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Köthe spaces modeled on spaces of C^∞ functions

by

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Abstract. The isomorphic classification problem for the Köthe models of some C^∞ function spaces is considered. By making use of some interpolative neighborhoods which are related to the linear topological invariant $\mathcal{D}_\varphi$ and other invariants related to the "quantity" characteristics of the space, a necessary condition for the isomorphism of two such spaces is proved. As applications, it is shown that some pairs of spaces which have the same interpolation property $\mathcal{D}_\varphi$ are not isomorphic.

We consider the isomorphic classification problem for the Köthe spaces which are modeled on the space of C^∞ functions on a domain with cusp which are flat at the cusp point (studied by Kondakov–Zahariuta [9]) and the spaces of infinitely differentiable Whitney functions on special compact sets in $\mathbb{R}$ and vanishing at the cusp-like limit point with all their derivatives (studied by Goncharov–Zahariuta [7]). In these papers the authors have shown that in some cases the spaces have a basis and they are isomorphic to the Köthe space $K(a)$ where

$$a_{n_1,n_2,k,p} = n_1^p n_2^p k^p r_k^{-\min(n_2,p)}, \quad r_k \to 0,$$

and

$$a_{n,k,p} = n^p h_k^{-p} d_k^{-\min(n,p)}, \quad h_k \to 0,\ d_k \to 0,$$

respectively. It was shown earlier (by Tidten [15], see also Goncharov–Zahariuta [6]) that these spaces depend on the thinness of the cusp in topological sense, namely, among them there is a continuum of pairwise non-isomorphic spaces.

In the present paper, we consider a generalization of these cases, namely we consider Köthe spaces of the form $K(a_{i,p}) = H(\kappa,\gamma,a)$ where

$$a_{i,p} = e^{(p+\gamma_i \min(\kappa(i),p))a_i}, \quad i,p \in \mathbb{N} = \{1,2,\ldots\},$$

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$$1 \le a_i \nearrow \infty, \quad \gamma = (\gamma_i), \quad \gamma_i \ge 1, \quad \kappa : \mathbb{N} \to \mathbb{N}.$$

We assume that the topology is defined as usual by the sequence of norms $(\|\cdot\|_p)$ where for $x = (\xi_i)$,

$$\|x\|_p = \sum_{i=1}^{\infty} |\xi_i| a_{i,p}$$

and

$$U_p = \{x \in H(\kappa,\gamma,a) : \|x\|_p \le 1\}$$

is the corresponding unit ball. Observe that $a_{i,p+1} \ge e a_{i,p}$ and also that $H(\kappa,\gamma,a)$ is a Schwartz space and if $(\ln i/i)$ is bounded, $H(\kappa,\gamma,a)$ is nuclear.

If $\sup_i \kappa(i) = K < \infty$, then $a_{i,p} = e^{\gamma_i \kappa(i) a_i} e^{p a_i}$ for $p \ge K$, so $H(\kappa,\gamma,a)$ is diagonally isomorphic to the infinite type power series space $E_\infty(a) = \Lambda_\infty(a) = K(e^{p a_i})$. If $\lim_{i\to\infty} \kappa(i) = \infty$, then $a_{i,p} \approx e^{p(1+\gamma_i)a_i}$, and the space $H(\kappa,\gamma,a)$ coincides with $E_\infty((\gamma_i a_i))$ as sets and the identity is an isomorphism. So the interesting case occurs when $\kappa^{-1}(n)$ is an infinite set for infinitely many n. If (γ_i) is bounded, say by Γ, then

$$e^{p a_i} \le a_{i,p} = e^{(p+\gamma_i \min(\kappa(i),p))a_i} \le e^{(p+\Gamma p)a_i}$$

so the space $H(\kappa,\gamma,a)$ coincides with $E_\infty(a)$ as sets and the identity is an isomorphism. Thus in the rest of the paper, we shall assume that $\limsup \gamma_i = \infty$.

In an arbitrary sequence space E, we denote the coordinate basis by $e = (e_i)$ where $e_i = (0,\ldots,0,1,0,\ldots)$, 1 being in the ith place. If $b = (b_i)$ is any sequence of nonnegative real numbers, we write

$$B^e(b) = B^e(b_i) = \Big\{x = (\xi_i) \in E : \sum_{i=1}^{\infty} |\xi_i| b_i \le 1\Big\}.$$

In particular, if $E = K(a_{i,p})$ is any Köthe space then $U_p = B^e(a_{i,p})$.

For two functions $f, g : \mathbb{R} \to \mathbb{R}$, $f(t) \prec g(t)$ means

$$\forall k > 0\ \exists t_0 : \quad t \ge t_0 \Rightarrow (f(t^k))^k \le g(t)$$

and $f(t) \precsim g(t)$ means

$$\exists k > 0\ \exists t_0 : \quad t \ge t_0 \Rightarrow f(t) \le g(t) + k.$$

If A is an arbitrary nonempty subset of a locally convex space, we denote the absolutely convex hull of A by $\Gamma(A)$ and the closed absolutely convex hull of A by $\overline{\Gamma}(A)$.

Problem. Give an isomorphic classification of spaces $H(\kappa,\gamma,a)$, that is, find necessary and sufficient conditions for $H(\kappa,\gamma,a) \cong H(\widetilde{\kappa},\widetilde{\gamma},\widetilde{a})$ (isomorphic).

In this paper by using appropriate linear topological invariants we give a necessary condition for isomorphism of these spaces. Then we give examples of spaces having the same interpolation property (i.e. condition $\mathcal{D}_\varphi$) which can be distinguished by this method.

Linear topological invariants. Linear topological invariants (such as approximative and diametral dimensions) as a tool for isomorphic classification of nonnormed linear topological spaces appeared in investigations of Pełczyński [14], Kolmogorov [8], Bessaga, Pełczyński and Rolewicz [2], Mityagin [10] *et al.*; they were also initiated by Gelfand [4]. These invariants can be called "quantity" invariants, because they are based on consideration of some measure characteristics such as entropy and diameters. Here we use the simplest characteristic of such kind. For a given linear space E and absolutely convex sets V and W in E, we consider

$$\beta_E(V,W) = \beta(V,W) = \sup\{\dim L : L \in \mathcal{F}, L \cap W \subset V\}$$

where $\mathcal{F}$ denotes the set of all finite-dimensional subspaces of E spanned on elements of V. Trivially, we have

$$(1) \qquad V_1 \subset V_2, W_1 \supset W_2 \Rightarrow \beta(V_1,W_1) \le \beta(V_2,W_2).$$

If T is a linear isomorphism, then clearly $\beta(T(V),T(W)) = \beta(V,W)$. Thus in the case of Fréchet spaces, the above-mentioned classical invariants can be described by the following fact.

Proposition 1. *Let X and Y be Fréchet spaces, and (U_p) and (V_p) be fundamental systems of zero neighborhoods in X and Y respectively. If $X \cong Y$, i.e. X and Y are isomorphic, then* $\forall p\ \exists p'\ \forall q'\ \exists q\ \exists C > 0\ \forall t > 0$:

$$\beta(tU_q, U_p) \le \beta\Big(tCV_{q'}, \frac{1}{C}V_{p'}\Big) \quad \textit{and} \quad \beta(tV_q, V_p) \le \beta\Big(tCU_{q'}, \frac{1}{C}U_{p'}\Big).$$

To get a new way for distinguishing nonisomorphic spaces with very slight differences in their structures (like those that will be considered here), following the ideas suggested in [19] and [20], we combine some "quantity" invariants with appropriate "quality" invariants. What we call a "quality" invariant is an invariant which appears if a certain general property of linear toplogical spaces is considered (e.g. montelity, nuclearity, quasinormability, etc.) or any other invariant property which is defined by an interpolation relation between seminorms. This way, one can get (see, for example, [21]) invariant characteristics of Köthe spaces considered in [17] and [18], which appeared by the influence of Mityagin's results [11], [12] on invariant characteristics of (non-Montel) power series spaces (see also [13] for more detailed consideration of these invariants).

If E is a sequence space with coordinate basis $e = (e_i)$, and $a = (a_i)$, $b = (b_i)$ are two sequences of nonnegative scalars, and $t, \tau > 0$, then

$$\beta(tB^e(b), \tau B^e(a)) = |\{i : b_i/a_i \le t/\tau\}| \tag{2}$$

where for a set S, $|S|$ denotes the cardinality of S if S is a finite set and $+\infty$ if S is an infinite set (see for example [3] and [21]).

It is also easy to see that

$$B^e(\max\{a_i, b_i\}) \subset B^e(a) \cap B^e(b) \subset 2B^e(\max\{a_i, b_i\}) = B^e(\tfrac{1}{2}\max\{a_i, b_i\}).$$

So it follows that

$$\begin{aligned} |\{i : a_i/c_i \le 1,\ b_i/c_i \le 1\}| &\le \beta(B^e(a) \cap B^e(b), B^e(c)) \\ &\le |\{i : a_i/c_i \le 2,\ b_i/c_i \le 2\}| \end{aligned} \tag{3}$$

and of course dropping any one of the two inequalities appearing in the right hand side expression possibly increases the corresponding cardinality.

Now we consider an appropriate "quality" invariant, defined by the interpolation property $\mathcal{D}_\varphi$. This property was considered by Vogt [16] and Tidten [15] and called DN_φ by them (see also $D_3(f)$ in the work of Apiola [1] as well as works of Goncharov and Zahariuta [5], [20] and [6]).

Let φ be a continuous, increasing function such that $\varphi(t) \ge t > 0$ for $t > 0$. A Fréchet space $(X, \|\cdot\|_p)$ is said to have *property* $\mathcal{D}_\varphi$ (written $X \in \mathcal{D}_\varphi$) if

$$\exists p\ \forall q\ \exists r\ \exists C > 0: \quad \|x\|_q \le \varphi(t)\|x\|_p + \frac{C}{t}\|x\|_r \quad \forall t > 0,\ \forall x \in X.$$

It is clear that property $\mathcal{D}_\varphi$ is a linear topological invariant, that is, if $X \cong Y$, i.e. X and Y are isomorphic Fréchet spaces, then $X \in \mathcal{D}_\varphi$ if and only if $Y \in \mathcal{D}_\varphi$.

The following proposition was proved in [6].

PROPOSITION 2. *Let $X = K(a_{i,p})$ be a Schwartz Köthe space. Then the following are equivalent*:

(i) *X has property $\mathcal{D}_\varphi$.*

(ii) $\exists p\ \forall q\ \exists r\ \exists C > 0: \quad a_{i,q}/a_{i,p} \le \varphi(Ca_{i,r}/a_{i,q})$.

Proof. (i)$\Rightarrow$(ii). By $\mathcal{D}_\varphi$ we have p. Given q we find r and C corresponding to $q+1$. Thus

$$\|x\|_{q+1} \le \varphi(t)\|x\|_p + \frac{C}{t}\|x\|_r \quad \forall t > 0,\ \forall x \in X.$$

Let $x = e_i$ and $t = 2Ca_{i,r}/a_{i,q+1}$. Then $\frac{1}{2}a_{i,q+1} \le \varphi(2Ca_{i,r}/a_{i,q+1})a_{i,p}$. Since $a_{i,q} \le \frac{1}{2}a_{i,q+1}$ for large i, we obtain

$$\frac{a_{i,q}}{a_{i,p}} \le \varphi\left(\frac{2Ca_{i,r}}{a_{i,q}}\right), \quad i \ge i_0.$$

By enlarging C, we can have the last inequality for all i.

(ii)$\Rightarrow$(i). Let p, q, r, C be as in (ii). Given $t > 0$, let

$$\mathbb{N}_1 = \{i \in \mathbb{N} : Ca_{i,r}/a_{i,q} \le t\}, \quad \mathbb{N}_2 = \{i \in \mathbb{N} : Ca_{i,r}/a_{i,q} > t\}.$$

Given $x = (\xi_i) \in X$, we have

$$\begin{aligned} \|x\|_q &= \sum_i |\xi_i| a_{i,q} = \sum_{i \in \mathbb{N}_1} |\xi_i| a_{i,q} + \sum_{i \in \mathbb{N}_2} |\xi_i| a_{i,q} \\ &\le \sum_{i \in \mathbb{N}_1} |\xi_i| a_{i,p}\varphi\left(C\frac{a_{i,r}}{a_{i,q}}\right) + \sum_{i \in \mathbb{N}_2} |\xi_i| \frac{C}{t} a_{i,r} \le \varphi(t)\|x\|_p + \frac{C}{t}\|x\|_r. \ \blacksquare \end{aligned}$$

Next we show that the space $H(\kappa, \gamma, a)$ has property $\mathcal{D}_\varphi$ for some φ. Let

$$N(t) = \sup\{\gamma_i : e^{a_i} \le t\}.$$

Since $\limsup \gamma_i = \infty$, we have $N(t) \nearrow \infty$ as $t \nearrow \infty$. We also have $\gamma_i \le N(e^{a_i})$. Let $\varphi(t) = t^{N(t)}$.

PROPOSITION 3. *$H(\kappa, \gamma, a)$ has property $\mathcal{D}_\varphi$.*

Proof. Let $p < q < r$ and $2(q-p) \le r - q$. We would like to have

$$\frac{a_{i,q}}{a_{i,p}} \le \varphi\left(\frac{a_{i,r}}{a_{i,q}}\right), \quad i \text{ large},$$

which is equivalent to

$$e^{(q-p+\gamma_i(\min(\kappa(i),q)-\min(\kappa(i),p)))a_i} \le e^{N(\ldots)(r-q+\gamma_i(\min(\kappa(i),r)-\min(\kappa(i),q)))a_i}.$$

Here $(\ldots)$ inside $N(\ldots)$ is $a_{i,r}/a_{i,q} = e^{(r-q+\ldots)a_i} \ge e^{a_i}$, so $N(\ldots) \ge \gamma_i \ge 1$. The above inequality takes the following forms:

- For $\kappa(i) \le p < q < r$:

$$q - p \le N(\ldots)(r-q),$$

which holds trivially.

- For $p < \kappa(i) \le q < r$:

$$q - p + \gamma_i(\kappa(i) - p) \le N(e^{(r-q)a_i})(r-q).$$

Indeed, now

$$\begin{aligned} q - p + \gamma_i(\kappa(i) - p) &\le q - p + \gamma_i(q-p) \le (q-p)2\gamma_i \\ &\le (r-q)N(e^{a_i}) \le (r-q)N(e^{(r-q)a_i}). \end{aligned}$$

- For $p < q < \kappa(i) \le r$:

$$q - p + \gamma_i(q-p) \le N(\ldots)(r - q + \gamma_i(\kappa(i) - q)).$$

This holds since

$$(q-p)(1+\gamma_i) \le (q-p)2\gamma_i \le N(\ldots)(r-q) \le N(\ldots)(r - q + \gamma_i(\kappa(i) - q)).$$

Finally,

- For $p < q < r < \kappa(i)$:

$$q - p + \gamma_i(q-p) \le N(\ldots)(r - q + \gamma_i(r-q)),$$

which holds trivially. ∎

We note that this φ may not be the "best" φ in general. For example, if $\gamma_i \nearrow \infty$ when $\kappa(i) \le p_0$ and (γ_i) is bounded when $\kappa(i) > p_0$, the space $H(\kappa, \gamma, a)$ has $\mathcal{D}_1$ (or DN), i.e. $\mathcal{D}_\varphi$ with $\varphi(t) = t$, but the above consideration gives a bigger function.

Remark. Our proof shows that if $\nu \subset \mathbb{N}$ is a subsequence such that $\{\kappa(i) : i \in \nu\}$ is bounded or $\kappa(i) \to \infty$ as $i \to \infty$, $i \in \nu$, then the subspace of $H(\kappa, \gamma, a)$ spanned by $\{e_i : i \in \nu\}$ has DN. The actual role of $\mathcal{D}_\varphi$ appears when $p \le \kappa(i) \le p_i$ where (p_i) is a sequence of positive integers such that $p_i \to \infty$. Let Π^∞ denote the set of all such sequences $P = (p_i)$. Given $p \in \mathbb{N}$ and $P \in \Pi^\infty$, we define

$$N_{p,P}(t) = \sup\{\gamma_i : p \le \kappa(i) \le p_i,\ e^{a_i} \le t\} \quad \text{and} \quad \varphi_{p,P}(t) = t^{N_{p,P}(t)}.$$

It is clear that $H(\kappa, \gamma, a) \in \bigcap \mathcal{D}_{\varphi_{p,P}}$ where the intersection is taken over all $p \in \mathbb{N}$ and $P = (p_i) \in \Pi^\infty$. It is an interesting question whether there is a smallest φ_0, i.e. such that $\bigcap \mathcal{D}_{\varphi_{p,P}} = \mathcal{D}_{\varphi_0}$.

In general, this question seems to have a negative answer even in the following slightly weaker form: does there exist a function φ_0 such that the space $H(\kappa, \gamma, a)$ belongs to the class $\mathcal{D}_{\varphi_0}$, but to no class $\mathcal{D}_\varphi$ with $\varphi \prec \varphi_0$? Let us describe some sufficient conditions for the existence of a smallest class $\mathcal{D}_{\varphi_0}$ (in the latter sense).

Let $M(t) \nearrow \infty$ as $t \nearrow \infty$. Then $\varphi_0(t) := t^{M(t)} \succ t$. Let ν_0 be a subsequence of $\mathbb{N}$ and $\gamma_i = M(e^{a_i})$ for $i \in \nu_0$, and $\gamma_i \le M(e^{a_i})$ otherwise. If for each $p \in \mathbb{N}$ and $P \in \Pi^\infty$ the set $\{i \in \nu_0 : p \le \kappa(i) \le p_i\}$ is infinite, then $H(\kappa, \gamma, a)$ belongs to $\mathcal{D}_{\varphi_0}$ and does not belong to any $\mathcal{D}_\varphi$ with $\varphi \prec \varphi_0$, in particular it does not belong to the class DN. If any subsequence $\{a_i : p \le \kappa(i) \le q,\ i \in \nu_0\}$, $p < q$, is nonlacunary (that is, there exists a constant c such that each interval $[t, ct]$ contains at least one point of this subsequence), then, moreover, $\mathcal{D}_{\varphi_p, P} = \mathcal{D}_{\varphi_0}$ for all $p \in \mathbb{N}$ and $P = (p_i) \in \Pi^\infty$.

Given a function φ as in the definition of property $\mathcal{D}_\varphi$ and $u > 0$, we define

$$\Phi(u) = \inf_{t>0}(\varphi(t) + u/t).$$

PROPOSITION 4. *Let $A, B > 0$. Then*

(i) $\Phi(AB) \le A \Rightarrow \varphi(B) \le A$ *and*

(ii) $\varphi(B) \le A \Rightarrow \Phi(AB) \le 2A$.

Proof. (i) Assume $A < \varphi(B)$. Given $t > 0$,

$$\begin{aligned} B \le t &\Rightarrow A < \varphi(B) \le \varphi(t) \le \varphi(t) + AB/t, \\ B/2 \le t < B &\Rightarrow A < \varphi(B/2) + A \le \varphi(t) + AB/t, \\ 0 < t < B/2 &\Rightarrow A < 2A < AB/t \le \varphi(t) + AB/t. \end{aligned}$$

Thus $A < \inf_{t>0}(\varphi(t) + AB/t) = \Phi(AB)$.

(ii) Assume $2A < \Phi(AB)$. Then $2A < \varphi(t) + AB/t$ for all $t > 0$. Taking $t = B$, we get $2A < \varphi(B) + A$, which gives $A < \varphi(B)$. ∎

Next we consider a function $f(t) = t^{M(t)}$ where $M(t) \nearrow \infty$ as $t \nearrow \infty$. In $H(\kappa, \gamma, a)$, we want to consider the "maximal" subspace which does not have property $\mathcal{D}_f$. More precisely, we consider the reverse inequality

$$\frac{a_{i,q}}{a_{i,p}} \ge f\left(\frac{a_{i,r}}{a_{i,q}}\right) \quad \text{where} \quad 2(q-p) \le r - q \text{ and } 2(q-p) \le M(e^{r-q}).$$

It follows immediately that if $\kappa(i) \le p$ or $r < \kappa(i)$, then the above inequality cannot hold. For $p < q < \kappa(i) \le r$, the inequality takes the form

$$M(e^{(r-q+\gamma_i(\kappa(i)-q))a_i})(r - q + \gamma_i(\kappa(i) - q)) \le (q-p)(1+\gamma_i).$$

This cannot hold, since

$$\text{LHS} \ge M(e^{(r-q)a_i})(1+\gamma_i) \ge M(e^{r-q})(1+\gamma_i) \ge 2(q-p)\gamma_i.$$

For $p < \kappa(i) \le q < r$, the inequality takes the form

$$M(e^{(r-q)a_i})(r-q) \le q - p + \gamma_i(\kappa(i) - p).$$

So we obtain

$$p < \kappa(i) \le q < r \text{ and } (r-q)M(e^{(r-q)a_i}) \le \gamma_i \Rightarrow \frac{a_{i,q}}{a_{i,p}} \ge f\left(\frac{a_{i,r}}{a_{i,q}}\right). \tag{4}$$

Conversely,

$$\frac{a_{i,q}}{a_{i,p}} \ge f\left(\frac{a_{i,r}}{a_{i,q}}\right) \Rightarrow p < \kappa(i) \le q < r \text{ and } M(e^{(r-q)a_i}) \le 2(q-p)\gamma_i. \tag{5}$$

In the next proposition we consider some specially defined neighborhoods which are connected with property $\mathcal{D}_\varphi$.

PROPOSITION 5. *Let $K(a_{i,p})$ be an arbitrary Köthe space. Let $f : (0,\infty) \to (0,\infty)$ be an increasing continuous function and $F(u) = \inf_{t>0}(f(t) + u/t)$. For $p < r$ let*

$$V = \overline{\Gamma}\left(\bigcup_{t>0}\left(\frac{1}{f(t)}U_p \cap tU_r\right)\right), \qquad V' = B^e\left(a_{i,p}F\left(\frac{a_{i,r}}{a_{i,p}}\right)\right).$$

Then $V' \subset V \subset (2+\varepsilon)V'$ for any $\varepsilon > 0$.

Proof. Let $x=(\xi_i)\in \frac{1}{f(t)}U_p\cap tU_r$ for some $t>0$. Then

$$\sum_i |\xi_i| a_{i,p} F\left(\frac{a_{i,r}}{a_{i,p}}\right) \le \sum_i |\xi_i| a_{i,p}\left(f(t)+\frac{1}{t}\frac{a_{i,r}}{a_{i,p}}\right) = f(t)\|x\|_p + \frac{1}{t}\|x\|_r \le 2.$$

So $x\in 2V'$, implying

$$\widetilde{V} := \Gamma\left(\bigcup_{t>0}\left(\frac{1}{f(t)}U_p\cap tU_r\right)\right)\subset 2V'.$$

Since $\widetilde{V}\supset \frac{1}{f(1)}U_p\cap U_r$, it follows that $\widetilde{V}$, and hence V', is a zero neighborhood. So

$$V=\overline{\widetilde{V}}\subset \widetilde{V}+\varepsilon V'\subset 2V'+\varepsilon V'=(2+\varepsilon)V'.$$

Conversely, let $x=(\xi_i)\in V'$, i.e. let $\sum_i|\xi_i|a_{i,p}F(a_{i,r}/a_{i,p})\le 1$. Let $t_i = |\xi_i|a_{i,p}F(a_{i,r}/a_{i,p})$. Then $\sum_i |t_i|\le 1$. Find α_i such that $|\alpha_i|=1$ and $\alpha_i|\xi_i| = \xi_i$. Set

$$u_i=\frac{\alpha_i}{a_{i,p}F(a_{i,r}/a_{i,p})}e_i.$$

We shall show that $u_i\in \frac{1}{f(t)}U_p\cap tU_r$ for some $t>0$. Then it will follow that

$$\sum_i t_iu_i\in \overline{\Gamma}\left(\bigcup_{t>0}\left(\frac{1}{f(t)}U_p\cap tU_r\right)\right)=V.$$

But $\sum_i t_iu_i=x$. So we get $V'\subset V$. Now

$$\|u_i\|_p=\frac{1}{F(a_{i,r}/a_{i,p})},\qquad \|u_i\|_r=\frac{a_{i,r}}{a_{i,p}F(a_{i,r}/a_{i,p})},$$

i.e.

$$u_i\in \frac{1}{F(a_{i,r}/a_{i,p})}U_p\cap \frac{a_{i,r}}{a_{i,p}F(a_{i,r}/a_{i,p})}U_r.$$

Now

$$\exists t>0:\ u_i\in\frac{1}{f(t)}U_p\cap tU_r \Leftrightarrow \exists v>0:\ u_i\in \frac{1}{v}U_p\cap f^{-1}(v)U_r.$$

Let $v=F(a_{i,r}/a_{i,p})$. It suffices to show that

$$\frac{a_{i,r}}{a_{i,p}F(a_{i,r}/a_{i,p})}\le f^{-1}(v),\quad \text{i.e.}\quad \frac{F^{-1}(v)}{v}\le f^{-1}(v),\quad \text{i.e.}\quad v\le F(vf^{-1}(v)).$$

That is,

$$\forall t>0:\quad v\le f(t)+\frac{1}{t}vf^{-1}(v),\ \text{i.e. } vt\le tf(t)+vf^{-1}(v).$$

But by considering two cases

$$v\le f(t)\Rightarrow vt\le f(t)t\le tf(t)+vf^{-1}(v),$$
$$v> f(t)\Rightarrow f^{-1}(v)>t\Rightarrow vt<vf^{-1}(v)\le tf(t)+vf^{-1}(v),$$

we obtain the result. ∎

Now we consider $\beta(V,W)$ where

$$V=\overline{\Gamma}\left(\bigcup_{t>0}\left(\frac{1}{f(t)}U_p\cap tU_r\right)\right)\cap e^\tau U_r,\quad W=U_q,\quad p<q<r.$$

This consideration involves a quality invariant, that is, measuring

$$\overline{\Gamma}\left(\bigcup_{t>0}\left(\frac{1}{f(t)}U_p\cap tU_r\right)\right)$$

through U_q, and a quantity invariant, that is, measuring $e^\tau U_r$ through U_q so that we cut some part of the space.

By Proposition 5 and (1), we have

$$\beta(V,W)\ge \beta(B^e(a_{i,p}F(a_{i,r}/a_{i,p}))\cap e^\tau B^e(a_{i,r}),B^e(a_{i,q}))$$

(by (2) and the left hand side of (3) we continue as)

$$\ge |\{i: F(a_{i,r}/a_{i,p})\le a_{i,q}/a_{i,p},\ a_{i,r}/a_{i,q}\le e^\tau\}|$$

(by Proposition 4(ii) with $A=\frac12 a_{i,q}/a_{i,p}$, $B=2a_{i,r}/a_{i,q}$ we have)

$$\ge \left|\left\{i: f(2a_{i,r}/a_{i,q})\le \tfrac12 a_{i,q}/a_{i,p},\ a_{i,r}/a_{i,q}\le e^\tau\right\}\right|$$
$$\ge |\{i: f(a_{i,r}/a_{i,q-1})\le a_{i,q-1}/a_{i,p},\ a_{i,r}/a_{i,q-1}\le e^\tau\}|$$

(and by (4))

$$\ge |\{i: p<\kappa(i)\le q-1,\ (r-q+1)M(e^{(r-q+1)a_i})\le\gamma_i,\ e^{(r-q+1)a_i}\le e^\tau\}|.$$

Similarly, if $C>1$ is a constant, then

$$\begin{aligned}\beta\left(CV,\frac{1}{C}W\right) &\le \beta\left(C\left(3B^e\left(a_{i,p}F\left(\frac{a_{i,r}}{a_{i,p}}\right)\right)\cap e^\tau B^e(a_{i,r})\right),\frac{1}{C}B^e(a_{i,q})\right)\\
&\le \left|\left\{i: F\left(\frac{a_{i,r}}{a_{i,p}}\right)\le 6C^2\frac{a_{i,q}}{a_{i,p}},\ \frac{a_{i,r}}{a_{i,q}}\le 6C^2e^\tau\right\}\right|\\
&\le \left|\left\{i: f\left(\frac{1}{6C^2}\frac{a_{i,r}}{a_{i,q}}\right)\le 6C^2\frac{a_{i,q}}{a_{i,p}},\ \frac{a_{i,r}}{a_{i,q}}\le 6C^2e^\tau\right\}\right|\\
&\precsim \left|\left\{i: f\left(\frac{a_{i,r-1}}{a_{i,q}}\right)\le \frac{a_{i,q}}{a_{i,p}},\ \frac{a_{i,r-1}}{a_{i,q}}\le e^\tau\right\}\right|\\
&\le |\{i: p<\kappa(i)\le q,\ M(e^{(r-1-q)a_i})\le 2(q-p)\gamma_i,\\
&\qquad\qquad e^{(r-1-q)a_i}\le e^\tau\}|.\end{aligned}$$

Now assume that $H(\kappa,\gamma,a)$ is isomorphic to $H(\widetilde{\kappa},\widetilde{\gamma},\widetilde{a})$. We denote the neighhborhoods in these spaces by U_p and $\widetilde{U}_p$ respectively. Then

$$\forall\,\widetilde{p}\ \exists p\ \forall q\ \exists \widetilde{q}\ \forall \widetilde{r}\ \exists r:\quad \widetilde{U}_{\widetilde{p}}\succ T(U_p)\succ T(U_q)\succ \widetilde{U}_{\widetilde{q}}\succ \widetilde{U}_{\widetilde{r}}\succ T(U_r)$$

where $U \succ V$ means that $U \supset cV$ for some $c \in \mathbb{R}$. We let

$$V = \overline{\Gamma}\Big(\bigcup_{t>0}\Big(\frac{1}{f(t)}U_p \cap tU_r\Big) \cap e^\tau U_r\Big), \qquad W = U_q,$$

$$\widetilde{V} = \overline{\Gamma}\Big(\bigcup_{t>0}\Big(\frac{1}{f(t)}\widetilde{U}_{\widetilde{p}} \cap t\widetilde{U}_{\widetilde{r}}\Big) \cap e^\tau \widetilde{U}_{\widetilde{r}}\Big), \qquad \widetilde{W} = \widetilde{U}_{\widetilde{q}}.$$

Then $\widetilde{V} \succ T(V)$ and $T(W) \succ \widetilde{W}$, so for some constant $C \geq 1$,

$$\beta(V, W) = \beta(T(V), T(W)) \leq \beta\Big(C\widetilde{V}, \frac{1}{C}\widetilde{W}\Big).$$

Observe that the above argument is valid for all M such that $M(t) \nearrow \infty$ as $t \nearrow \infty$. Thus we have the following theorem.

THEOREM. *Let $M : (0, \infty) \to (0, \infty)$ be any function such that $M(t) \nearrow \infty$ as $t \nearrow \infty$. Assume $H(\kappa, \gamma, a)$ is isomorphic to $H(\widetilde{\kappa}, \widetilde{\gamma}, \widetilde{a})$. Then*

(i) $\forall \widetilde{p}\ \exists p\ \forall q\ \exists \widetilde{q}\ \forall \widetilde{r}\ \exists r$:

$$|\{i : p < \kappa(i) \leq q,\ (r-q)M(e^{(r-q)a_i}) \leq \gamma_i,\ e^{(r-q)a_i} \leq e^\tau\}|$$
$$\precsim |\{j : \widetilde{p} < \widetilde{\kappa}(j) \leq \widetilde{q},\ M(e^{(\widetilde{r}-\widetilde{q})\widetilde{a}_j}) \leq 4(\widetilde{q}-\widetilde{p})\widetilde{\gamma}_j,\ e^{(\widetilde{r}-\widetilde{q})\widetilde{a}_j} \leq e^\tau\}|,$$

(ii) $\forall p\ \exists \widetilde{p}\ \forall \widetilde{q}\ \exists q\ \forall r\ \exists \widetilde{r}$:

$$|\{i : \widetilde{p} < \widetilde{\kappa}(i) \leq \widetilde{q},\ (\widetilde{r}-\widetilde{q})M(e^{(\widetilde{r}-\widetilde{q})\widetilde{a}_i}) \leq \widetilde{\gamma}_i,\ e^{(\widetilde{r}-\widetilde{q})\widetilde{a}_i} \leq e^\tau\}|$$
$$\precsim |\{j : p < \kappa(j) \leq q,\ M(e^{(r-q)a_j}) \leq 4(q-p)\gamma_j,\ e^{(r-q)a_j} \leq e^\tau\}|.$$

Applications. Now we consider some examples of the above-mentioned spaces which are distinguished by the Theorem.

EXAMPLE 1. Let $a_i = \widetilde{a}_i = \ln i$ and $\kappa = \widetilde{\kappa}$. Let $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3$ be a decomposition of $\mathbb{N}$ into three disjoint, infinite subsets such that $\mathbb{N}_1$ is thinner than $\mathbb{N}_2$ in the following sense:

$$\lim_{n\to\infty} \frac{|\{i \in \mathbb{N}_1 : p < \kappa(i) \leq q,\ i \leq n\}|}{|\{i \in \mathbb{N}_1 \cup \mathbb{N}_2 : p < \kappa(i) \leq q,\ i \leq n^{1/m}\}|} = 0$$

for every $m > 0$ and $p < q$. Also assume that for all $p < q$ and $\widetilde{p} < \widetilde{q}$ we have

$$|\{i \in \mathbb{N}_1 \cup \mathbb{N}_2 : p < \kappa(i) \leq q,\ i \leq s^2\}| \succsim |\{i \in \mathbb{N}_1 \cup \mathbb{N}_2 : \widetilde{p} < \kappa(i) \leq \widetilde{q},\ i \leq s\}|$$

as $s \to \infty$. Let $M_3, M_2, M_1 : (0, \infty) \to (0, \infty)$ be three functions which increase to ∞ as $t \nearrow \infty$ and assume that $kM_3(t^k) \leq M_2(t)$ for every k asymptotically as $t \nearrow \infty$ and $M_2(t) \leq M_1(t)$. We define

$$\gamma_i = \begin{cases} M_1(e^{a_i}), & i \in \mathbb{N}_1, \\ M_2(e^{a_i}), & i \in \mathbb{N}_2, \\ M_3(e^{a_i}), & i \in \mathbb{N}_3, \end{cases} \quad \text{and} \quad \widetilde{\gamma}_i = \begin{cases} M_1(e^{a_i}), & i \in \mathbb{N}_1, \\ M_3(e^{a_i}), & i \in \mathbb{N}_2 \cup \mathbb{N}_3. \end{cases}$$

By our remark after Proposition 3 it follows that under some additional conditions, for each φ, $H(\kappa, \gamma, a) \in \mathcal{D}_\varphi$ if and only if $H(\kappa, \widetilde{\gamma}, a) \in \mathcal{D}_\varphi$. The spaces $H(\kappa, \gamma, a)$ and $H(\kappa, \widetilde{\gamma}, a)$ are not isomorphic. In fact, we assume $H(\kappa, \gamma, a) \cong H(\kappa, \widetilde{\gamma}, a)$ and apply the Theorem with $M = M_3$. The middle inequality

$$(r-q)M(e^{(r-q)a_i}) \leq \gamma_i$$

on the left hand side of (i) is true for all large enough $i \in \mathbb{N}_1 \cup \mathbb{N}_2$ and for at most finitely many $i \in \mathbb{N}_3$. The middle inequality

$$M(e^{(\widetilde{r}-\widetilde{q})\widetilde{a}_j}) \leq 2(\widetilde{q}-\widetilde{p})\widetilde{\gamma}_j$$

on the right hand side of (i) is true for all large enough $j \in \mathbb{N}_1$ and for at most finitely many $j \in \mathbb{N}_2 \cup \mathbb{N}_3$. Thus we have $\forall \widetilde{p}\ \exists p\ \forall q\ \exists \widetilde{q}\ \forall \widetilde{r}\ \exists r$:

$$|\{i \in \mathbb{N}_1 \cup \mathbb{N}_2 : p < \kappa(i) \leq q,\ i \leq e^{\tau/(r-q)}\}|$$
$$\precsim |\{j \in \mathbb{N}_1 : \widetilde{p} < \kappa(j) \leq \widetilde{q},\ j \leq e^{t/(\widetilde{r}-\widetilde{q})}\}|.$$

Now with $m = 3(r-q)/(\widetilde{r}-\widetilde{q})$, we obtain

$$|\{j \in \mathbb{N}_1 : \widetilde{p} < \kappa(j) \leq \widetilde{q},\ j \leq e^{\tau/(\widetilde{r}-\widetilde{q})}\}|$$
$$\precsim |\{j \in \mathbb{N}_1 \cup \mathbb{N}_2 : \widetilde{p} < \kappa(j) \leq \widetilde{q},\ j \leq e^{\tau/(3(r-q))}\}|.$$

Thus we obtain

$$|\{i \in \mathbb{N}_1 \cup \mathbb{N}_2 : p < \kappa(i) \leq q,\ i \leq e^{\tau/(r-q)}\}|$$
$$\precsim |\{j \in \mathbb{N}_1 \cup \mathbb{N}_2 : \widetilde{p} < \kappa(j) \leq \widetilde{q},\ j \leq e^{\tau/(3(r-q))}\}|.$$

But this contradicts our assumption above.

EXAMPLE 2. As before, let $a_i = \widetilde{a}_i = \ln i$ and $\kappa = \widetilde{\kappa}$. Let $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3$ and $\mathbb{N} = \mathbb{N}_1 \cup \widetilde{\mathbb{N}}_2 \cup \widetilde{\mathbb{N}}_3$ be two decompositions of $\mathbb{N}$ into three disjoint, infinite subsets such that $\mathbb{N}_1$ and $\widetilde{\mathbb{N}}_2$ are both thinner than $\mathbb{N}_2$ in the sense of the previous example, i.e. for all $p < q$ and $m > 1$ we have

$$\lim_{n\to\infty} \frac{|\{i \in \mathbb{N}_1 : p < \kappa(i) \leq q,\ i \leq n\}|}{|\{i \in \mathbb{N}_2 : p < \kappa(i) \leq q,\ i \leq n^{1/m}\}|} = 0$$

and

$$\lim_{n\to\infty} \frac{|\{i \in \widetilde{\mathbb{N}}_2 : p < \kappa(i) \leq q,\ i \leq n\}|}{|\{i \in \mathbb{N}_2 : p < \kappa(i) \leq q,\ i \leq n^{1/m}\}|} = 0.$$

We also assume that for all $p < q$ and $\widetilde{p} < \widetilde{q}$ we have

$$|\{i \in \mathbb{N}_2 : p < \kappa(i) \leq q,\ i \leq s^2\}|$$
$$\succsim |\{i \in \mathbb{N}_2 : \widetilde{p} < \kappa(i) \leq \widetilde{q},\ i \leq s\}| \quad \text{as } s \to \infty.$$

We let $M_3 = M, M_2, M_1$ be as in the previous example, and define

$$\gamma_i = \begin{cases} M_1(e^{a_i}), & i \in \mathbb{N}_1, \\ M_2(e^{a_i}), & i \in \mathbb{N}_2, \\ M_3(e^{a_i}), & i \in \mathbb{N}_3, \end{cases} \quad \text{and} \quad \widetilde{\gamma}_i = \begin{cases} M_1(e^{a_i}), & i \in \mathbb{N}_1, \\ M_2(e^{a_i}), & i \in \widetilde{\mathbb{N}}_2, \\ M_3(e^{a_i}), & i \in \widetilde{\mathbb{N}}_3. \end{cases}$$

The spaces $H(\kappa,\gamma,a)$ and $H(\kappa,\widetilde{\gamma},a)$ are not isomorphic. For if $H(\kappa,\gamma,a)\cong H(\kappa,\widetilde{\gamma},a)$ then by the Theorem we would have $\forall\widetilde{p}\ \exists p\ \forall q\ \exists\widetilde{q}\ \forall\widetilde{r}\ \exists r$:

$$\begin{aligned}&|\{i\in\mathbb{N}_1\cup\mathbb{N}_2 : p<\kappa(i)\le q,\ i\le e^{\tau/(r-q)}\}|\\&\qquad\precsim|\{j\in\mathbb{N}_1\cup\widetilde{\mathbb{N}}_2 : \widetilde{p}<\kappa(j)\le\widetilde{q},\ j\le e^{\tau/(\widetilde{r}-\widetilde{q})}\}|.\end{aligned}$$

But the left hand side is

$$\ge|\{i\in\mathbb{N}_2 : p<\kappa(i)\le q,\ i\le e^{\tau/(r-q)}\}|$$

and the right hand side is equal to

$$\begin{aligned}&|\{j\in\mathbb{N}_1 : \widetilde{p}<\kappa(j)\le\widetilde{q},\ j\le e^{\tau/(\widetilde{r}-\widetilde{q})}\}|\\&\qquad+|\{j\in\widetilde{\mathbb{N}}_2 : \widetilde{p}<\kappa(j)\le\widetilde{q},\ j\le e^{\tau/(\widetilde{r}-\widetilde{q})}\}|\\&\qquad\precsim|\{j\in\mathbb{N}_2 : \widetilde{p}<\kappa(j)\le\widetilde{q},\ j\le e^{\tau/(3(r-q))}\}|\\&\qquad+|\{j\in\mathbb{N}_2 : \widetilde{p}<\kappa(j)\le\widetilde{q},\ j\le e^{\tau/(3(r-q))}\}|.\end{aligned}$$

So we obtain

$$\begin{aligned}&|\{i\in\mathbb{N}_2 : p<\kappa(i)\le q,\ i\le e^{\tau/(r-q)}\}|\\&\qquad\precsim 2|\{j\in\mathbb{N}_2 : \widetilde{p}<\kappa(j)\le\widetilde{q},\ j\le e^{\tau/(3(r-q))}\}|\end{aligned}$$

which contradicts our assumptions.

In our third example the densities of the distributions of κ and $\widetilde{\kappa}$ are different.

Example 3. We consider $\kappa(i)=i-[\![\sqrt{i}]\!]^2+1$ and $\widetilde{\kappa}(i)=i-2^{[\![\log_2 i]\!]}+1$. So

$$\begin{array}{rcccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots & 15 & 16 & 17 & \ldots\\ \kappa: & \downarrow\\ & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & \ldots & 7 & 1 & 2 & \ldots\end{array}$$

and

$$\begin{array}{rcccccccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots & 15 & 16 & 17 & \ldots & 31 & 32 & \ldots\\ \widetilde{\kappa}: & \downarrow\\ & 1 & 1 & 2 & 1 & 2 & 3 & 4 & 1 & 2 & \ldots & 8 & 1 & 2 & \ldots & 16 & 1 & \ldots\end{array}$$

We let $a_i=\widetilde{a}_i=\log_2 i$, and choose $\gamma_i=\widetilde{\gamma}_i$ and $M(t)$ such that the middle inequality on both sides in the statement of the theorem always holds, e.g., $\gamma_i=M(i)^2$. Thus if $H(\kappa,\gamma,a)\cong H(\widetilde{\kappa},\gamma,a)$ we have $\forall\widetilde{p}\ \exists p\ \forall q\ \exists\widetilde{q}\ \forall\widetilde{r}\ \exists r$:

$$|\{i : p<\kappa(i)\le q,\ i\le 2^{\tau/(r-q)}\}|\precsim|\{j : \widetilde{p}<\widetilde{\kappa}(j)\le\widetilde{q},\ j\le 2^{\tau/(\widetilde{r}-\widetilde{q})}\}|.$$

So we obtain

$$\sqrt{2^{\tau/(r-q)}}\precsim\log_2 2^{\tau/(\widetilde{r}-\widetilde{q})},$$

which is a contradiction. So the spaces $H(\kappa,\gamma,a)$ and $H(\widetilde{\kappa},\gamma,a)$ are not isomorphic.

These examples show that in some special cases where the distributions of γ and $\widetilde{\gamma}$ are considerably different or the densities of κ and $\widetilde{\kappa}$ are considerably different, these spaces can be distinguished by this invariant. However, we still have the following question.

Question. Assume $a_i=\widetilde{a}_i$. Let $\mathbb{N}=\mathbb{N}_1\cup\mathbb{N}_2$ be a decomposition of $\mathbb{N}$ into two infinite disjoint subsets and let κ, $\widetilde{\kappa}$ be such that for $p,q\in\mathbb{N}$, $p<q$,

$$\lim_{n\to\infty}\frac{|\{i : p<\kappa(i)\le q,\ i\le n\}|}{n}>0$$

and

$$\lim_{n\to\infty}\frac{|\{i\in\mathbb{N}_k : p<\widetilde{\kappa}(i)\le q,\ i\le n\}|}{n}>0,\qquad k=1,2.$$

Let $M_1(t),M_2(t):(0,\infty)\to(0,\infty)$ increase to ∞ as $t\to\infty$, and assume $kM_1(t^k)\le M_2(t)$ for each k asymptotically as $t\nearrow$. Define

$$\gamma_i=M_2(e^{a_i})\quad\text{and}\quad\widetilde{\gamma}_i=\begin{cases}M_1(e^{a_i}) & \text{if } i\in\mathbb{N}_1,\\ M_2(e^{a_i}) & \text{if } i\in\mathbb{N}_2.\end{cases}$$

Are the spaces $H(\kappa,\gamma,a)$ and $H(\widetilde{\kappa},\widetilde{\gamma},a)$ isomorphic? If they are not, can we distinguish them by using the above invariant or some modifications of it?

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Stochastic continuity and approximation

by

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Abstract. This work is concerned with the study of stochastic processes which are continuous in probability, over various parameter spaces, from the point of view of approximation and extension. A stochastic version of the classical theorem of Mergelyan on polynomial approximation is shown to be valid for subsets of the plane whose boundaries are sets of rational approximation.

In a similar vein, one can obtain a version in the context of continuity in probability of the theorem of Arakelyan on the uniform approximation of continuous functions on a closed set by entire functions.

Locally bounded processes continuous in probability are characterized via operators from L^1-spaces to spaces of continuous functions. This characterization is utilized in a discussion of the problem of extension of the parameter space.

Introduction. The notion of a stochastic process which is continuous in probability (stochastically continuous in [16]) arises in numerous contexts in probability theory (see [4], [7], [8], [16], [28]). Indeed, the Poisson process is continuous in probability, and this notion plays a role in the study of generalizations of this process and, from a broader point of view, in the theory of processes with independent increments [16]. For instance, R. K. Getoor [15] showed that the Brownian escape process, in dimension at least three, is continuous in probability and has independent increments. The recent work of X. Fernique [13] on random right-continuous functions with left-hand limits (so-called cadlag functions) involves continuity in probability in an essential way.

The study of processes continuous in probability as a generalization of the notion of a continuous function began with the approximation theorems of K. Fan [11], [8, Thms. VI.III.III, VI.III.IV] and D. Dugué [8, Thm. VI.III.V] on the unit interval. These results were generalized to convex domains in higher dimensions in [18], where the problem was raised of describing all compact sets in the complex plane on which every random function continuous in probability can be uniformly approximated in probability by random

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