples already show that the norms $|\cdot|_K$ are in general not submultiplicative. Even for the case of the unit disk D the algebra $H(D)$ is not a Fréchet algebra; cf. [7] or Proposition 3.11. Therefore it is surprising that $H(G)$ is indeed a Fréchet algebra for the case where G contains the closed unit disk. Instead of constructing a suitable family of submultiplicative seminorms we use a general result proved in [14]. For the case $G = D_r$ with $r > 1$ it is a consequence of the integral representation (see [15])

$$(7)\qquad f * g(z) = \frac{1}{2\pi i} \int_{|t|=\sqrt{|z|}} f(t) g\left(\frac{z}{t}\right) \frac{dt}{t}$$

for $|z| < r$ and $f, g \in H(D_r)$ that $|f * g|_{D_s} \leq |f|_{D_{\sqrt{s}}} |g|_{D_{\sqrt{s}}}$ for all $1 < s < r$.

2.8. THEOREM. *Let G be an admissible domain containing the closed unit disk. Then $H(G)$ and $H_+(G)$ are Fréchet algebras with respect to the topology of compact convergence.*

Proof. By Theorem 1 in [14] a complete unital B_0-algebra A is a Fréchet algebra iff for every entire function $\phi(x) = \sum_{n=0}^{\infty} b_n x^n$ and for every element $g \in A$ the series $\phi(g) = \sum_{n=0}^{\infty} b_n g^n$ is convergent. Let $g = \lambda\gamma + f \in H_+(G)$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(G)$. Choose $\mu > 0$ such that $|\mu\lambda| < 1$ and $\|\mu f\|_{\mathbb{N}} < 1$. Let K be an arbitrary compact subset of G. For $z \in K$ we obtain

$$|\phi(g)(z)| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} |b_n| \mu^{-n} |\mu\lambda|^{n-k} |(\mu f)^k|_K$$
$$\leq \sum_{n=0}^{\infty} |b_n| 2^n \mu^{-n} \sum_{k=0}^{\infty} |(\mu f)^k|_K.$$

By the proof of Theorem 2.4 we know that $\sum_{k=0}^{\infty}|\mu^k f^k|_K < \infty$, and $\sum_{n=0}^{\infty}|b_n|2^n\mu^{-n}|$ converges since ϕ is an entire function. ∎

In Corollary 2.5 we characterized the maximal modular ideals of $H(G)$. If G is bounded we are able to give deeper insight into the ideal structure of $H(G)$. For this purpose we recall that for an ideal M in an algebra A the radical of M is the ideal $\mathrm{rad}(M) := \{a \in A : a^n \in M \text{ for some } n \in \mathbb{N}\}$. Note that $\mathrm{rad}(M) = M$ for a prime ideal M.

2.9. THEOREM. *Let G be a bounded admissible domain containing the closed unit disk and M be an ideal. Then*

$$\mathrm{rad}(\overline{M}) = \bigcap_{n\in\mathbb{N}_0,\, M\subset\ker(\delta_n)} \ker(\delta_n). \tag{8}$$

Further, a closed prime ideal is the kernel of some δ_n and is thus maximal.

Proof. Clearly $\overline{M}$ is contained in the intersection since δ_n is continuous. The intersection of maximal ideals is a radical ideal and thus $\mathrm{rad}(\overline{M})$ is contained in the set on the right hand side. For the other inclusion let $B := \{n \in \mathbb{N}_0 : M \subset \ker(\delta_n)\}$. Now let $f \in H(G)$ with $\delta_n(f) = 0$ for all $n \in B$. Since $f(z) = \sum_{n\notin B} a_n z^n$ has convergence radius $r > 1$ there exists $k \in \mathbb{N}$ such that the ball of radius r^k contains G. Hence the kth power $f^k(z) = \sum_{n=0}^{\infty} a_n^k z^n$ converges in G. It follows that $f^k = \sum_{n\notin B} a_n^k z^n \in \overline{M}$ since $z^n \in \overline{M}$ for all $n \notin B$ by Lemma 2.2. If M is a closed prime ideal then $M = \mathrm{rad}(\overline{M})$. So M must be contained in some $\ker(\delta_n)$ (otherwise we have $M = H(G)$ by (8)). Lemma 2.2 completes the proof. ∎

We finish this section with an example which shows that the conclusion of Theorem 2.9 does not necessarily hold for unbounded domains G. Indeed we construct a closed prime ideal which is neither contained in some $\ker(\delta_n)$ nor maximal:

2.10. EXAMPLE. Let $G = \mathbb{C} \setminus \{n \in \mathbb{N} : n \geq 2\}$ and consider the set of all entire functions $E := H(\mathbb{C}) \subset H(G)$. Obviously E is a proper ideal which is not contained in some $\ker(\delta_n)$. Further, E is prime: Suppose there exist $f, g \in H(G)$, both not entire, with $f * g \in E$. Denote by α resp. β the singularity of f resp. g with the least modulus. Then clearly $\alpha\beta \neq \alpha'\beta'$ for all other singularities α' of f and β' of g. According to a theorem of Borel (see [6]), $\alpha\beta$ is a singularity of $f * g$, contradicting $f * g \in E$. To see that E is closed we observe that, on E, the topology of uniform convergence on compact subsets of G is the same as the topology of uniform convergence on compact subsets of $\mathbb{C}$, by the maximum modulus principle. Since E is complete in the latter topology, it is complete in the former and is therefore closed in $H(G)$. Moreover, E is not maximal since E is contained in the ideal of all functions in $H(G)$ which have residue 0 at $z = 2$. ∎

3. The second case: $1 \in G^c$ is non-isolated. In [7] Brooks showed that the closed maximal ideals of $H(D)$ (D is the unit disk) are exactly the kernels of the multiplicative functionals. This result was extended in [10] to the case of an admissible domain with $1 \in G^c$ non-isolated (hence G is simply connected) using non-trivial results of Arakelyan [3]. We give a quite elementary approach using only some simple properties of the algebra $H(G)$. It may be useful to formulate this in a more general setting:

Let A be a commutative topological (Hausdorff) algebra over a field K. We assume that A contains an infinite family of distinct points $z_i \in A$ with the following properties:

(a) $z_i z_i = z_i \neq 0$ for all $i \in I$.
(b) $a z_i \in K \cdot z_i$ for all $a \in A$, $i \in I$.
(c) The linear span P of $\{z_i : i \in I\}$ is dense in A.

This concept generalizes the definition of a topological algebra with an orthogonal basis which was discussed by several authors ([2], [11], [12], [13]); cf. the end of this section. Note that by property (b) a linear functional $\delta_i : A \to K$ is induced via the formula $a z_i = \delta_i(a) z_i$. Further, the linear span P of the set $\{z_i : i \in I\}$ is a dense ideal. Hence A can never be a Banach algebra (if A is unital). Of course, in the case of $H(G)$ the elements z_i correspond to the monomials z^i and if G is simply connected property (c) is satisfied.

First we need two lemmas which are similar to 2.1 and 2.2.

3.1. LEMMA. *δ_i is a continuous multiplicative functional and $z_i z_j = 0$ for all $i \neq j$.*

Proof. Let a_k be a net converging to 0. By continuity of multiplication $a_k z_i = \delta_i(a_k) z_i \to 0$ and therefore $\delta_i(a_k) \to 0$. Let us prove the multiplicativity: we have $a z_i = \delta(a) z_i$ and $b z_i = \delta_i(b) z_i$. So $\delta_i(ab) z_i = ab z_i = a z_i b z_i = \delta_i(a)\delta_i(b) z_i$. For the last statement note that $z_j z_i = \delta_i(z_j) z_i = \delta_j(z_i) z_j$. If $\delta_i(z_j) = 0$ we are ready. Suppose that $\delta_i(z_j) \neq 0$. Then $\delta_j(z_i) \neq 0$ and therefore $z_i = \alpha z_j$ for some $\alpha \neq 0$. Hence $\alpha z_j = z_i = z_i^2 = \alpha^2 z_j$. So $\alpha = 1$ and $z_i = z_j$, a contradiction. ∎

3.2. LEMMA. *Let M be an ideal. Then either $M \subset \ker(\delta_i)$ or $z_i \in M$.*

Proof. If M is not contained in $\ker(\delta_i)$ then there exists $a \in M$ with $\delta_i(a) \neq 0$. Since $a z_i \in M$ and $a z_i = \delta_i(a) z_i$ it follows that $z_i \in M$. ∎

3.3. THEOREM. *Let M be an ideal of A. Then the following statements are equivalent:*

(a) *M is a prime ideal which is contained in a closed ideal.*
(b) *M is a closed prime ideal.*

(c) *M is a closed maximal ideal.*
(d) *There exists $i \in I$ with $M = \ker(\delta_i)$.*

If A has a unit element e the closed maximal ideals are generated by th elements $e - z_i$, $i \in I$.

Proof. It suffices to prove (a)⇒(d). Let J be a closed ideal containin M. If J is not contained in some $\ker(\delta_i)$ then $z_i \in J$ for all $i \in I$ b Lemma 3.2. Since J is closed we obtain $J = A$, a contradiction. Assum now that $J \subset \ker(\delta_i)$ for some $i \in I$. Let $a \in \ker(\delta_i)$. Now $\delta_i(a) =$ (implies $az_i = 0 \in M$. Since M is a prime ideal either $a \in M$ or $z_i \in M$ But the latter is impossible since $\delta_i(z_i) = 1$. For the last statement assum that $M = \ker(\delta_i)$ for some $i \in I$. Clearly $(e - z_i) \in M$. If $a \in M$ the $a(e - z_i) \in M$. But $a(e - z_i) = a - \delta_i(a) z_i = a$. ■

The first three lines of the last proof also show the "if" part of th following result (which was proved in [7, Theorem 4.2] in a more complicate way for the algebra $H(D)$).

3.4. COROLLARY. *If M is an ideal then $\overline{M}$ is a proper ideal if and onl if M is contained in some $\ker(\delta_i)$.*

3.5. THEOREM. *Suppose that A contains a unit element and let M b a closed ideal and $B := \{i \in I : \delta_i(a) = 0$ for all $a \in M\}$. Then M = $\bigcap_{i \in B} \ker(\delta_i) =: M_B$.*

Proof. The inclusion $M \subset M_B$ is trivial. Now let $a \in M_B$. By Lemm 3.2 we know that $z_i \in M$ for all $i \in I \setminus B$. Let $(p_k)_{k \in I}$ be a net in P (th linear span of z_i, $i \in I$) converging to the unit element e. Then ap_k converge to a by continuity of the multiplication. But ap_k is in M since $\delta_i(ap_k) =$ for all $i \in B$ and therefore ap_k is a finite sum of elements $z_i \in M$. ■

3.6. PROPOSITION. *If A contains a unit element then $\delta_i(a) = 0$ for a $i \in I$ implies $a = 0$. In particular, A is semisimple.*

Proof. Let $(p_i)_i$ be a net in P converging to the unit element e.] p_i is of the form $c_1 z_1 + \ldots + c_n z_n$ for some $c_1, \ldots, c_n \in K$ then ap_i = $c_1 \delta_1(a) z_1 + \ldots + c_n \delta_n(a) z_n = 0$. Hence $a = \lim ap_i = 0$. ■

3.7. PROPOSITION. *The unit element e is not an element of P and n element of P is invertible.*

Proof. If $e = c_1 z_1 + \ldots + c_n z_n$ then $z_{n+1} = z_{n+1} e = 0$, a contradictio If $a = c_1 z_1 + \ldots + c_n z_n$ is invertible then there exists $b \in A$ such tha $e = ab = (c_1 z_1 + \ldots + c_n z_n) b \in P$, a contradiction. ■

In the following we want to prove that each multiplicative function on $H(G)$ (assuming $1 \in G^c$ non-isolated) is continuous. The next exampl shows that this property cannot be expected in the general setting:

3.8. EXAMPLE. Let $A := \{f :$ the function $f : \mathbb{N} \to \mathbb{C}$ bounded$\}$ be endowed with the topology of pointwise convergence. Define z_i by $z_i(k) = 0$ if $k \neq i$ and $z_i(i) = 1$. Then A has properties (a)–(c). But each point in $\beta\mathbb{N} \setminus \mathbb{N}$ induces a discontinuous multiplicative functional (where $\beta\mathbb{N}$ is the Stone–Čech compactification of $\mathbb{N}$). ■

The next result is needed for proving the continuity of multiplicative functionals; cf. Theorem 3.10.

3.9. THEOREM. *Let G be a simply connected domain containing the open unit disk D, with $1 \notin G$. Then*

(a) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(G)$. If α is a complex number such that $n + \alpha \neq 0$ for all $n \in \Lambda(f) := \{n \in \mathbb{N}_0 : a_n \neq 0\}$ then*

$$h_\alpha(z) = \sum_{n \in \Lambda(f)} \frac{a_n}{n + \alpha} z^n$$

defines a function in $H(G)$.

(b) *The series*

$$l(z) := \sum_{n=0}^{\infty} \frac{z^n}{n+1}$$

defines a function in $H(G)$ and for each $\alpha \in \mathbb{C}$ there exists a function $g_\alpha \in H(G)$ such that $g_\alpha * (l + \alpha\gamma)(z) = \sum_{n \in \Lambda_\alpha} z^n$, where $\Lambda_\alpha := \{n \in \mathbb{N}_0 : \alpha + 1/(n+1) \neq 0\}$.

Proof. It is easy to see that $G_1 := G \setminus [0, 1]$ is simply connected. Let L be a logarithm on G_1 and define $z^\alpha := \exp(\alpha L(z))$ on G_1. Then $F_1(z) = z^{\alpha-1} f(z)$ is a holomorphic function on G_1. Let F_2 be a primitive of F_1, i.e. $F_2' = F_1$ on G_1. Note that $F_1(z) = \sum_{n \in \Lambda(f)} a_n z^{n+\alpha-1}$ for all $z \in G_1 \cap D$. Hence we can assume that

$$F_2(z) = \sum_{n \in \Lambda(f)} \frac{a_n}{n + \alpha} z^{n+\alpha}$$

for all $z \in G_1 \cap D$. Now put $h_\alpha := z^{-\alpha} F_2(z)$ and observe that h_α is holomorphic on D and G.

For (b), put $a_n = 1$ for all $n \in \mathbb{N}_0$ and $\alpha = 1$ in (a). It follows that l is in $H(G)$. Note that l is invertible since $l^{-1}(z) = \sum_{n=0}^{\infty} (n+1) z^n = \gamma'(z)$ is in $H(G)$. Thus the assertion is true for $\alpha = 0$. Let us consider $l + \alpha\gamma$ for $\alpha \neq 0$. Note that Λ_α is either $\mathbb{N}_0$ or $\mathbb{N}_0 \setminus \{n_0\}$ for some $n_0 \in \mathbb{N}_0$. Hence $g(z) := \sum_{n \in \Lambda_\alpha} (n+1) z^n$ is in $H(G)$. Now (a) implies that $g_\alpha(z) := \sum_{n \in \Lambda_\alpha} b_n z^n$ is in $H(G)$, where

$$\widetilde{\alpha} := 1 + \frac{1}{\alpha} \quad \text{and} \quad b_n = \frac{1}{\alpha} \cdot \frac{n+1}{n + \widetilde{\alpha}} = \left(\frac{1}{n+1} + \alpha \right)^{-1}.$$

It follows that $g_\alpha * (l + \alpha\gamma)(z) = \sum_{n \in \Lambda_\alpha} z^n$. ■

3.10. THEOREM. *Let G be an admissible domain with $1 \in G^c$ no isolated. Then each multiplicative functional on $H(G)$ is continuous a has the form δ_n for some $n \in \mathbb{N}_0$.*

Proof. Let M be the kernel of a multiplicative functional and l l defined as above. Then $l + \alpha\gamma$ is in M for some $\alpha \in \mathbb{C}$. If $\Lambda_\alpha = \mathbb{N}_0$ then l Theorem 3.9, $l + \alpha\gamma$ is invertible, a contradiction. Hence $\Lambda_\alpha = \mathbb{N}_0 \setminus \{n_0\}$ f some $n_0 \in \mathbb{N}_0$. By Theorem 3.9 there exists $g_\alpha \in H(G)$ such that $\gamma - z^{n_0}$ $\sum_{n\in\Lambda_\alpha} z^n = g_\alpha * (l + \alpha\gamma) \in M$. It follows that $M \subset \ker(\delta_{n_0})$. ∎

We now show that $H(G)$ is not a Fréchet algebra.

3.11. PROPOSITION. *Let G be an admissible domain with $1 \in G^c$ no isolated. Then $H(G)$ is not a Fréchet algebra.*

Proof. Let Δ_A be the set of all continuous multiplicative functionals the algebra A. If A is a Fréchet algebra then the spectrum $\sigma(f)$ is equal $\{h(f) : h \in \Delta_A\}$. Note that $f(z) = \exp(z)$ is not invertible. Hence $0 \in \sigma($ On the other hand, $\Delta_A = \{\delta_n : n \in \mathbb{N}_0\}$, hence $0 \notin \{h(f) : f \in \Delta_A\}$. ∎

Let A be a topological algebra. A sequence $(x_n)_{n\in\mathbb{N}}$ is called a *basis* for each $x \in A$ there exists a unique sequence of scalars $(\alpha_n)_{n\in\mathbb{N}}$ such th $x = \sum_{n=1}^\infty \alpha_n x_n$. A basis $(x_n)_{n\in\mathbb{N}}$ is called *orthogonal* if $x_n x_m = 0$ for $m \neq n$ and $x_n^2 = x_n$ for all $n \in \mathbb{N}$. With Lemma 1.1 in [12] it is easy see that a topological algebra with an orthogonal basis satisfies properti (a)–(c) of the beginning of this section. Lemma 3.1 shows that an orthogor basis in a topological algebra is actually a Schauder basis (Theorem 1.1 [11]) since $x = \sum_{n=1}^\infty \alpha_n x_n$ implies $\delta_m(x)x_m = xx_m = \alpha_m x_m$, i.e., th $\delta_m(x) = \alpha_m$ for all m. Moreover, it follows that a topological algebra wi an orthogonal basis is semisimple (Corollary 1.5 in [12]): if x is in the radi then $\alpha_n = \delta_n(x) = 0$ for all n and hence $x = \sum_{n=1}^\infty \alpha_n x_n = 0$. Recall th a topological (Hausdorff) algebra is a *LC-algebra* if the induced topology locally convex.

3.12. THEOREM. *Let A be a unital complete LC-algebra with an unc ditional orthogonal basis $(x_n)_{n\in\mathbb{N}}$. Then each multiplicative functional $H(G)$ is continuous and has the form δ_n for some $n \in \mathbb{N}$.*

Proof. Let γ be the unit element. Then $\gamma = \sum_{n=1}^\infty \alpha_n x_n$ for suital α_n. Since $\gamma x_n = x_n$ we infer that $\alpha_n = 1$ for all n. By Lemma 1 in [1 $\sum_{n=1}^\infty \alpha_n x_n$ converges for any bounded sequence of scalars $(\alpha_n)_{n\in\mathbb{N}}$. Hen

$$l = \sum_{n=1}^\infty \frac{1}{n+1} x_n \in A \quad \text{and} \quad g_\alpha = \sum_{n\in\Lambda_\alpha} \frac{n+1}{\alpha(n+1+1/\alpha)} x_n \in A,$$

where $\Lambda_\alpha := \{n \in \mathbb{N} : \alpha + 1/(n+1) \neq 0\}$. Since $g_\alpha(l + \alpha\gamma) = \sum_{n\in\Lambda_\alpha} x_n$ can now proceed as in the proof of Theorem 3.10. ∎

Finally, we want to show that $H(\mathbb{C}_-)$ does not have an orthogonal basis with respect to Hadamard multiplication. Suppose that $(x_n)_{n\in\mathbb{N}}$ is an orthogonal basis. It is easy to see that the set $\{x_n : n \in \mathbb{N}\}$ is uniquely determined by the properties (a)–(c). Hence $\{x_n : n \in \mathbb{N}\} = \{z^n : n \in \mathbb{N}_0\}$. So $\gamma = \sum_{n=1}^\infty x_n = \sum_{n=0}^\infty z^{\phi(n)}$ for a bijective mapping $\phi : \mathbb{N}_0 \to \mathbb{N}_0$. Hence $\gamma(-1) = \sum_{n=0}^\infty (-1)^{\phi(n)}$ converges, a contradiction.

4. The algebra H_1. We now turn to the third case: $1 \in G^c$ is isolated. Then clearly $H(G)$ contains $H_1 := \{f \in H(\widehat{\mathbb{C}} \setminus \{1\}) : f(\infty) = 0\}$ as a subalgebra (where $f \in H_1$ is identified by $f|G$). Roughly speaking, the structure of $H(G)$ can be derived from the ground model H_1 which has already been discussed in [10]. It was shown that H_1 is topologically and algebraically isomorphic to the algebra E_0 of all entire functions of zero exponential type. By results of Rashevskiĭ the multiplicative functionals can be completely characterized. In the following we obtain this characterization by more direct and elementary methods which may be interesting in their own right. An important observation is that the algebra H_1 is generated by the element $q_2 := (1-z)^{-2}$ (cf. formula (9)), where $q_n(z) := (1-z)^{-n} = \sum_{k=0}^\infty \binom{k+n-1}{n-1} z^k$ for $n \in \mathbb{N}$. It follows that a continuous multiplicative functional δ is determined by the value $\alpha := \delta(q_2)$ (note that $\delta(q_1) = \delta(\gamma) = 1$). For later reasons this multiplicative functional will be denoted by $\delta_{\alpha-1}$. An elementary calculation yields the equality

$$(9) \qquad q_n = \frac{1}{n-1}[q_2 * q_{n-1} + (n-2)q_{n-1}] = \frac{1}{n-1}[q_2 - q_1] * q_{n-1} + q_{n-1}$$

for all $n \geq 2$. More generally, we want to show in the next lemma that $p_\alpha := q_2 - \alpha\gamma$ is a generating element for each $\alpha \in \mathbb{C}$. Using (9) we find that

$$(10) \qquad p_\alpha * q_{n-1} = (n-1)q_n - (\alpha + n - 2)q_{n-1} \quad \text{for all } n \geq 2.$$

In the following we need the binomial coefficients $\binom{\beta}{n} := \beta(\beta-1)\ldots(\beta-(n-1))/n!$ and $\binom{\beta}{0} := 1$ for $\beta \in \mathbb{C}$. Then $\binom{\alpha+n-2}{n-1} = \alpha(\alpha+1)\ldots(\alpha+(n-2))/(n-1)!$.

4.1. LEMMA. *Let $p_\alpha := q_2 - \alpha\gamma$. Then for each $n \geq 2$ there exist coefficients $c_0(n), \ldots, c_{n-1}(n) \in \mathbb{C}$ such that*

$$(11) \qquad q_n = c_{n-1}(n) q_{n-1} * p_\alpha + \ldots + c_2(n) q_2 * p_\alpha + c_1(n) p_\alpha + c_0(n) q_1.$$

Proof. For $n = 2$ we have $q_2 = p_\alpha + \alpha q_1$. Also note that

$$q_n = \frac{1}{n-1} p_\alpha * q_{n-1} + \frac{\alpha+n-2}{n-1} q_{n-1}$$

by (10). Now an easy induction yields the statement. ∎

Moreover, it is obvious that $c_{n-1}(n) = 1/(n-1)$ and for $n-1 > k$ one has the recursion formula

$$c_k(n) = \frac{\alpha+n-2}{n-1} c_k(n-1).$$

This leads to

$$c_k(n) = \frac{(\alpha+k+1)\dots(\alpha+(n-2))}{(k+2)\dots(n-1)} c_k(k+2)$$

for $n \geq 2$ and $k < n-1$. We thus have

$$c_k(n) = \frac{(\alpha+k)\dots(\alpha+(n-2))}{k\dots(n-1)} \quad \text{for } k \geq 1 \quad \text{and} \quad c_0(n) = \binom{\alpha+n-2}{n-1}.$$

Note that

$$\left|\frac{\alpha+n-2}{n-1}\right| \leq |\alpha|+1.$$

By the recursion formula it follows that $|c_k(n)| \leq (|\alpha|+1)^{n-k-1}$ for $n > k$.

Every element $f \in H_1$ has a unique representation $f(z) = \sum_{n=1}^\infty a_n q_n(z)$, where $\sum_{n=1}^\infty a_n z^n$ is an entire function. Define

$$\delta_\alpha(f) := \sum_{n=1}^\infty a_n \binom{\alpha+n-1}{n-1}. \tag{12}$$

This number exists for all $f \in H_1$ and $\alpha \in \mathbb{C}$ since $\left|\binom{\alpha+n-1}{n-1}\right| \leq (|\alpha|+1)^{n-1}$ and therefore $|\delta_\alpha(f)| \leq \sum_{n=1}^\infty |a_n|(|\alpha|+1)^{n-1} < \infty$. Clearly $\delta_\alpha : H_1 \to \mathbb{C}$ defined by (12) is a linear functional with $\delta_\alpha(\gamma) = \delta_\alpha(q_1) = 1$.

4.2. Theorem. *Let I be an ideal of H_1 which contains p_α. Then I is generated by p_α and I is the kernel of the continuous multiplicative functional $\delta_{\alpha-1} : H_1 \to \mathbb{C}$. If ϕ is a multiplicative functional then ϕ is continuous and $\phi = \delta_{\alpha-1}$ for $\alpha := \phi(q_2)$. Hence the multiplicative functionals are exactly the functionals δ_α with $\alpha \in \mathbb{C}$.*

Proof. Let $f(z) = \sum_{n=1}^\infty a_n q_n(z)$. Put $d_k := \sum_{n=k+1}^\infty a_n c_k(n)$, which is well defined since $|d_k| \leq \sum_{n=k+1}^\infty |a_n c_k(n)| \leq \sum_{n=k+1}^\infty |a_n|(|\alpha|+1)^{n-k-1}$. We now claim that $g(z) := \sum_{k=0}^\infty d_k z^k$ is an entire function. Indeed, let $k_0 \in \mathbb{N}$ be so large that $|\alpha| \leq k_0$. For all $k \geq k_0$ we infer that

$$|c_k(n)| \leq \frac{|\alpha+k|\dots|\alpha+(n-2)|}{k\dots(n-1)} \leq 2^{n-k-1}.$$

This yields $|d_k| \leq \sum_{n=k+1}^\infty |a_n c_k(n)| \leq \sum_{n=k+1}^\infty |a_n| 2^{n-k-1}$ and thus, for $r > 1$,

$$\sum_{k=k_0}^\infty |d_k| r^k \leq \sum_{k=k_0}^\infty r^k \sum_{n=k+1}^\infty |a_n| 2^{n-k-1} \leq \sum_{k=k_0}^\infty \frac{1}{2^{k+1}} \sum_{n=k+1}^\infty |a_n|(2r)^n, \tag{13}$$

which is finite since $\sum_{n=1}^\infty a_n z^n$ is an entire function. Hence we have proved that $\sum_{k=1}^\infty d_k q_k$ is an element of H_1. With (11) we obtain

$$\begin{aligned} f(z) &= \Big(\sum_{n=2}^\infty \sum_{k=1}^{n-1} a_n c_k(n) q_k * p_\alpha\Big) + \delta_{\alpha-1}(f) q_1 \\ &= \Big(\sum_{k=1}^\infty d_k q_k\Big) * p_\alpha + \delta_{\alpha-1}(f) q_1. \end{aligned} \tag{14}$$

Let now I be an ideal containing p_α. By (14), $f - \delta_{\alpha-1}(f) \in I$ for each $f \in H(G)$. It follows that $\ker(\delta_{\alpha-1}) \subset I$ and therefore I is a maximal ideal and $\ker(\delta_{\alpha-1}) = I$. Thus $\ker(\delta_{\alpha-1})$ is an ideal and as already mentioned $\delta_{\alpha-1}(\gamma) = 1$. Hence $\delta_{\alpha-1}$ is multiplicative. Now let ϕ be a multiplicative functional. Define $\alpha := \phi(q_2)$. Then $\phi(p_\alpha) = \phi(q_2 - \alpha\gamma) = 0$. Hence $p_\alpha \in \ker(\phi) =: I$ and therefore I is the kernel of $\delta_{\alpha-1}$ by the first part of the proof. Let us now show that each δ_α is indeed multiplicative: Since p_α is not invertible it is contained in some maximal ideal I_α. As above it follows that $I_\alpha = \ker(\delta_{\alpha-1})$.

We now show that $\delta_{\alpha-1}$ is continuous. To this end let $f_l \to 0$ with $f_l(z) = \sum_{n=1}^\infty a_n^l q_n(z)$. We define $h(z) = \sum_{n=1}^\infty \binom{\alpha+n-2}{n-1} z^n \in H(D_{(|\alpha|+1)^{-1}})$ and $\widetilde{f}_l(z) = \sum_{n=1}^\infty a_n^l z^n \in H(\mathbb{C})$. From $f_l \to 0$ uniformly on every compact set $K \subset \widehat{\mathbb{C}} \setminus \{1\}$ it follows that $\widetilde{f}_l \to 0$ uniformly on every compact set $\widetilde{K} \subset \mathbb{C}$ and $\delta_{\alpha-1}(f_l) = \sum_{n=1}^\infty a_n^l \binom{\alpha+n-2}{n-1} = \widetilde{f}_l * h(1)$ converges to 0 by the continuity of the Hadamard multiplication. ∎

We now give an elementary approach to the Gelfand transform of H_1. For this purpose let us recall some definitions from the theory of entire functions (see [5]). An entire function f is said to be of *exponential type* τ if $\limsup_{r\to\infty} \log(M(r,f))/r \leq \tau$, where $M(r,f) := \max_{|z|=r} |f(z)|$ is the maximum modulus of f. Equivalent to this definition is the one that for every $\varepsilon > 0$ and sufficiently large $|z|$ we have $|f(z)| \leq \exp((\tau+\varepsilon)|z|)$. Of special interest are functions of exponential type zero. We just mention the following property:

A function of exponential type zero is either constant or surjective.

This can be seen as follows: If f omits the value 0 then f is of the form $f = \exp(\varphi)$ with an entire function φ. Since f is of exponential type zero this leads to $M(r, \operatorname{Re}\varphi) = o(r)$. It follows that $\varphi = \text{const}$.

4.3. Lemma. *Let $p_n(\alpha) = (\alpha+1)\dots(\alpha+n-1)$ for $n \geq 2$. Then for each $\alpha \geq 0$,*

$$|p_n(\alpha)| \leq \Big(\alpha + \frac{n}{2}\Big)^n \leq \Big(\frac{n}{2}\Big)^n \sum_{k=0}^n \frac{(2\alpha)^k}{k!}. \tag{15}$$

Proof. Apply the arithmetic-geometric inequality to the numbers $1, \alpha+1, \ldots, \alpha+n-1$. This gives

$$(1 \cdot (\alpha+1) \cdot \ldots \cdot (\alpha+n-1))^{1/n} \le \frac{1}{n}\left((n-1)\alpha + 1 + \frac{n(n-1)}{2}\right) \le \alpha + \frac{n}{2}$$

for $n \ge 2$. Now the first inequality is obvious. For the second note that

$$\left(\alpha + \frac{n}{2}\right)^n = \left(\frac{n}{2}\right)^n \left(1 + \frac{2\alpha}{n}\right)^n \quad \text{and} \quad \frac{n!}{k!(n-k)!n^k} \le \frac{1}{k!}. \ ∎$$

4.4. THEOREM. *Let $f(z) = \sum_{n=1}^\infty a_n z^n$ be a power series with convergence radius $r > e/2$ and*

$$\widehat{f}(\alpha) := \sum_{n=1}^\infty a_n \binom{\alpha+n-1}{n-1} = a_1 + \sum_{n=2}^\infty a_n \frac{p_n(\alpha)}{(n-1)!}. \tag{16}$$

Then $\widehat{f}$ is an entire function of exponential type at most e/r.

Proof. Lemma 4.3 gives

$$|\widehat{f}(\alpha)| \le |a_1| + \sum_{n=2}^\infty \sum_{k=0}^n \frac{|a_n|}{(n-1)!}\left(\frac{n}{2}\right)^n \frac{(2|\alpha|)^k}{k!}.$$

Put

$$b_n := \frac{1}{(n-1)!}\left(\frac{n}{2}\right)^n.$$

Then

$$\frac{b_n}{b_{n+1}} = 2\left(1 - \frac{1}{n+1}\right)^{n+1} \to 2/e < 1.$$

It follows that the power series $\sum_{n=1}^\infty a_n b_n z^n$ has convergence radius at least $2r/e$, which is by assumption strictly larger than one. Let $1 < R < 2r/e$. Then there exists $k_0 \in \mathbb{N}$ such that $\sum_{n=k}^\infty |a_n| b_n R^n \le 1$ for all $k \ge k_0$. It follows that $\sum_{n=k}^\infty |a_n| b_n \le R^{-k}$ since $R \ge 1$. Finally, we have

$$\sum_{k=k_0}^\infty \sum_{n=k}^\infty |a_n| b_n \frac{(2|\alpha|)^k}{k!} \le \sum_{k=k_0}^\infty \frac{1}{k!}\left(\frac{2|\alpha|}{R}\right)^k \le e^{2|\alpha|/R}. \tag{17}$$

Hence $f(z)$ has type less than or equal to $2/R$ for any $R < 2r/e$ and the proof is complete. ∎

4.5. COROLLARY. *Let $f \in H_1$ with $f(z) = \sum_{j=1}^\infty a_j q_j$. Then the Gelfand transform $\widehat{f}$ defined by* (16) *is of zero exponential type and $\widehat{f}(n)$ is the nth coefficient of the Taylor expansion of $f(z)$ at $z = 0$.*

Proof. Theorem 4.4 yields the first statement. For the second note that $q_n(z) = \sum_{k=0}^\infty \binom{k+n-1}{n-1} z^k$. Now it is straightforward to determine the Taylor coefficients. ∎

4.6. COROLLARY. *An element $f \in H_1$ is invertible if and only if there exists $\lambda \ne 0$ with $f = \lambda\gamma$.*

Proof. For the "if" part, suppose that f is not a scalar multiple of γ. By Corollary 4.5, $\widehat{f} : \mathbb{C} \to \mathbb{C}$ is a non-constant function of exponential type zero and is therefore surjective. So $\widehat{f}(\alpha) = 0$ for some $\alpha \in \mathbb{C}$. Hence $\delta_\alpha(f) = 0$, a contradiction to the invertibility. The converse is trivial. ∎

Theorem 4.4 can be deduced from more general and deeper results in the theory of entire functions; see e.g. [5, p. 171]. Corollary 4.5 can be interpreted as one part of the theorem of Wigert, mentioned in the introduction. The other part says that the Gelfand transform from H_1 to the algebra E_0 is actually surjective. The results in Theorem 4.2 can already be found in [16], where the following deeper result was proved:

An ideal $I \subset E_0$ is closed and non-trivial iff there exists a sequence (finite or countable) $(\alpha_n)_n$ with $|\alpha_n|/n \to \infty$ such that I is the set of all $\widehat{f} \in E_0$ with $\widehat{f}(\alpha_n) = 0$ for all n.

From this we deduce

4.7. THEOREM. *Let $I \subset H_1$ be an ideal. Then the following statements are equivalent*:

(a) *I is a closed prime ideal.*
(b) *I is a closed maximal ideal.*
(c) *I is the kernel of some $\delta_{\alpha-1}$.*

Proof. It suffices to show (a)⇒(c). By Theorem 4.2 we just need to prove $p_\alpha \in I$. Now let $\widehat{I}$ be the corresponding closed prime ideal in E_0. Further let $(\alpha_n)_n$ be the sequence in the above result of Rashevskiĭ and $\widehat{f} \in \widehat{I}$. Clearly $(\alpha_n)_n$ consists of at least one element, say α_1. If n is the order of the zero of $\widehat{f}$ at α_1 then $\widehat{g}(z) := \widehat{f}(z)/(z-\alpha_1)^n \in E_0$ does not vanish at α_1 and thus $\widehat{g} \notin \widehat{I}$. Since $\widehat{f}(z) = \widehat{g}(z)(z-\alpha_1)^n \in \widehat{I}$ and $\widehat{I}$ is prime we find that $(z-\alpha_1) \in \widehat{I}$ and thus $(\alpha_n)_n$ consists just of the element α_1. With $\alpha := \alpha_1 + 1$ we find that $\sum_{n=0}^\infty (n-\alpha_1) z^n = q_2(z) - \alpha\gamma(z) = p_\alpha(z) \in I$. ∎

5. The algebra H_k. Throughout this section k denotes a fixed positive integer. We let by ξ be the kth root of 1 given by $\exp(2i\pi/k)$, and $A_k := \{\xi^j : j = 0, \ldots, k-1\}$ be the set of all kth roots of 1. Similarly to the algebra H_1 we consider $H_k := \{f \in H(\widehat{\mathbb{C}} \setminus A_k) : f(\infty) = 0\}$. Define $\gamma_j(z) := \gamma(z/\xi^j)$; we also write $\gamma_\zeta(z) := \gamma(z/\zeta)$ for $\zeta \in A_k$. Then $\gamma_j \in H_k$ for each $j = 0, \ldots, k-1$. Each $f \in H_k$ has a unique decomposition

$$f = \sum_{j=0}^{k-1} \gamma_j * f_j \quad \text{with } f_j \in H_1 \tag{18}$$

by Laurent expansion. As pointed out in [10], H_k is topologically and linearly isomorphic to the topological direct sum of the closed subspaces $\gamma_j * H_1$, $j = 0, \ldots, k-1$. Note that no problems occur in considering H_1 as a subspace of H_k because of removability of singularities. For each $\zeta \in A_k$ there is a natural continuous algebra homomorphism $T_\zeta : H_k \to H_1$ defined by

$$T_\zeta(f) = T_\zeta\Big(\sum_{j=0}^{k-1} \gamma_j * f_j\Big) := \sum_{j=0}^{k-1} \zeta^j f_j; \tag{19}$$

cf. Lemma 2 in [10]. Note that $T_\zeta|H_1$ is just the identity and that $T_\zeta(\gamma_j) = \zeta^j \cdot \gamma$ for $j = 0, \ldots, k-1$. The next two theorems will be the basis for lifting results from H_1 to H_k.

5.1. Theorem. *For every prime ideal M of H_k there exists $\zeta \in A_k$ such that $T_\zeta(M)$ is a (proper) prime ideal of H_1. If M is closed then so is $T_\zeta(M)$.*

Proof. For $\zeta \in A_k$ define $g_\zeta = \sum_{j=0}^{k-1} \zeta^j \gamma_j$. Recall that $\sum_{l=0}^{k-1} (\xi^j)^l = 0$ for each $j = 1, \ldots, k-1$, where $\xi = \exp(2i\pi/k)$. Then

$$\sum_{l=0}^{k-1} g_{\xi^l} = k\gamma + \sum_{l=0}^{k-1}\sum_{j=1}^{k-1} (\xi^l)^j \gamma_j = k\gamma + \sum_{j=1}^{k-1} \gamma_j \Big(\sum_{l=0}^{k-1} (\xi^j)^l\Big) = k\gamma. \tag{20}$$

We infer that if M is an ideal then there exists $\zeta \in A_k$ with $g_\zeta \notin M$ which will be fixed in the following. Now let $f = \sum_{i=0}^{k-1} \gamma_i * f_i$ be an arbitrary element of the ideal M. Note that $g_\zeta * f$ is in M and an easy calculation (using the identity $\gamma_i * \gamma_j = \gamma_{i+j}$) shows that

$$g_\zeta * f = \sum_{j=0}^{k-1}\sum_{i=0}^{k-1} \zeta^j \gamma_j * \gamma_i * f_i = \Big(\sum_{i=0}^{k-1} \zeta^{-i} f_i\Big) * g_\zeta. \tag{21}$$

Since M is a prime ideal and $g_\zeta \notin M$ we infer that the first factor of the last product in (21) is in M. It follows that $T_{\zeta^{-1}}(M) \subset M \cap H_1$. On the other hand we have $M \cap H_1 = T_{\zeta^{-1}}(M \cap H_1) \subset T_{\zeta^{-1}}(M)$ and thus $T_{\zeta^{-1}}(M) = M \cap H_1$. In particular it is obvious that it is a proper prime ideal in H_1 (for if $\gamma \in M \cap H_1$ then $\gamma \in M$). Let us show that $M \cap H_1$ is closed in H_1 if M is closed in H_k: let $(f_n)_n$ be a sequence in $M \cap H_1$ converging in H_1 to an element f. Then $(f_n)_n$ converges to f also in H_k and since M is closed in H_k we infer that $f \in M$ and thus $f \in M \cap H_1$. Hence $M \cap H_1$ is closed. ∎

5.2. Theorem. *Let M be a maximal (and closed resp.) ideal of H_k. Then there exists $\zeta \in A_k$ such that $T_\zeta(M)$ is a maximal (and closed resp.) ideal of H_1.*

Proof. By Theorem 5.1 there exists $\zeta \in A_k$ such that $T_\zeta(M)$ is a proper ideal. Let J be an ideal which contains $T_\zeta(M)$. It follows that $M \subset T_\zeta^{-1}(J) \neq H_k$. By maximality we obtain equality and therefore $J = T_\zeta(M)$. ∎

Our first conclusion from the above is:

5.3. Theorem. *For every mutiplicative functional ϕ on H_k there exist $\alpha \in \mathbb{C}$ and $\zeta \in A_k$ with $\phi = \delta_\alpha \circ T_\zeta$. In particular, each multiplicative functional is continuous. Further, each closed prime ideal M of H_k is contained in the kernel of some $\delta_\alpha \circ T_\zeta$. Hence the closed maximal ideals of H_k are exactly the kernels of the multiplicative functionals.*

Proof. Let $f = \sum_{j=0}^{k-1} \gamma_j * f_j \in H_k$. Since $\phi|_{H_1} = \delta_\alpha$ for some $\alpha \in \mathbb{C}$ it follows that $\phi(f) = \sum_{j=0}^{k-1} \phi(\gamma_j)\delta_\alpha(f_j)$. Hence ϕ is continuous. Now let M be a closed prime ideal in H_k. By Theorem 5.1 there exists $\zeta \in A_k$ such that $T_\zeta(M)$ is a closed prime ideal in H_1. By Theorem 4.7 it follows that $T_\zeta(M)$ is the kernel of some δ_α. Thus M is contained in the kernel of $\delta_\alpha \circ T_\zeta$. To complete the proof observe that $\delta_\alpha \circ T_\zeta$ is a multiplicative functional for all $\alpha \in \mathbb{C}$ and $\zeta \in A_k$. ∎

5.4. Remark. Although each closed prime ideal is maximal for the algebra H_1 this is no longer true for H_k with $k > 1$. For example, $I := \ker(T_\xi)$ is a closed prime ideal since T_ξ is continuous and multiplicative and H_1 is an integral domain (E_0 is one). But I is not maximal by Theorem 5.3.

Now we are going to characterize invertibility in H_k. This will be the key for the characterization of invertible elements in the case where $1 \in G^c$ is isolated. Invertibility in Hadamard algebras has a very interesting analytic interpretation: Let $f(z) = \sum_{n=0}^\infty a_n z^n$ define a holomorphic function on a domain G. Under what conditions does $f^{-1}(z) = \sum_{n=0}^\infty a_n^{-1} z^n$ define a function which is also holomorphic on G? In fact, this question was the starting point for the investigation of Hadamard algebras (see [1] and [9]).

5.5. Corollary. *An element $f \in H_k$ is not invertible if and only if there exist $\alpha \in \mathbb{C}$ and $\zeta \in A_k$ such that $\delta_\alpha \circ T_\zeta(f) = 0$.*

Proof. If f is not invertible then f is contained in a maximal ideal M. By Theorem 5.1 there exists $\zeta \in A_k$ such that $T_\zeta(M)$ is a maximal ideal in H_1. Hence $T_\zeta(f)$ is not invertible in H_1. From the proof of Corollary 4.6 it follows that there exists $\alpha \in \mathbb{C}$ with $0 = \widehat{T_\zeta(f)}(\alpha) = \delta_\alpha \circ T_\zeta(f)$. The converse is clear. ∎

5.6. Theorem. *An element $f \in H_k$ is invertible if and only if there exist complex numbers $c_0, \ldots, c_{k-1}$ such that $f = \sum_{j=0}^{k-1} c_j \gamma_j$ and $\sum_{j=0}^{k-1} c_j \zeta^j \neq 0$ for all $\zeta \in A_k$.*

Theorem 5.6 can be expressed in the following way: $f \in H_k$ *is invertible iff* f *is a linear combination of the* γ_j *and* $\delta_n(f) \neq 0$ *for all* $n \in \mathbb{N}_0$.

Proof. Let $f = \sum_{j=0}^{k-1} \gamma_j * f_j$ be invertible. Then $g := T_\zeta(f) = \sum_{j=0}^{k-1} \zeta^j f_j$ is invertible in H_1 (since $g^{-1} = T_\zeta(f^{-1})$). But invertible elements in H_1 are non-zero multiples of γ. Hence there exists $d_\zeta \neq 0$ with $T_\zeta(f) = d_\zeta \gamma$. For each $\alpha \in \mathbb{C}$ we have $\delta_\alpha \circ T_\zeta(f) = \sum_{j=0}^{k-1} \zeta^j \delta_\alpha(f_j) = d_\zeta$. This is a linear system for $\delta_\alpha(f_j)$. Since the coefficient matrix is Vandermonde there exists a *unique* solution $c_0, \ldots, c_{k-1}$. It follows that $\delta_\alpha(f_j) = c_j$ independent of α and therefore $f_j = c_j \gamma$ for $j = 0, \ldots, k-1$.

For the converse let $f = \sum_{j=0}^{k-1} c_j \gamma_j$ as in the theorem. Then $\delta_\alpha \circ T_\zeta(f) = \sum_{j=0}^{k-1} c_j \zeta^j \neq 0$. By Corollary 5.5, f is invertible. ■

6. The third case: $1 \in G^c$ is isolated. In this section we assume that G is an admissible domain such that 1 is isolated in G^c. As explained in the introduction there exists $k \in \mathbb{N}$ such that $\widetilde{G} := G \cup A_k$ is an admissible domain containing the closed unit disk and there exists a topological linear isomorphism

$$T : H(G) \to H_k \oplus H(\widetilde{G}), \quad Tf = f_1 + f_2. \tag{22}$$

$H(\widetilde{G})$ is an ideal of $H(G)$, where $f \in H(\widetilde{G})$ is identified with $f|G$. Note again that considering $H(\widetilde{G})$ as a subspace of $H(G)$ causes no trouble since all singularities $\zeta \in A_k$ are removable for $f \in H(\widetilde{G})$. First we characterize the multiplicative functionals on $H(G)$.

6.1. THEOREM. *Let G be an admissible domain with $1 \in G^c$ isolated. Then each multiplicative functional $\phi : H(G) \to \mathbb{C}$ is continuous. Further, either there exists $n \in \mathbb{N}_0$ such that $\phi = \delta_n$ or there exist $\alpha \in \mathbb{C}$ and $\zeta \in A_k$ such that $\phi(f) = \delta_\alpha \circ T_\zeta(f_1)$, where $f = f_1 + f_2 \in H_k \oplus H(\widetilde{G})$.*

Proof. Clearly $\psi := \phi|H(\widetilde{G})$ is a multiplicative functional on $H(\widetilde{G})$. If $\psi \neq 0$ then Theorem 2.6 yields $\psi = \delta_n|H(\widetilde{G})$ for some $n \in \mathbb{N}_0$. Let now $f \in H(G)$ with $\phi(f) = 0$. As $f * \exp \in H(\widetilde{G})$ we obtain $\delta_n(f * \exp) = \phi(f * \exp) = 0$. It follows that $\delta_n(f) = 0$ and therefore $\phi = \delta_n$. Suppose now $\psi = 0$. Let $p : H(G) \to H_k$ be the projection on H_k. Since p is an algebra homomorphism and $\psi = 0$ it follows that $\phi(f) = \phi(f_1) = \phi \circ p(f)$. Now Theorem 5.3 applied to $\phi \circ p$ completes the proof. ■

According to Corollary 2.5, Theorem 3.10 and the above we have for each admissible domain G the following:

Every multiplicative functional on $H(G)$ is continuous.

We are now able to generalize the invertibility criterion given in [10] for the case $G = D_r \setminus A_k$ with $r > 1$ to all admissible domains G with $1 \in G^c$ isolated.

6.2. THEOREM. *Let G be an admissible domain with $1 \in G^c$ isolated. For $f = f_1 + f_2 \in H_k \oplus H(\widetilde{G})$ and $f_2(z) = \sum_{n=0}^\infty a_n z^n$ the following statements are equivalent:*

(a) *f is invertible.*

(b) *$\phi(f) \neq 0$ for all (continuous) multiplicative functionals ϕ.*

(c) *$\delta_\alpha \circ T_\zeta(f_1) \neq 0$ for all $\alpha \in \mathbb{C}$, and $\zeta \in A_k$ and $\delta_n(f) \neq 0$ for all $n \in \mathbb{N}_0$.*

(d) *f_1 is invertible in H_k and $\delta_n(f) \neq 0$ for all $n \in \mathbb{N}_0$.*

(e) *There exist $c_0, \ldots, c_{k-1} \in \mathbb{C}$ such that $f_1 = \sum_{j=0}^{k-1} c_j \gamma_j$ and $\sum_{j=0}^{k-1} c_j \zeta^j \neq 0$ for all $\zeta \in A_k$ and $a_n \neq -\sum_{j=0}^{k-1} c_j \xi^{-nj}$ for all $n \in \mathbb{N}_0$.*

Proof. (a)⇒(b) and (b)⇒(c) are clear; cf. Theorem 6.1. Corollary 5.5 yields (c)⇔(d). For (d)⇔(e) use Theorem 5.6 and note that $a_n = \delta_n(f_2) \neq -\delta_n(f_1) = -\sum_{j=0}^{k-1} c_j \xi^{-nj}$ since $\delta_n(\gamma_j) = \delta_n(1/(1 - z/\xi^j)) = \xi^{-nj}$. Hence it remains to prove, for example, (d)⇒(a). We consider $g := f_2 * f_1^{-1}$, which is an element of $H(\widetilde{G})$. By Theorem 2.7 it follows that $(\gamma + g)$ is invertible since $\delta_n(g) = \delta_n(f_2) \cdot \delta_n(f_1^{-1}) \neq -1$. Hence there exists $h \in H(\widetilde{G})$ such that $(\gamma + h) * (\gamma + g) = \gamma$. On multiplying by $f_1^{-1} * f_1$ we obtain $\gamma = (f_1^{-1} + f_1^{-1} * h) * (f_1 + f_2)$. ■

6.3. PROPOSITION. *Let G be an admissible domain with $1 \in G^c$ isolated and $f \in H(G)$. Then the spectrum $\sigma(f)$ is equal to the set $\{\phi(f) : \phi$ is a multiplicative functional$\}$. The spectrum of $\sum_{j=0}^{k-1} c_j \gamma_j + f_2$ is a finite or countable set.*

Proof. The first statement is a consequence of Theorem 6.2(a)⇔(b). The second follows from the fact that ϕ is equal either to some δ_n or $\delta_\alpha \circ T_\zeta$. But $\delta_\alpha \circ T_\zeta(f_1) = \sum_{j=0}^{k-1} c_j \zeta^j$ for all $\alpha \in \mathbb{C}$ and $\zeta \in A_k$ which is independent of α. ■

6.4. PROPOSITION. *Let G be an admissible domain with $1 \in G^c$ isolated. Then $H(G)$ is not a Fréchet algebra.*

Proof. Since H_1 is a closed subspace of $H(G)$ it suffices to show that the statement is true for H_1. In a commutative semisimple Fréchet algebra (different from $\mathbb{C}$) there always exist invertible elements which are not multiples of the unit element (consider $\exp(a)$ with $a \notin \mathbb{C}$). This contradicts Corollary 4.6. ■

Together with Theorem 2.8 and Proposition 3.11 we have the following general result for Hadamard algebras: *$H(G)$ is a Fréchet algebra iff $1 \in G$.*

Our knowledge about invertibility enables us to generalize the results on *formal* invertibility given in [10].

6.5. DEFINITION. Let G be an admissible domain. For $f \in H(G)$ define $\Lambda(f) := \{n \in \mathbb{N}_0 : a_n \neq 0\}$. The element $f_{-1} := \sum_{n\in\Lambda(f)} a_n^{-1} z^n$ is called the *formal inverse* of f provided it is an element of $H(G)$. We call f *formally invertible* if the formal inverse exists. For $B \subset \mathbb{N}_0$ define $\gamma_B(z) := \sum_{n\in B} z^n$. For $B = \emptyset$ this means $\gamma_B = 0$.

Note that f is formally invertible iff there exists $h \in H(G)$ with $f * h = \gamma_{\Lambda(f)}$. If $\gamma_{\Lambda(f)} \in H(G)$ then $\gamma_{\Lambda(f)}$ has an analytic continuation beyond the unit circle. According to a theorem of Szegö (see e.g. [17]), $\gamma_{\Lambda(f)}$ is of the form $P(z)/(1-z^k)$ with a polynomial P. This applied to $\gamma_f := \gamma_{\mathbb{N}_0\setminus\Lambda(f)}$ and polynomial division gives a decomposition $\gamma_f := P_1 + P_2/(1-z^k)$ with a polynomial P_1, and a polynomial P_2 of degree lower than k. Let $\overline{\gamma_f}$ denote $P_2/(1-z^k)$. Observe that $\overline{\gamma_f}$ is a linear combination of the functions γ_j and $\gamma_f = \overline{\gamma_f} + P_1$ is an element of $H_k \oplus H(\widetilde{G})$.

6.6. THEOREM. *Let $k \geq 1$ and $f = f_1 + f_2 \in H_k \oplus H(\widetilde{G})$ and $\xi = \exp(2\pi i/k)$. Then the following statements are equivalent:*

(a) *f is formally invertible.*

(b) *$f + c\gamma_f$ is invertible for all $c \neq 0$.*

(c) *There exists $c \neq 0$ such that $f + c\gamma_f$ is invertible.*

(d) *There exist $c_0, \ldots, c_{k-1}$ such that $f_1 = \sum_{j=0}^{k-1} c_j\gamma_j$. Further, $\gamma_f \in H(G)$ and $\delta_\alpha \circ T_\zeta(f_1) = 0$ implies $\delta_\alpha \circ T_\zeta(\overline{\gamma_f}) \neq 0$ for all $\alpha \in \mathbb{C}$ and $\zeta \in A_k$.*

(e) *There exist $c_0, \ldots, c_{k-1}$ such that $f_1 = \sum_{j=0}^{k-1} c_j\gamma_j$. Further, $\gamma_f \in H(G)$ and $\delta_n(f_1) = 0$ implies $\delta_n(\overline{\gamma_f}) \neq 0$ for all $n \in \mathbb{N}_0$.*

(f) *There exist $c_0, \ldots, c_{k-1}$ such that $f_1 = \sum_{j=0}^{k-1} c_j\gamma_j$ and for each $m = 0, \ldots, k-1$ with $\sum_{j=0}^{k-1} c_j\xi^{-jm} = 0$ there exists n_0 such that $a_{m+kn} = 0$ for all $n \geq n_0$.*

Proof. (a)⇒(b). Let f be formally invertible and $c \neq 0$. Since $f * \gamma_f = 0$ it follows that $(f + c\gamma_f) * (f_{-1} + c^{-1}\gamma_f) = \gamma$. (b)⇒(c) is trivial. For (c)⇒(a) a glance at the power series shows that f_{-1} is the difference of $(f + c\gamma_f)^{-1}$ and $c^{-1}\gamma_f$ and therefore it is in $H(G)$.

For (b)⇒(d) note that $f + \gamma_f$ is invertible and obviously $\gamma_f \in H(G)$. Let $p : H(G) \to H_k$ be the canonical projection. Since $f + \gamma_f = f_1 + \overline{\gamma_f} + P_1 + f_2$ and $0 \neq \delta_\alpha \circ T_\zeta \circ p(f + \gamma_f) = \delta_\alpha \circ T_\zeta(f_1 + \overline{\gamma_f})$ we find that $f_1 + \overline{\gamma_f}$ is invertible in H_k. Thus by Theorem 5.6, $f_1 + \overline{\gamma_f}$ is a linear combination of the functions γ_j and since $\overline{\gamma_f}$ is also such a combination, so is f_1. Now if $\delta_\alpha \circ T_\zeta(f_1) = 0$ then clearly $\delta_\alpha \circ T_\zeta(\overline{\gamma_f}) \neq 0$. Otherwise it follows that $\delta_\alpha \circ T_\zeta(f_1 + \overline{\gamma_f}) = 0$.

For (d)⇒(e) consider δ_n as a multiplicative functional on H_k. By Theorem 5.3 there exists $\zeta \in A_k$ and $\alpha \in \mathbb{C}$ such that $\delta_n = \delta_\alpha \circ T_\zeta$. Now apply

assumption (d). For (e)⇒(f) choose n_0 strictly larger than the degree of the polynomial P_1 (see above). Then $\delta_r(\overline{\gamma_f}) = \delta_r(\gamma_f)$ for all $r \geq n_0$. Let m be as in (f). Then $r := m + kn$ is larger than n_0 (since $k \geq 1$) for all $n \geq n_0$. Now suppose that

$$\delta_r(f_1) = \sum_{j=0}^{k-1} c_j\delta_r(\gamma_j) = \sum_{j=0}^{k-1} c_j\xi^{-rj} = \sum_{j=0}^{k-1} c_j\xi^{-mj} = 0. \tag{23}$$

By (e) we infer that $\delta_r(\overline{\gamma_f}) = \delta_r(\gamma_f) \neq 0$. Hence $r \in \mathbb{N}_0 \setminus \Lambda(f)$ and therefore $\delta_r(f) = 0$. Since $\delta_r(f_1) = 0$ we finally obtain $0 = \delta_r(f) = \delta_r(f_2) = a_r$.

For (f)⇒(a) we choose a polynomial Q such that for $g := f_1 + g_2$ with $g_2(z) := f_2(z) + Q(z) =: \sum_{n=0}^{\infty} \widetilde{a}_n z^n$ and each $m = 0, \ldots, k-1$ with $\sum_{j=0}^{k-1} c_j\xi^{-jm} = 0$ it follows that $\widetilde{a}_{m+kn} = 0$ for *all* $n \in \mathbb{N}_0$. Obviously it suffices to show that $g = f + Q$ is formally invertible. For this purpose we show that $g + \gamma_g$ is invertible. By the choice of Q we have $\Lambda(g) = \Lambda(f_1)$ and since $\Lambda(f_1)$ is periodic it follows from the proof of Szegö's theorem that $\gamma_g = \overline{\gamma_g}$. Now clearly $\delta_n(g + \gamma_g) \neq 0$ for all $n \in \mathbb{N}_0$. Moreover, $f_1 + \gamma_g$ is invertible in H_k by Theorem 5.6. Thus $\delta_\alpha \circ T_\zeta \circ p(g + \gamma_g) = \delta_\alpha \circ T_\zeta(f_1 + \gamma_g) \neq 0$. An application of Theorem 6.2(b) shows that $g + \gamma_g$ is invertible. ∎

We conclude with a description of the closed maximal ideals if G is bounded.

6.7. THEOREM. *Let G be a bounded admissible domain with $1 \in G^c$ isolated and let M be a closed maximal ideal of $H(G)$. If M does not contain $H(\widetilde{G})$ then M is the kernel of some δ_n. If M contains $H(\widetilde{G})$ then M is the kernel of some $\delta_\alpha \circ T_\zeta \circ p$.*

Proof. In the first case $\widetilde{M} := M \cap H(\widetilde{G})$ is a closed prime ideal in $H(\widetilde{G})$ and by Theorem 2.9 it is the kernel of some δ_n. Now let $f \in M$. Since $f * \exp \in \widetilde{M}$ we obtain $\delta_n(f * \exp) = 0$ and thus $\delta_n(f) = 0$.

In the second case there exists a closed maximal ideal I of H_k with $M = I + H(\widetilde{G})$; cf. Proposition 5 in [10]. But I is the kernel of some multiplicative functional $\delta_\alpha \circ T_\zeta$ by Theorem 5.3. ∎

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