[9] S. Janson and M. H. Taibleson, *I teoremi di rappresentazione di Calderón*, Rend. Sem. Mat. Univ. Politec. Torino 39 (1981), 27–35.
[10] V. M. Kokilašvili [V. M. Kokilashvili], *Maximal inequalities and multipliers in weighted Triebel–Lizorkin spaces*, Soviet Math. Dokl. 19 (1978), 272–276.
[11] J. Peetre, *On spaces of Triebel–Lizorkin type*, Ark. Mat. 13 (1975), 123–130.
[12] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Springer, Berlin, 1989.
[13] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[14] —, *Characterizations of Besov–Hardy–Sobolev spaces: A unified approach*, J. Approx. Theory 52 (1988), 162–203.
[15] —, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

Department of Mathematics & Statistics
University of Canterbury
Christchurch 1, New Zealand
E-mail: hqb@math.canterbury.ac.nz

Institute of Mathematics
University of Wrocław
50-384 Wrocław, Poland
E-mail: mpal@math.uni.wroc.pl

Department of Mathematics
Washington University
St. Louis, Missouri 63130
U.S.A.
E-mail: mitch@math.wustl.edu

Received March 27, 1995 (3443)

STUDIA MATHEMATICA 119 (3) (1996)

A note on a formula for the fractional powers of infinitesimal generators of semigroups

by

CELSO MARTINEZ and MIGUEL SANZ (València)

Abstract. If $-A$ is the generator of an equibounded C_0-semigroup and $0 < \operatorname{Re}\alpha < m$ (m integer), its fractional power A^α can be described in terms of the semigroup, through a formula that is only valid if a certain function $K_{\alpha,m}$ is nonzero. This paper is devoted to the study of the zeros of $K_{\alpha,m}$.

1. Introduction. A closed linear operator $A : D(A) \subset X \to X$ in a Banach space X is nonnegative if $\varrho(A) \supset]-\infty, 0[$ and there exists a constant $M \ge 0$ such that

$$\|\lambda(\lambda + A)^{-1}\| \le M, \qquad \lambda > 0.$$

If $-A$ is the generator of an equibounded C_0-semigroup $\{P_t : t > 0\}$ in X, then A is a nonnegative operator with a dense domain.

The fractional power, A^α, with base a nonnegative operator A and complex exponent $\alpha \in \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, has been widely studied (see [1, 3–5, 8, 9]) and has the property that if $0 < \operatorname{Re}\alpha < m$, with m integer, then $D(A^m) \subset D(A^\alpha)$ and

$$A^\alpha\phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1}[(\lambda + A)^{-1}A]^m \phi \, d\lambda, \qquad \phi \in D(A^m)$$

(where $\lambda^\alpha = e^{\alpha \log \lambda}$ with $\log \lambda \in \mathbb{R}$). This formula was obtained by H. Komatsu and, furthermore, he proved in [4, Th. 2.10] that if the operator A is densely defined, then the domain $D(A^\alpha)$ consists of elements $\phi \in X$ for which

$$\lim_{N\to\infty} \int_0^N \lambda^{\alpha-1}[A(\lambda + A)^{-1}]^m \phi \, d\lambda$$

1991 *Mathematics Subject Classification*: Primary 47A60; Secondary 47D03.
Work partially supported by DGCYT, grant PS88-0115, Spain.

exists and A^α is defined as

$$A^\alpha\phi = \lim_{N\to\infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^N \lambda^{\alpha-1}[A(\lambda+A)^{-1}]^m \phi\, d\lambda, \qquad \phi\in D(A^\alpha).$$

This formula can be rewritten as

$$A^\alpha = \operatorname*{s\text{-}lim}_{N\to\infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^N \lambda^{\alpha-1}[A(\lambda+A)^{-1}]^m\, d\lambda.$$

In the particular case where $-A$ is the generator of an equibounded C_0-semigroup, Komatsu described the fractional power A^α by means of the semigroup P_t. He proved in [4, Th. 4.4] that if $0<\operatorname{Re}\alpha<m$, then

$$A^\alpha = \operatorname*{s\text{-}lim}_{\varepsilon\to 0} \frac{1}{K_{\alpha,m}} \int_\varepsilon^\infty t^{-\alpha-1}(1-P_t)^m\, dt, \tag{1}$$

where $K_{\alpha,m} = \int_0^\infty t^{-\alpha-1}(1-e^{-t})^m\, dt$.

But this assertion is only correct when $K_{\alpha,m}$ is nonzero. It is obvious that it holds when α is a real number, but it is not true if α is complex; for example, $K_{\alpha,2}=0$ for $\alpha = 1\pm i(2n\pi/\log 2)$ $(n\ge 1)$.

The aim of this paper is to study the behaviour of the function $K_{\alpha,m}$; thus, we will know if it is correct to apply (1) to obtain the fractional power A^α.

We prove that the set $\{\operatorname{Re}\alpha : K_{\alpha,3}=0\}$ is dense in $[0,1]\cup[2,3]$ and likewise the closure of $\{\operatorname{Re}\alpha : K_{\alpha,3}\ne 0\}$ is the interval $[0,3]$. On the other hand, we obtain general representations for $K_{\alpha,m}$, deducing from them two interesting properties:

1. Given $\alpha\in\mathbb{C}_+$, there exists an integer $m>\operatorname{Re}\alpha$ such that $K_{\alpha,m}\ne 0$.
2. If $0<\operatorname{Re}\alpha<m$ and $|\operatorname{Im}\alpha|<\pi/\log m$, then $K_{\alpha,m}\ne 0$.

If $0<\operatorname{Re}\alpha<1$ and $m\ge 1$, we obtain, after simple calculations, the known formula

$$K_{\alpha,m} = \sum_{1\le k\le m} \binom{m}{k}(-1)^k k^\alpha \Gamma(-\alpha), \tag{2}$$

where Γ is the Euler function (see [4, pp. 103–104]).

Analytically, identity (2) is valid on the strip $B=\{\alpha\in\mathbb{C} : 0<\operatorname{Re}\alpha< m\}$. The symbol $H_{\alpha,m}$ will denote the value $\sum_{1\le k\le m}\binom{m}{k}(-1)^{k+1}k^\alpha$. As $K_{\alpha,m} = -\Gamma(-\alpha)H_{\alpha,m}$, we have $H_{\alpha,m}=0$ for $\alpha = 1,\ldots,m-1$. These identities can also be easily obtained by observing that

$$H_{p,m} = (-1)^{p-1}\frac{d^p}{dt^p}\Big|_{t=0}(1-e^{-t})^m = 0 \qquad (p=1,\ldots,m-1). \tag{3}$$

2. Zeros of $K_{\alpha,1}$, $K_{\alpha,2}$ and $K_{\alpha,3}$. By integrating by parts it is evident that $K_{\alpha,1} = -\Gamma(-\alpha)$ and therefore $K_{\alpha,1}$ is nonzero. On the other hand, it is clear that the zeros of $K_{\alpha,2}$ are the complex numbers $1\pm i(2n\pi/\log 2)$ $(n\ge 1)$.

The behaviour of the zeros of $K_{\alpha,3}$ is completely different, as the following proposition shows.

PROPOSITION 2.1. *The function $K_{\alpha,3}$ satisfies*

$$\overline{\{\operatorname{Re}\alpha : K_{\alpha,3}=0\}} = [0,1]\cup[2,3].$$

Proof. The proof is based on the fact that for any couple of complex numbers z_1,z_2 we have the simple equivalence

$$1+z_1+z_2=0 \Leftrightarrow |1+z_1| = |z_2|, |1+z_2| = |z_1| \text{ and } \operatorname{Im}(z_1+z_2)=0.$$

On the other hand, we will also use the known fact that if c is an irrational number, then the set $\{pc+q : p,q\in 2\mathbb{Z}\}$ is dense in $\mathbb{R}$.

For simplicity, we split the proof into two parts:

Part 1. Let α be such that $\operatorname{Re}\alpha\in[0,3]$. We define

$$b=\operatorname{Re}\alpha-1,\quad \tau=\operatorname{Im}\beta\in\mathbb{R},\quad z_1=-2\cdot 2^b\cdot 2^{i\tau} \quad\text{and}\quad z_2 = 3^b\cdot 3^{i\tau}.$$

Firstly, let us prove that $H_{\alpha,3}=0$ if and only if $b\in[-1,0]\cup[1,2]$ and moreover there exist integers p,q with the same parity such that

$$\pi(p\log 3 - q\log 2) = (\log 2)\arccos\left((-1)^q\frac{4\cdot 4^b - 9^b - 1}{2\cdot 3^b}\right) - (\log 3)\arccos\left((-1)^p\frac{4\cdot 4^b-9^b+1}{4\cdot 2^b}\right) \tag{4}$$

(where arccos denotes the main determination of the arc cosine function in the interval $[0,\pi]$).

If $H_{\alpha,3}=0$, then $1+z_1+z_2=0$ and thus $|1+z_1|=|z_2|$ and $|1+z_2|=|z_1|$, that is,

$$9^b = 1+4\cdot 4^b - 4\cdot 2^b\cos(\tau\log 2) \quad\text{and}\quad 4\cdot 4^b = 1+9^b+2\cdot 3^b\cos(\tau\log 3),$$

whence

$$-1\le \frac{4\cdot 4^b - 9^b+1}{4\cdot 2^b}\le 1 \quad\text{and}\quad -1\le\frac{4\cdot 4^b-9^b-1}{2\cdot 3^b}\le 1.$$

It is easy to see, by rearranging the above inequalities in order to form the squares $(2\cdot 2^b\pm 1)^2$ and $(3^b\pm 1)^2$, that they are equivalent to the relation

$$0\le 3^b-2\cdot 2^b+1 = \frac{1}{3}H_{\operatorname{Re}\alpha,3}\le 2.$$

Hence, by elementary considerations (or, more directly, through formula (8) that we will prove below in Proposition 3.2) we can conclude that $b\in[-1,0]\cup[1,2]$.

If p, q are integers such that $\tau\log 2 - p\pi \in [0,\pi]$ and $\tau\log 3 - q\pi \in [0,\pi]$, then we have

$$\tau\log 2 - p\pi = \arccos\left((-1)^p\frac{4\cdot 4^b - 9^b + 1}{4\cdot 2^b}\right),$$
$$\tau\log 3 - q\pi = \arccos\left((-1)^q\frac{4\cdot 4^b - 9^b - 1}{2\cdot 3^b}\right).$$

By eliminating τ from the last two expressions, we get (4).

Now, let us see that p and q have the same parity. Indeed, the equality $\operatorname{Im}(z_1+z_2)=0$ means that

$$2\cdot 2^b\sin(\tau\log 2) = 3^b\sin(\tau\log 3),$$

that is,

$$(5)\qquad 2\cdot 2^b(-1)^p\left[1-\left(\frac{4\cdot 4^b-9^b+1}{4\cdot 2^b}\right)^2\right]^{1/2} = 3^b(-1)^q\left[1-\left(\frac{4\cdot 4^b-9^b-1}{2\cdot 3^b}\right)^2\right]^{1/2}.$$

And, since

$$2\cdot 2^b\left[1-\left(\frac{4\cdot 4^b-9^b+1}{4\cdot 2^b}\right)^2\right]^{1/2} = 3^b\left[1-\left(\frac{4\cdot 4^b-9^b-1}{2\cdot 3^b}\right)^2\right]^{1/2}$$

for any $b\in[-1,2]$ (where both members are zero only if $b=0$ or $b=1$ and these values of b are not possible since $H_{1+i\tau,3}\neq 0$ and $H_{2+i\tau,3}\neq 0$ for any $\tau\in\mathbb{R}$), it follows from (6) that p and q have the same parity.

Conversely, if there exist $b\in[-1,0]\cup[1,2]$ and two integers p,q with the same parity such that (4) holds, then by considering

$$\tau = \frac{1}{\log 2}\left[\arccos\left((-1)^p\frac{4\cdot 4^b-9^b+1}{4\cdot 2^b}\right)+p\pi\right] = \frac{1}{\log 3}\left[\arccos\left((-1)^q\frac{4\cdot 4^b-9^b-1}{2\cdot 3^b}\right)+q\pi\right],$$

it is obvious that $|1+z_1|=|z_2|$ and $|1+z_2|=|z_1|$ and direct calculations show that $\operatorname{Im}(z_1+z_2)=0$.

Part 2. Let us prove that $\overline{\{\operatorname{Re}\alpha : K_{\alpha,3}=0\}}=[0,1]\cup[2,3]$.

Given $b_0\in]-1,0[\cup]1,2[$ and an open interval I around b_0, the image of I under the function

$$f(b) = (\log 2)\arccos\frac{4\cdot 4^b-9^b-1}{2\cdot 3^b} - (\log 3)\arccos\frac{4\cdot 4^b-9^b+1}{4\cdot 2^b}$$

has a nonempty interior, since the analytic function f is not constant. On the other hand, the set $\{\pi(p\log 3 - q\log 2) : p\in 2\mathbb{Z},\ q\in 2\mathbb{Z}\}$ is dense in $\mathbb{R}$.

So, there will exist even integers p_1, q_1 such that $f(b_1)=\pi(p_1\log 3 - q_1\log 2)$ for some $b_1\in I$. From this and part 1, the result follows. ∎

The behaviour of the zeros of $K_{\alpha,1}, K_{\alpha,2}$ and $K_{\alpha,3}$ shows that the zeros of $K_{\alpha,m}$ are not predictable, which seriously limits the possibilities for applying formula (1).

3. Representations of $K_{\alpha,m}$ and consequences. A first interpretation of $H_{\alpha,m}$ is suggested by relation (3) between $H_{p,m}$ and the derivatives of $(1-e^{-t})^m$. In fact, for any noninteger $\alpha\in B$, the value of $-H_{\alpha,m}$ equals the value at $t=0$ of Weyl's fractional integral of the $-\alpha$th order:

$$\frac{1}{\Gamma(-\alpha)}\int_t^\infty (s-t)^{-\alpha-1}(1-e^{-s})^m\,ds,$$

associated with the function $(1-e^{-t})^m$. That is, in the language of fractional differential calculus, $H_{\alpha,m}$ would be the value of Weyl's fractional derivative of the αth order of the function $(1-e^{-t})^m$ at $t=0$.

Although $K_{\alpha,m}$ could be zero, we can prove, in the following proposition, that there always exists an integer $m>\operatorname{Re}\alpha$ such that $K_{\alpha,m}\neq 0$.

PROPOSITION 3.1. *Let m and n be integers such that $n>m>\operatorname{Re}\alpha$. Then*

$$(6)\qquad K_{\alpha,m}-K_{\alpha,n} = (\alpha+1)\sum_{m+1\le p\le n}\frac{1}{p}K_{\alpha+1,p}.$$

Therefore, the series $\sum_{p\ge m+1}(1/p)K_{\alpha+1,p}$ is convergent and

$$(7)\qquad K_{\alpha,m} = (\alpha+1)\sum_{p\ge m+1}\frac{1}{p}K_{\alpha+1,p}.$$

Accordingly, given any $\alpha\in\mathbb{C}_+$ there exists an integer $m>\operatorname{Re}\alpha$ such that $K_{\alpha,m}\neq 0$.

Proof. By writing

$$(1-e^{-t})^m-(1-e^{-t})^n = (1-e^{-t})^m e^{-t}\sum_{0\le q\le n-m-1}(1-e^{-t})^q$$

and integrating by parts, (6) is obtained.

As $n\to\infty$, $K_{\alpha,n}\to 0$ by the dominated convergence theorem. Taking limits in (6) as $n\to\infty$ we get (7).

The last assertion of the statement is shown by induction. For $0<\operatorname{Re}\alpha<1$, we know that $K_{\alpha,1}\neq 0$, and if $\operatorname{Re}\alpha=1$, then, according to Proposition 2.1, $K_{\alpha,3}\neq 0$. Thus, the statement is true for $0<\operatorname{Re}\alpha\le 1$. Assuming that it is valid for all α with $0<\operatorname{Re}\alpha\le h$, if β is a complex number such that $h < \operatorname{Re}\beta \le h+1$, then, by the induction hypothesis and (7), there exists an integer $m > \operatorname{Re}\beta - 1$ such that

$$K_{\beta-1,m} = \beta \sum_{p \ge m+1} \frac{1}{p} K_{\beta,p} \ne 0,$$

and therefore some of the summands must be nonzero, as we wanted to prove. ∎

The following representation of $K_{\alpha,m}$ provides a horizontal strip on the real axis where $K_{\alpha,m}$ has no zeros.

PROPOSITION 3.2. *The following identity holds:*

$$(8) \quad H_{\alpha,m} = m \sum_{0\le k\le m-1} \binom{m-1}{k} (-1)^k (k+1)^{\alpha-1}$$
$$= (-1)^{m-1} m(\alpha-1)\dots(\alpha-m+1) \int_{[0,1]^{m-1}} (1+x_1+\dots+x_{m-1})^{\alpha-m}\,dx_1\dots dx_{m-1}.$$

Therefore, $K_{\alpha,m}$ has no zeros if $|\operatorname{Im}\alpha| \le \pi/\log m$.

Proof. The first identity is evident. The second is a particular case of the general formula

$$(9) \quad \sum_{0\le k\le n} \binom{n}{k} (-1)^k (k+a)^\beta$$
$$= (-1)^n \beta(\beta-1)\dots(\beta-n+1) \int_{[0,1]^n} (a+x_1+\dots+x_n)^{\beta-n}\,dx_1\dots dx_n,$$

where $\beta \in \mathbb{C}$, $a > 0$, and n is a positive integer. This formula can be proved by induction on n. The case $n = 1$ is an immediate consequence of Barrow's formula. If we assume that the formula is true for any integer k, $1 \le k \le n$, then from the identity

$$\sum_{0\le k\le n} \binom{n}{k} (-1)^k (k+a)^\beta - \sum_{0\le k\le n+1} \binom{n+1}{k} (-1)^k (k+a)^\beta = \sum_{0\le k\le n} \binom{n}{k} (-1)^k (k+a+1)^\beta,$$

we conclude that (9) is also valid for $n+1$.

Finally, if α is noninteger and $|\operatorname{Im}\alpha| \le \pi/\log m$, then $\operatorname{Im}[(1+x_1+\dots+x_{m-1})^{\alpha-m}]$ has a constant sign for any $(x_1,\dots,x_{m-1}) \in \,]0,1[^{m-1}$ and thus, by (8), we obtain $H_{\alpha,m} \ne 0$, and so $K_{\alpha,m} \ne 0$. ∎

$H_{\alpha,m}$ can also be represented by the value of a certain determinant, which is a sort of "exponential version" of Vandermonde's determinant. This new representation itself is very interesting, although we were not able to obtain information on the zeros of $H_{\alpha,m}$ from it.

PROPOSITION 3.3. *We have the following identity:*

$$H_{\alpha,m} = \frac{(-1)^{m-1}}{(m-1)!(m-2)!\dots 2!} \det \begin{bmatrix} 1 & 2 & 3 & \dots & m \\ 1 & 2^2 & 3^2 & \dots & m^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{m-1} & 3^{m-1} & \dots & m^{m-1} \\ 1 & 2^\alpha & 3^\alpha & \dots & m^\alpha \end{bmatrix}.$$

Proof. Analytically, it is sufficient to prove the identity for those values of α such that the above matrix is invertible. We will call this matrix $T_m(\alpha)$ and let $v \in \mathbb{C}^m$ be the vector with components $v_k = (-1)^{k+1}\binom{m}{k}$, $k = 1,\dots,m$. The identities $H_{p,m} = 0$ for $p = 1,\dots,m-1$, along with the very definition of $H_{\alpha,m}$, can be expressed as $T_m(\alpha)v = H_{\alpha,m}e_m$, where e_m is the m-tuple with a 1 in the mth place and zeros elsewhere. Then $v = H_{\alpha,m}[T_m(\alpha)]^{-1}e_m$ and by equating the mth component of the two members, we obtain

$$(-1)^{m+1} = H_{\alpha,m} \frac{\det T_{m-1}(m-1)}{\det T_m(\alpha)},$$

which proves the statement, taking into account that $T_{m-1}(m-1)$ is immediately reduced to a Vandermonde determinant. ∎

Acknowledgments. The authors are very grateful to Prof. J. Kisyński for his interesting and constructive suggestions.

References

[1] A. V. Balakrishnan, *Fractional powers of closed operators and semigroups generated by them*, Pacific J. Math. 10 (1960), 419–437.

[2] H. Berens, P. L. Butzer and U. Westphal, *Representation of fractional powers of infinitesimal generators of semigroups*, Bull. Amer. Math. Soc. 74 (1968), 191–196.

[3] H. Komatsu, *Fractional powers of operators*, Pacific J. Math. 19 (1966), 285–346.

[4] —, *Fractional powers of operators, II. Interpolation spaces*, ibid. 21 (1967), 89–111.

[5] —, *Fractional powers of operators, III. Negative powers*, J. Math. Soc. Japan 21 (1969), 205–220.

[6] O. E. Landford and W. Robinson, *Fractional powers of generators of equicontinuous semigroups and fractional derivatives*, J. Austral. Math. Soc. Ser. A 46 (1989), 473–504.

[7] J. L. Lions et J. Peetre, *Sur une classe d'espaces d'interpolation*, Publ. Math. Inst. Hautes Etudes Sci. 19 (1964), 5–68.

[8] C. Martinez and M. Sanz, *Fractional powers of non-densely defined operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 18 (1991), 443–454.

[9] C. Martinez, M. Sanz and L. Marco, *Fractional powers of operators*, J. Math. Soc. Japan 40 (1988), 331–347.

[10] J. D. Stafney, *Integral representations of fractional powers of infinitesimal generators*, Illinois J. Math. 20 (1976), 124–133.
[11] U. Westphal, *An approach to fractional powers of operators via fractional differences*, Proc. London Math. Soc. (3) 29 (1974), 557–576.

Departament de Matemàtica Aplicada
Universitat de València
46100 Burjassot, València, Spain
E-mail: celso.martinez@uv.es
miguel.angel.sanz@uv.es

Received October 25, 1995 (3552)
Revised version December 18, 1995

STUDIA MATHEMATICA 119 (3) (1996)

On uniqueness of G-measures and g-measures

by

AI HUA FAN (Cergy-Pontoise)

Abstract. We give a simple proof of the sufficiency of a log-lipschitzian condition for the uniqueness of G-measures and g-measures which were studied by G. Brown, A. H. Dooley and M. Keane. In the opposite direction, we show that the lipschitzian condition together with positivity is not sufficient. In the special case where the defining function depends only upon two coordinates, we find a necessary and sufficient condition. The special case of Riesz products is discussed and the Hausdorff dimension of Riesz products is calculated.

1. Introduction and main statements. The G-measures were constructed by G. Brown and A. H. Dooley ([2]) and they generalized to some extent the g-measures constructed previously by M. Keane ([8]). Typical G-measures are the Riesz products defined by

$$\mu = \prod_{n=1}^{\infty}(1 + r_n \cos 2\pi m_1 \dots m_n x)$$

($-1 \le r_n \le 1, m_n \ge 2$ integers) (see [5]). The special case where $r_n = r$ and $m_n = m$ provides typical examples of g-measures. For these two constructions, a major question is to know when we have a unique G-measure or g-measure. This is the subject of the present work.

Here are the definitions of G-measures and g-measures, and the results that will be proved in the sequel.

Let $\{X_j\}_{j\ge1}$ be a sequence of finite abelian groups of orders $\{m_j\}_{j\ge1}$. We shall denote by X their infinite product $\prod_{j=1}^{\infty} X_j$ and by Γ their infinite direct sum $\bigoplus_{j=1}^{\infty} X_j$. Then X is a totally disconnected compact metric group, and Γ is viewed as a countable subgroup of X that acts on X. More precisely, for $\gamma \in \Gamma$ and $x \in X$, the action is $\gamma x = \gamma\cdot x = (\gamma_1+x_1, \gamma_2+x_2, \dots)$ (recall that $\gamma_j = 0$ for j sufficiently large). For $n \ge 1$, we shall denote by Γ_n the finite product $\prod_{j=1}^{n} X_j$, which can be viewed as a subgroup of Γ. For a

1991 *Mathematics Subject Classification*: 28A35, 28D05, 43A05.
Key words and phrases: G-measures, g-measures, ergodic measures, Riesz products, quasi-invariance, dimension of measures.