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Invariance properties of homomorphisms on algebras of holomorphic functions with the Hadamard product

by

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Abstract. Let $H(G_1)$ be the set of all holomorphic functions on the domain G_1. Two domains G_1 and G_2 are called *Hadamard-isomorphic* if $H(G_1)$ and $H(G_2)$ are isomorphic algebras with respect to the Hadamard product. Our main result states that two admissible domains are Hadamard-isomorphic if and only if they are equal.

Introduction. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be power series with positive radii of convergence. Then the *Hadamard product* of f and g is defined by $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. Let G be an open domain of $\mathbb{C}$ containing 0 and let $H(G)$ be the set of all holomorphic functions on G. We call a domain G *admissible* if for all $f, g \in H(G)$ the Hadamard product $f*g$ extends to a (unique) function of $H(G)$, i.e. $H(G)$ is a commutative algebra. Examples of admissible domains are the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, or more generally $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ for $r \geq 1$, $\mathbb{C} \setminus \{1\}$ and so-called α-starlike regions like $\mathbb{C}_- := \{z \in \mathbb{C} : z \notin [1, \infty)\}$ (see [3] for details). By the famous Hadamard multiplication theorem a domain G is admissible if and only if the complement G^c of G is a multiplicative semigroup (cf. e.g. [5]).

The aim of this paper is to study homomorphisms on $H(G)$. Let us call two domains G_1 and G_2 *Hadamard-isomorphic* if G_1 and G_2 are admissible and $H(G_1)$ and $H(G_2)$ are isomorphic algebras with respect to the Hadamard product. Our main result states that two admissible domains G_1, G_2 with $1 \in G_1^c$ are Hadamard-isomorphic if and only if they are equal. This stands in sharp contrast to the following classical result: $H(G_1)$ and $H(G_2)$ are isomorphic algebras with respect to the *pointwise multiplication* if and only if G_1 is biholomorphically equivalent to G_2. Indeed, we give a complete description of all isomorphisms for admissible domains G with $1 \in G^c$. Roughly speaking, if G is an admissible domain differ-

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ent from $\mathbb{D}$, an isomorphism Φ induces a permutation $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$(1) \qquad \Phi\Big(\sum_{n=0}^{\infty} a_n z^n\Big) = \sum_{n=0}^{\infty} a_n z^{\varphi(n)}$$

for all z in a suitable neighborhood of 0 *and* φ must be of a rather special form depending on the following property of the domain: if k denotes the cardinality of $\{z \in G^c : |z| = 1\}$ then φ permutes only the equivalence classes $\{kn + j : n \in \mathbb{N}_0\}$ for $j = 0, 1, \ldots, k-1$ and $n \geq n_0$. For example, if $G = \mathbb{C}_-$ we have $k = 1$ and φ is therefore equal to the identity for large $n \in \mathbb{N}_0$. The proof is based on function-theoretic theorems of Szegö and Pólya–Carlson.

In the first section we show that boundedness of the domain is an invariant for isomorphisms. Further we prove that the continuity of a linear mapping Φ satisfying $\Phi(z^n) = z^{\varphi(n)}$ implies that $\varphi(n)/n$ is bounded. In the next section we determine the isomorphisms of $H(\mathbb{D}_r)$ for $r \in [1, \infty]$. Examples show that our results are sharp. In Section 3 we prove that the cardinality k of $\{z \in G^c : |z| = 1\}$ is an invariant. Section 4 contains the above-mentioned description of isomorphisms. In the last Section 5 we prove our main result about Hadamard-isomorphic domains.

1. Elementary properties of homomorphisms. Let G be an admissible domain. We define the coefficient functionals $\delta_n : H(G) \to \mathbb{C}$ by $\delta_n(f) := a_n$ (where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < 1$). The algebra $H(G)$ is a completely metrizable locally convex vector space (i.e. a Fréchet space), where the norms are given by $|f|_K := \sup_{z\in K} |f(z)|$ for an arbitrary compact subset K of G. It can be shown that the Hadamard multiplication is continuous, hence $H(G)$ is a so-called B_0-algebra. If $1 \in G^c$ then $H(G)$ has a unit element given by

$$(2) \qquad \gamma(z) := \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for all } |z| < 1.$$

In [8] it is proved that each multiplicative functional on $H(G)$ is continuous. An application of the closed graph theorem shows that each homomorphism $\Phi : H(G_1) \to H(G_2)$ is continuous (cf. [7]). An *isomorphism* Φ has the property that $\Phi(z^n)$ is again a monomial (cf. Theorems 1.1 and 2.1 which have been proved in [7]). Example 1.2 shows that this is not true for arbitrary homomorphisms.

1.1. THEOREM. *Let G_1 and G_2 be admissible domains and let $\Phi : H(G_1) \to H(G_2)$ be a homomorphism. If the range of Φ is dense or contains all monomials then $\Phi(z^n)$ is either* 0 *or a monomial, i.e. has the form z^m for some $m \in \mathbb{N}_0$.*

1.2. EXAMPLE. The assumption concerning the range cannot be omitted. Let G be an admissible domain. Define a linear mapping $\Phi : H(G) \to H(G)$ by $\Phi(f) = \delta_1(f)\gamma$. Then Φ is multiplicative and unital but $\Phi(z) = \gamma$ is not a monomial.

1.3. THEOREM. *Let $\Phi : H(G_1) \to H(G_2)$ be a continuous linear mapping with $\Phi(z^n) = z^{\varphi(n)}$, $\varphi(n) \in \mathbb{N}_0$, for all $n \in \mathbb{N}_0$. Define $r_i := \sup\{|z| : z \in G_i\}$ for $i = 1, 2$. If $G_2 \neq \mathbb{D}$ then $\varphi(n)/n$ is bounded and $r_2^{\alpha} \leq r_1$, where $\alpha := \limsup \varphi(n)/n$.*

Proof. We assume first that $r_1 < \infty$. Let $(a_n)_n$ be a sequence of positive numbers with $a_n \to r_1$. Then $p_n(z) := (z/a_n)^n$ converges uniformly to 0 on every compact set in G. The continuity of Φ implies that $\Phi(p_n)(z) = a_n^{-n} z^{\varphi(n)} \to 0$. Let $z_0 \in G_2$ with $|z_0| > 1$. Then $a_n^{-n}|z_0|^{\varphi(n)} \to 0$ and therefore there exists $n_0 \in \mathbb{N}$ such that $a_n^{-n}|z_0|^{\varphi(n)} \leq 1$ for all $n \geq n_0$. It follows that $\varphi(n)/n \leq \log a_n / \log|z_0| \to \log r_1/\log|z_0|$ and $\alpha \leq \log r_1/\log|z_0|$. Hence $|z_0|^{\alpha} \leq r_1$ since $\log|z_0| > 0$. Thus we have proved the inequality $r_2^{\alpha} \leq r_1$.

Assume now that $r_1 = \infty$ and let $(a_n)_n$ be a sequence of positive numbers with $a_n \to \infty$. Then $p_n(z) := (z/a_n)^n$ converges uniformly to 0 on every compact set in G_1. Let $z_0 \in G_2$ with $|z_0| > 1$. The continuity of Φ yields $a_n^{-n}|z_0|^{\varphi(n)} \to 0$. Then for sufficiently large n we have $a_n^{-n}|z_0|^{\varphi(n)} \leq 1$, which is equivalent to

$$(3) \qquad \frac{\varphi(n)\log|z_0|}{na_n} \leq \frac{\log a_n}{a_n} \to 0.$$

If $\limsup_{n\to\infty} \varphi(n)/n = \infty$ then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \varphi(n_k)/n_k = \infty$. Define now $a_n := n$ if $n \notin \{n_1, n_2, \ldots\}$ and $a_{n_k} := \sqrt{\varphi(n_k)/n_k}$. Clearly $a_n \to \infty$ but $\varphi(n)/(na_n) \not\to 0$, which contradicts the above statement. Hence the proof is complete. ∎

1.4. LEMMA. *Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be an injection. Then there exists an infinite sequence $n_k \in \mathbb{N}_0$ with $\varphi(n_k) \geq n_k$.*

Proof. Assume contrary to the assertion that there exists $n' \in \mathbb{N}$ with $\varphi(n) < n$ for all $n > n'$. Consider the set $N' := \{0, \ldots, n'\}$. Since $\varphi(n'+1) \in N'$ there exists $m \in N'$ with $\varphi(m) > n'$ and we choose m such that $\varphi(m)$ is maximal. Moreover, $\varphi(\{n'+1, \ldots, \varphi(m)\}) \subset \{0, \ldots, \varphi(m)\}$ and thus $\varphi(\{0, \ldots, \varphi(m)\}) \subset \{0, \ldots, \varphi(m)\}$. Since φ is an injection we have $\varphi(\{0, \ldots, \varphi(m)\}) = \{0, \ldots, \varphi(m)\}$. Now obviously $\varphi(\varphi(m)+1) \geq \varphi(m)+1$, a contradiction. ∎

1.5. THEOREM. *Let $\Phi : H(G_1) \to H(G_2)$ be an isomorphism. Then G_1 is bounded iff G_2 is. Further, $r_2 := \max\{|z| : z \in G_2\} = \max\{|z| : z \in G_1\} =: r_1$. If G_1 is a bounded domain different from $\mathbb{D}$ then $\lim \varphi(n)/n = 1$.*

Proof. We claim that $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ is injective: suppose that $\varphi(n_1) = \varphi(n_2)$. Then $\Phi(z^{n_1} * z^{n_2}) = z^{\varphi(n_1)} * z^{\varphi(n_2)} = z^{\varphi(n_1)} \neq 0$. On the other hand, $n_1 \neq n_2$ implies $z^{n_1} * z^{n_2} = 0$, hence $\Phi(z^{n_1} * z^{n_2}) = 0$, a contradiction. Lemma 1.4 now shows that $1 \leq \limsup \varphi(n)/n =: \alpha$. Let now G_1 be bounded. If $G_2 \neq \mathbb{D}$ we have $r_2 \leq r_2^\alpha \leq r_1$ by Theorem 1.3. Hence G_2 is bounded and $r_2 \leq r_1$. If $G_2 = \mathbb{D}$ we have $r_2 \leq r_1$ since G_1 contains $\mathbb{D}$. Finally, the same argument applied to Φ^{-1} yields $r_1 = r_2$. For the last statement note that $r_1^\alpha = r_2^\alpha \leq r_1$ by Theorem 1.3, hence $\alpha = 1$ since $r_1 > 1$. ■

1.6. Lemma. *Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be a bijection. Then* $\limsup \varphi(n)/n < \infty$ *if and only if* $\liminf \varphi^{-1}(n)/n > 0$.

Proof. The easy proof is omitted. ■

The next example shows that Theorem 1.3 cannot be improved.

1.7. Example. Let $r > 1$ and $G_1 := \{z \in \mathbb{C} : |z| < r\} \setminus [1, r)$ and $G_2 := \{z \in \mathbb{C} : |z| < \sqrt{r}\} \setminus ((-\sqrt{r}, -1] \cup [1, \sqrt{r}))$. Then $H(G_1)$ and $H(G_2)$ are unital algebras. Define $\Phi(f)(z) := f(z^2) = \sum_{n=0}^\infty a_n z^{2n}$, which is well-defined since $z \in G_2$ implies $z^2 \in G_1$. Then Φ is an injective homomorphism but not unital since $\Phi(\gamma) = 1/(1 - z^2)$. Note that $\varphi(n) = 2n$ and that $\lim \varphi(n)/n = 2$.

2. Isomorphisms of $H(\mathbb{D}_r)$ for $r \in [1, \infty]$. The results of this section (except for Theorem 2.1) are independent of the rest of the paper. Most of them are quite elementary but we include them for completeness. The main result, Theorem 2.6, characterizes the isomorphisms of $H(\mathbb{D}_r)$. The following theorem can be found in [7].

2.1. Theorem. *Let G_1 and G_2 be admissible domains and let $\Phi : H(G_1) \to H(G_2)$ be a homomorphism with $\Phi(z^n) = z^{\varphi(n)}$ for $n \in \mathbb{N}_0$. Then $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ is injective and*

$$\Phi(f)(z) = \sum_{n=0}^{\infty} a_n z^{\varphi(n)} \tag{4}$$

for all z in a suitable neighborhood of 0, where $f \in H(G_1)$ is given by $f(z) = \sum_{n=0}^\infty a_n z^n$ for all z in a neighborhood of 0. If the range of Φ is dense then φ is a bijection.

Proof. The proof of Theorem 1.5 shows that φ is injective. Suppose now that there exists $m \in \mathbb{N}_0$ with $\varphi(n) \neq m$ for all $n \in \mathbb{N}_0$. Then $\Phi(f) * z^m = 0$ for all $f \in H(G)$. Hence $\Phi(f) \in \ker(\delta_m)$ for all $f \in H(G)$. Thus the range is not dense in $H(G)$, a contradiction. For the representation (4) note that $f * \exp(z)$ is an entire function and Φ is continuous (cf. [7] for details). ■

2.2. Theorem. *Let $\Phi : H(G_1) \to H(G_2)$ be a homomorphism with $\Phi(z^n) = z^{\varphi(n)}$ for all $n \in \mathbb{N}_0$. If $G_2 \neq \mathbb{D}$ then $\Phi(f)$ is entire for every entire function f.*

Proof. Let $f(z)$ be entire. By Theorem 2.1 we know that $\Phi(f)$ is locally equal to $\sum_{n=0}^\infty a_n z^{\varphi(n)}$, where $f(z) = \sum_{n=0}^\infty a_n z^n$. It suffices to show that the series converges for each $z \in \mathbb{C}$. Let $\alpha := \limsup \varphi(n)/n$, which is finite by Theorem 1.3. Then there exists $n_0 \in \mathbb{N}$ such that $\varphi(n)/n \leq \alpha + 1$ and therefore $r^{\varphi(n)/n} \leq r^{\alpha+1}$ for all $n \geq n_0$. Since the convergence radius of f is infinite there exists $n_1 \in \mathbb{N}$ such that $|a_n|^{1/n} \leq 1/r^{\alpha+2}$ for all $n \geq n_1$. Now let $r > 1$ be arbitrary and $|z| \leq r$. Then

$$|a_n z^{\varphi(n)}| \leq (|a_n|^{1/n} r^{\varphi(n)/n})^n \leq \left(\frac{r^{\alpha+1}}{r^{\alpha+2}}\right)^n = \left(\frac{1}{r}\right)^n. \tag{5}$$

Since $\sum_n r^{-n}$ converges the proof is complete. ■

The assumption $G_2 \neq \mathbb{D}$ is essential in Theorems 1.3 and 2.2 as the following example shows:

2.3. Example. Let $1 < r \leq \infty$ and let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be an injective function. For $f \in H(\mathbb{D}_r)$ define $\Phi(f) := \sum_{n=0}^\infty a_n z^{\varphi(n)}$, which is an element of $H(\mathbb{D})$: clearly $|z^{\varphi(n)}| \leq 1$ for $|z| \leq 1$ and $|a_n| \leq \theta^n$ with $1/r < \theta < 1$ for large $n \in \mathbb{N}$ by the convergence radius formula. It follows that $\Phi : H(\mathbb{D}_r) \to H(\mathbb{D})$ is a homomorphism for an *arbitrary* injection φ.

2.4. Lemma. *Let $1 < r < \infty$ and $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be an injection with $\alpha := \liminf_{n\to\infty} \varphi(n)/n < \infty$. Then $\Phi : H(\mathbb{D}_r) \to H(\mathbb{D}_s)$ defined by $\Phi(f) = \sum_{n=0}^\infty a_n z^{\varphi(n)}$ is a homomorphism for any $s > 1$ with $s^\alpha \leq r$. In particular, if $r = s > 1$ then $\Phi : H(\mathbb{D}_r) \to H(\mathbb{D}_r)$ is a homomorphism for any injection $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ with $\lim \varphi(n)/n = 1$.*

Proof. Let $|z| \leq s_1 < s$ and $s_1 > 1$. Note that $\limsup s_1^{\varphi(n)/n} \leq s_1^\alpha < s^\alpha \leq r$ since $s_1 > 1$. Choose $\varepsilon > 0$ with $s_1^\alpha < (1-\varepsilon)(r-\varepsilon)$. Then there exists $n_0 \in \mathbb{N}_0$ such that $s_1^{\varphi(n)/n} \leq (1-\varepsilon)(r-\varepsilon)$ for all $n \geq n_0$. Since $\limsup |a_n|^{1/n} \leq 1/r$ there exists n_1 such that $|a_n| \leq 1/(r-\varepsilon)^n$ for all $n \geq n_1$. The estimation for $s_1^{\varphi(n)}$ yields $|a_n z^{\varphi(n)}| \leq (1-\varepsilon)^n$ for $|z| \leq s_1$ and $n \geq \max\{n_0, n_1\}$. Hence $\sum_n |a_n| \cdot |z|^{\varphi(n)}$ converges for all $|z| \leq s_1$. ■

2.5. Lemma. *Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be an injection with $\liminf_{n\to\infty} \varphi(n)/n > 0$. Then $\Phi : H(\mathbb{D}) \to H(\mathbb{D})$ defined by $\Phi(f) = \sum_{n=0}^\infty a_n z^{\varphi(n)}$ is a homomorphism.*

Proof. Let $|z| \leq r < 1$ and let $\alpha := \liminf_{n\to\infty} \varphi(n)/n$. Then there exists $n_0 \in \mathbb{N}$ with $\alpha/2 < \varphi(n)/n$ for all $n \geq n_0$. Now $|z| \leq r < 1$ implies $|z|^{\varphi(n)/n} \leq r^{\alpha/2}$. Since f has at least convergence radius 1 there exists $n_1 \in \mathbb{N}$ such that $|a_n|^{1/n} \le r^{-\alpha/4}$ (note that $r^{\alpha/4} < 1$). We infer that

$$(6) \qquad |a_n z^{\varphi(n)}| \le (|a_n|^{1/n} |z|^{\varphi(n)/n})^n \le (r^{-\alpha/4} r^{\alpha/2})^n = (r^{\alpha/2})^n.$$

Since $r^{\alpha/2} < 1$ we have found a convergent majorant. ■

2.6. THEOREM. *If $1 < r < \infty$ then $\Phi : H(\mathbb{D}_r) \to H(\mathbb{D}_r)$ is an isomorphism if and only if it is induced by a bijection φ with $\lim_{n\to\infty} \varphi(n)/n = 1$. If $r = 1$ or ∞ it is an isomorphism iff $0 < \liminf_{n\to\infty} \varphi(n)/n \le \limsup_{n\to\infty} \varphi(n)/n < \infty$.*

Proof. The necessity of the first statement follows from Theorem 1.5. For the converse note that Φ is a homomorphism by Lemma 2.4. Moreover, the inverse permutation φ^{-1} satisfies $\varphi^{-1}(n)/n \to 1$ and induces a homomorphism which is the inverse of Φ. For the case $r = \infty$ the necessity follows from Theorem 1.3 applied to φ and φ^{-1} and from Lemma 1.6. The converse follows from the proof of Theorem 2.2 applied to φ and φ^{-1} (cf. also Lemma 1.6).

It remains to consider the case $r = 1$. Lemma 2.5 applied to φ and φ^{-1} yields the sufficiency (cf. Lemma 1.6). For necessity suppose $\liminf \varphi(n)/n = 0$. Then there exists a subsequence $n_k \to \infty$ with $\lim \varphi(n_k)/n_k = 0$. We define $b_k := (\log(n_k/\varphi(n_k)))^{-1}$, which converges to zero. Consider $f(z) := \sum_{k=0}^{\infty} (1 + b_k)^{n_k} z^{n_k}$, which is an element of $H(\mathbb{D})$. Now

$$(7) \qquad \Phi(f)(z) = \sum_{k=0}^{\infty} \left[1 + \left(\log\left(\frac{\varphi^{-1}(n_k)}{n_k}\right)\right)^{-1}\right]^{\varphi^{-1}(n_k)} z^{n_k}.$$

We will show that the radius of convergence of the latter series is 0, which yields a contradiction. We have to consider

$$(8) \qquad \limsup_{k\to\infty} \left[1 + \left(\log\left(\frac{\varphi^{-1}(n_k)}{n_k}\right)\right)^{-1}\right]^{\varphi^{-1}(n_k)/n_k}.$$

It is easy to see that $\lim \varphi^{-1}(n_k)/n_k = \infty$. Now $\lim_{x\to\infty}[1 + 1/\log x]^x = \infty$ gives the desired contradiction. If $\liminf \varphi(n)/n = \infty$ then $\liminf \varphi^{-1}(n)/n = 0$. The above reasoning applied to φ^{-1} shows that $\liminf \varphi(n)/n < \infty$. Finally, Lemma 1.6 yields $\limsup \varphi(n)/n < \infty$. ■

The next example shows that a homomorphism is in general not determined by the values of $\Phi(z^n)$: the constructed homomorphism Φ satisfies $\Phi(z^n) = 0$ for all $n \in \mathbb{N}_0$, but $\Phi \ne 0$. On the other hand, an isomorphism is determined by the values $\Phi(z^n)$ (cf. Theorem 2.1).

2.7. EXAMPLE. Let $G_1 := \mathbb{D}_r \setminus \{1\}$ (where $r > 1$ is a fixed number) and let G_2 be an admissible domain with $1 \in G_2^c$. Let $T : H(G_1) \to \mathbb{C}$ be the functional where $T(f)$ is defined as the residue of f at the point $z = 1$, and define $\Phi : H(G_1) \to H(G_2)$ by $\Phi(f) = -T(f)\gamma$. It can be shown that Φ is a unital homomorphism with $\Phi(z^n) = 0$ for all $n \in \mathbb{N}_0$ but $\Phi \ne 0$.

3. Some invariants of Hadamard isomorphisms. In the first section we have seen that boundedness of the domain is an invariant of isomorphisms (with respect to the Hadamard product). From [7] we take the following result:

3.1. THEOREM. *Let G_1 and G_2 be Hadamard-isomorphic domains. Then:*

(a) *G_1 is simply connected iff G_2 is simply connected.*
(b) *$1 \in G_1^c$ iff $1 \in G_2^c$.*
(c) *$1 \in G_1^c$ is non-isolated iff $1 \in G_2^c$ is non-isolated.*
(d) *$1 \in G_1^c$ is isolated iff $1 \in G_2^c$ is isolated.*

In the sequel we assume that 1 is in G^c. It is clear that $A := \{z \in G^c : |z| = 1\}$ is a unital subsemigroup of the unit circle. If it is not finite then A is dense in the unit circle and G is the unit disk. If A is a finite semigroup then it is a subgroup, equal to the set of all kth roots of unity for a suitable $k \in \mathbb{N}$, and will be denoted by A_k in the sequel. Suppose that $G_1, G_2 \ne \mathbb{D}$ are Hadamard-isomorphic domains with $1 \in G_i^c$ for $i = 1, 2$. We want to prove that $k_1 = k_2$, i.e., that the number of kth roots of unity is an invariant. As a preparation we need

3.2. THEOREM. *Let $G_1, G_2 \ne \mathbb{D}$ be admissible domains with $1 \in G_i^c$ and let k_i be the cardinality of $A_i := \{z \in G_i^c : |z| = 1\}$ for $i = 1, 2$. Let $\xi := \exp(2\pi i/k_1)$ and $\gamma_j(z) := \gamma(z/\xi^j) \in H(G_1)$ for $j = 0, \ldots, k_1 - 1$. If $\Phi : H(G_1) \to H(G_2)$ is a homomorphism then there exist a natural number $r \le k_2$ and a polynomial $p(z)$ such that $\Phi(\gamma_j) = p(z)/(1 - z^r)$, in particular $\Phi(\gamma_j)$ is a rational function for $j = 0, \ldots, k_1$.*

Proof. Note that $\Phi(\gamma) = \Phi(\gamma) * \Phi(\gamma)$. It follows that the Taylor coefficients of $\Phi(\gamma)$ are either 0 or 1. Let $\Phi(\gamma_j) = \sum_{n=0}^{\infty} b_n z^n$. Since $\gamma_j^{k_1} = \gamma$ we infer that $\Phi(\gamma) = (\Phi(\gamma_j))^{k_1}$. Hence $b_n^{k_1}$ are either equal to 0 or 1 for all $n \in \mathbb{N}_0$, i.e. that the coefficients b_n are either 0 or k_1th roots of unity. Since $G_2 \ne \mathbb{D}$ a theorem of Szegö [6, p. 227] shows that there exist $r \in \mathbb{N}$ and a polynomial $p(z)$ such that $\Phi(\gamma_j) = p(z)/(1 - z^r)$. We can assume that $r \in \mathbb{N}$ is minimal with this property. Each pole of $g(z) := p(z)/(1 - z^r)$ is of first order and it must be contained in A_2 since g is holomorphic on G_2. Hence there exists a polynomial $q(z)$ with $g(z) = q(z)/(1 - z^{k_2})$. Since r was minimal with this property we infer that $r \le k_2$. ■

3.3. THEOREM. *Let $G_1, G_2 \neq \mathbb{D}$ be admissible domains and $1 \in G_i^c$. Let k_i be the cardinality of $A_i := \{z \in G_i^c : |z| = 1\}$ for $i = 1, 2$. If $\Phi : H(G_1) \to H(G_2)$ is an isomorphism then $k_1 = k_2$.*

Proof. By Theorem 3.2 there exist $r \le k_2$ and a polynomial $p(z)$ such that $\Phi(\gamma_1) = p(z)/(1 - z^r)$. By polynomial division there exist polynomials p_1 and p_2 with $p(z) = p_1(z)(1-z^r)+p_2(z)$ and the degree of p_2 at most $r-1$. Let $p_2(z) = c_0 + c_1 z + \ldots + c_{r-1}z^{r-1}$. Since $\Phi(\gamma_1) = p_1(z) + p_2(z)/(1 - z^r)$ there exists $n_0 \in \mathbb{N}$ such that the Taylor expansion of $\Phi(\gamma_1)$ is periodic for all $n \ge n_0$ and the coefficients are given by $c_0, \ldots, c_{r-1}$. In particular, $c_0, \ldots, c_{r-1}$ are k_1th roots of unity (cf. the proof of Theorem 3.2). We claim that

$$\{1, \xi, \ldots, \xi^{k_1-1}\} = \{c_0, \ldots, c_{r-1}\}. \tag{9}$$

For this we consider $f_N(z) := \sum_{n=N}^{\infty} \xi^{-n} z^n$ for large $N \in \mathbb{N}$. Then the Taylor coefficients of $\Phi(f_N)$ are either zero or equal to some c_j for $j = 0, \ldots, r-1$ since φ only permutes the Taylor coefficients of f_N (cf. Theorem 2.1). Now (9) implies $k_1 \le r \le k_2$. The same argument applied to Φ^{-1} yields $k_2 \le k_1$. ∎

4. Isomorphisms of $H(G)$ for $G \neq \mathbb{D}$ and $1 \in G^c$. Let $G \neq \mathbb{D}$ be an admissible domain with $1 \in G^c$ and let k be the cardinality of $\{z \in G^c : |z| = 1\}$. Throughout this section we denote by ξ the kth root of 1 given by $\exp(2i\pi/k)$ and by $A_k := \{\xi^j : j = 0, \ldots, k-1\}$ the set of all kth roots of 1. Then $H_k := \{f \in H(\widehat{\mathbb{C}} \setminus A_k) : f(\infty) = 0\}$ is a unital subalgebra (with respect to Hadamard multiplication) of $H(G)$. We need some properties of H_k. Define $\gamma_j(z) := \gamma(z/\xi^j) \in H_k$ for each $j = 0, \ldots, k-1$. Each $f \in H_k$ has a unique decomposition

$$f = \sum_{j=0}^{k-1} \gamma_j * f_j \qquad \text{with } f_j \in H_1 \tag{10}$$

by Laurent expansion. As pointed out in [4], H_k is topologically and linearly isomorphic to the topological direct sum of the closed subspaces $\gamma_j * H_1$, $j = 0, \ldots, k-1$. For each $\zeta \in A_k$ there is a natural continuous algebra homomorphism $T_\zeta : H_k \to H_1$ defined by

$$T_\zeta(f) = T_\zeta\Big(\sum_{j=0}^{k-1} \gamma_j * f_j\Big) := \sum_{j=0}^{k-1} \zeta^j f_j \tag{11}$$

(cf. Lemma 2 of [4]). Note that $T_\zeta|H_1$ is just the identity and that $T_\zeta(\gamma_j) = \zeta^j\gamma$ for $j = 0, \ldots, k-1$. Now suppose that $g \in H_1$ and define $\widehat{g} : \mathbb{C} \to \mathbb{C}$ as the Gelfand transform (cf. [8]). Then $\widehat{g}$ is an entire function of exponential type zero interpolating the Taylor coefficients of g, i.e., that $g(z) = \sum_{n=0}^{\infty} \widehat{g}(n) z^n$.

This result is known as the theorem of Wigert (cf. [8] or [1, p. 8]). We want to determine an analogue for functions in H_k. Theorem 4.2 gives a description of the power series of $f \in H_k$ in terms of the Gelfand transform of $T_\zeta(f) \in H_1$. The next lemma is a technical preparation.

4.1. LEMMA. *Let $f \in H_k$. Then*

$$f = \sum_{j=0}^{k-1} \gamma_j * f_j \quad \textit{with} \quad f_j = \frac{1}{k} \sum_{\zeta \in A_k} \zeta^{-j} T_\zeta(f).$$

Proof. Define $g := \sum_{j=0}^{k-1} \gamma_j * f_j$ with f_j as in the statement. We have to show that $g = f$. It follows that $T_\eta(g) = \sum_{j=0}^{k-1} \eta^j f_j$ and

$$T_\eta(g) = \frac{1}{k} \sum_{j=0}^{k-1} \sum_{\zeta \in A_k} (\eta\zeta^{-1})^j T_\zeta(f) = \frac{1}{k} \sum_{\zeta \in A_k} T_\zeta(f) \sum_{j=0}^{k-1} (\eta\zeta^{-1})^j = T_\eta(f) \tag{12}$$

since $\sum_{j=0}^{k-1} (\eta\zeta^{-1})^j$ is 0 for $\eta \neq \zeta$ and equal to k for $\eta = \zeta$. This gives $T_\eta(g - f) = 0$ for all $\eta \in A_k$, leading to a system of k linear equations for $(g - f)_j$ as in (10). Since the coefficient matrix is Vandermonde we infer that $f - g = 0$.

4.2. THEOREM. *Let $f \in H_k$. Then $f(z) = \sum_{n=0}^{\infty} \widehat{T_{\xi^{-n}}(f)}(n) z^n$.*

Proof. Let $f = \sum_{j=0}^{k-1} \gamma_j * f_j$ as in Lemma 4.1. Then

$$f = \frac{1}{k} \sum_{j=0}^{k-1} \sum_{\zeta \in A_k} \gamma_j * \zeta^{-j} T_\zeta(f). \tag{13}$$

Since $\gamma_j(z) = 1/(1-z/\xi^j) = \sum_{n=0}^{\infty} \xi^{-jn} z^n$ and $T_\zeta(f)(z) = \sum_{n=0}^{\infty} \widehat{T_\zeta(f)}(n) z^n$ we obtain

$$f(z) = \frac{1}{k} \sum_{n=0}^{\infty} \sum_{\zeta \in A_k} \sum_{j=0}^{k-1} (\zeta^{-1}\xi^{-n})^j \widehat{T_\zeta(f)}(n) z^n. \tag{14}$$

The last sum is equal to $\sum_{n=0}^{\infty} \widehat{T_{\xi^{-n}}(f)}(n) z^n$ since $\sum_{j=0}^{k-1} (\zeta^{-1}\xi^{-n})^j = 0$ for $\zeta \neq \xi^{-n}$. ∎

4.3. THEOREM. *Let $G_1, G_2 \neq \mathbb{D}$ be admissible domains with $1 \in G_i^c$ for $i = 1, 2$ and let k be the cardinality of $\{z \in G_2^c : |z| = 1\}$. If $\Phi : H(G_1) \to H(G_2)$ is an isomorphism then there exist $n_0 \in \mathbb{N}$ and pairwise distinct $b_0, \ldots, b_{k-1} \in \mathbb{Z}$ such that $\varphi(kn + j) = kn + b_j$ for all $nk + j \ge n_0$ and $j = 0, \ldots, k-1$.*

Proof. Let $q(z) := z/(1 - z)^2 = \sum_{n=0}^{\infty} nz^n$. Then $\Phi(q) = \sum_{n=0}^{\infty} nz^{\varphi(n)}$. Since the coefficients of $\Phi(q)$ are integers the theorem of Pólya–Carlson [6,

p. 231] (note that $G_2 \neq \mathbb{D}$) shows there exist $M, N \in \mathbb{N}$ and a polynomial $p(z)$ (with integral coefficients) such that $\Phi(q) = p(z)/(1-z^M)^N$. By polynomial division there exist polynomials p_1 and p_2 such that $\Phi(q) = p_1(z) + p_2(z)/(1-z^M)^N$ and the degree of p_2 is strictly lower than that of the denominator. It follows that $q_2 := \Phi(q) - p_1$ vanishes at infinity, hence $q_2 \in H_{k_2} = H_k$ by Theorem 3.3. Then $h_\zeta := T_\zeta(q_2)$ is a rational function in the algebra H_1 for each $\zeta \in A_k$. It follows that the Gelfand transform $\widehat{h}_\zeta : \mathbb{C} \to \mathbb{C}$ is a polynomial (cf. Satz 1.3.X of [1]). Moreover, the nth Taylor coefficient of q_2 is given by $\widehat{h_{\xi^{-n}}}(n)$ (cf. Theorem 4.2).

On the other hand, we know that

$$\Phi(q) = q_2 + p_1 \quad \text{and} \quad \Phi(q) = \sum_{n=0}^{\infty} \varphi^{-1}(n) z^n.$$

Hence there exists $n_0 \in \mathbb{N}$ such that $\varphi^{-1}(n) = \widehat{h}_{\xi^{-n}}(n)$ for all $n \geq n_0$. Then

$$\varphi^{-1}(nk+j) = \widehat{h_{\xi^{-j}}}(nk+j) \qquad \text{for all } nk+j \geq n_0. \tag{15}$$

Since $\widehat{h_{\xi^{-j}}}$ is a polynomial and $\limsup \varphi^{-1}(n)/n < \infty$ (apply Theorem 1.3) we infer that $\widehat{h_{\xi^{-j}}}$ is a polynomial of degree at most 1. Hence for each $j = 0, \ldots, k-1$ there exist coefficients $a_j, b_j \in \mathbb{C}$ such that $\varphi^{-1}(nk+j) = a_j(nk) + b_j$ for all $nk + j \geq n_0$. Note that $a_j \geq 0$. Moreover, $\varphi^{-1}((n+1)k + j) - \varphi^{-1}(nk+j) \in \mathbb{Z}$ implies $a_j k \in \mathbb{Z}$. It follows that $b_j \in \mathbb{Z}$. For each j define $p_j := \Phi^{-1}(\sum_{n=0}^{n_0-1} z^{kn+j})$. It follows that

$$\Phi^{-1}\left(\frac{z^j}{1-z^k}\right) - p_j = \sum_{n=n_0}^{\infty} z^{\varphi^{-1}(kn+j)} = \sum_{n=n_0}^{\infty} z^{a_j kn + b_j} \tag{16}$$
$$= z^{b_j + a_j k n_0} \frac{1}{1 - z^{a_j k}}.$$

Hence $z^{b_j + a_j k n_0}/(1 - z^{a_j k})$ is in $H(G_1)$. Consequently, $a_j k$ is smaller than $k_1 = k$ (cf. Theorem 3.3), i.e., $0 \leq a_j \leq 1$. Since $\varphi^{-1}(nk+j)/(kn+j) \to a_j$ for $j = 0, \ldots, k-1$ we infer that $1 \leq \limsup \varphi^{-1}(n)/n \leq 1$. It follows that $a_j = 1$ for $j = 0, \ldots, k-1$.

Let us show that the b_j's are distinct. Suppose that there exists $d \in \mathbb{Z}$ with $b_i - b_j = dk$ for some $i \neq j$ and $d \in \mathbb{Z}$. Choose $n \in \mathbb{N}$ such that $n + d \geq n_0$. Then $nk + b_i = (n+d)k + b_j$, a contradiction to the injectivity of φ^{-1} (preserving disjoint classes). Finally, it is easy to see that φ is of the same form as φ^{-1}. ∎

4.4. THEOREM. *Let $G \neq \mathbb{D}$ be an admissible domain with $1 \in G^c$ and let k be the cardinality of $\{z \in G^c : |z| = 1\}$. Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be an injection such that there exist $n_0 \in \mathbb{N}$ and pairwise distinct $b_0, \ldots, b_{k-1} \in \mathbb{Z}$ such that $\varphi(kn+j) = kn + b_j$ for all $n \geq n_0$ and $j = 0, \ldots, k-1$. Then there exists a homomorphism $\Phi : H(G) \to H(G)$ with $\Phi(z^n) = z^{\varphi(n)}$. If φ is a bijection then Φ is an isomorphism.*

Proof. Choose $n_0 \in \mathbb{N}$ so large that $\varphi(nk+j) = nk + b_j$ for all $n \geq n_0$ and $b_j - j > -n_0 k$ for $j = 0, \ldots, k-1$. Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be an element of $H(G)$. We want to prove that $\Phi(f)(z) := \sum_{m=0}^{\infty} a_m z^{\varphi(m)}$ defines an element in $H(G)$. Note that $f * \frac{z^j}{1-z^k} \in H(G)$ for $j = 0, \ldots, k-1$ and $q(z) := \sum_{n=0}^{kn_0-1} a_{nk+j} z^{nk+j} \in H(G)$. Hence

$$f * \frac{z^j}{1-z^k} - q(z) = \sum_{n=kn_0}^{\infty} a_{nk+j} z^{nk+j} \tag{17}$$

is a holomorphic function on G which has a zero of order at least kn_0 at $z = 0$. Since $b_j - j > -n_0 k$ we can multiply with $z^{b_j - j}$ and obtain

$$\sum_{j=0}^{k-1} z^{b_j - j} \sum_{n=kn_0}^{\infty} a_{nk+j} z^{nk+j} = \sum_{j=0}^{k-1} \sum_{n=kn_0}^{\infty} a_{nk+j} z^{nk+b_j} \in H(G). \tag{18}$$

The latter is equal to $\sum_{m=n_0 k}^{\infty} a_m z^{\varphi(m)}$, hence $\sum_{m=0}^{\infty} a_m z^{\varphi(m)}$ is in $H(G)$. Clearly Φ defines a linear and multiplicative mapping.

If φ^{-1} is the inverse function of φ then there exist $c_j \in \mathbb{Z}$ and $n_1 \in \mathbb{N}$ such that $\varphi^{-1}(nk+j) = nk + c_j$ for all $n \geq n_1$. Thus φ^{-1} induces a homomorphism Φ^{-1} by the above proof. It is easy to see that $\Phi^{-1} \circ \Phi = \Phi \circ \Phi^{-1} = \mathrm{id}$. ∎

5. Hadamard-isomorphic domains

5.1. THEOREM. *Let G_1 and G_2 be admissible domains and $1 \in G_1^c$. If G_2 is Hadamard-isomorphic to G_1 then $G_1 = G_2$.*

Proof. By Theorem 3.1(b) we know that $1 \in G_2^c$. First assume that $G_1 = \mathbb{D}$. By Theorem 1.3 we obtain $\max\{|z| : z \in G_2\} \leq r_1 = 1$. Since G_2 contains $\mathbb{D}$ we have $G_2 = \mathbb{D}$. For the general case, let k be the cardinality of $\{z \in G_1^c : |z| = 1\}$ and let $a \in G_1^c$. Then $f(z) = z^j/(1-(z/a)^k)$ is in $H(G_1)$. Let $\Phi : H(G_1) \to H(G_2)$ be an isomorphism and, according to Theorem 4.3, let $\varphi(kn+j) = kn + b_j$ for $nk + j \geq n_0$ and put $p(z) := \sum_{n=0}^{n_0-1} \Phi(z^{kn+j}/a^{nk})$. Then

$$\Phi(f) - p(z) = \Phi\left(\sum_{n=n_0}^{\infty} \frac{z^{nk+j}}{a^{nk}}\right) = \sum_{n=n_0}^{\infty} \frac{z^{nk+b_j}}{a^{nk}} = z^{b_j + n_0 k} \frac{1}{1-(z/a)^k}. \tag{19}$$

It follows that $a \in G_2^c$ since otherwise $\Phi(f)$ would have a pole at $z = a$. Hence $G_1^c \subset G_2^c$ and equality follows by symmetry. ∎

5.2. THEOREM. *Let G_2 be an admissible domain and $\mathbb{D}_r$ the open ball of radius $1 \le r \le \infty$. If G_2 is Hadamard-isomorphic to $\mathbb{D}_r$ then $G_2 = \mathbb{D}_r$.*

Proof. The case $r = 1$ follows from Theorem 5.1. Assume that $1 < r < \infty$. Let $\Phi : H(\mathbb{D}_r) \to H(G_2)$ be an isomorphism. By Theorem 1.5 we know that $G_2 \subset \mathbb{D}_r$ and $\lim \varphi(n)/n = 1$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < r$ then $\Phi(f)$ has convergence radius at least r (cf. the proof of Lemma 2.4). Hence each function $\Phi(f)$ can be considered as a function on $\mathbb{D}_r$. Suppose that there exists $z_0 \in \mathbb{D}_r \setminus G_2$. Then $g(z) := 1/(z - z_0) \in H(G_2)$ cannot be of the form $\Phi(f)$ for some $f \in H(G_2)$, a contradiction. Hence $\mathbb{D}_r \subset G_2$.

Let us consider the case $r = \infty$. Then $\Phi(f)$ is entire by Theorem 2.2. Hence G_2 must be equal to $\mathbb{C}$. ∎

Finally, assume that G is an admissible domain with $1 \in G$. Then $H(G)$ is a Fréchet algebra (see [8]) without unit element and G contains the closed unit disk. If G is bounded then $\lim \varphi(n)/n = 1$ by Theorem 1.5. This is also true for unbounded domains as the following result shows.

5.3. THEOREM. *Let G_1 and G_2 be admissible domains with $1 \in G_2$. If $\Phi : H(G_1) \to H(G_2)$ is an isomorphism then $\varphi(n)/n$ converges to 1 and $r_1 = r_2$ where $r_i := \inf\{|w| : w \in G_i^c\}$ for $i = 1, 2$.*

Proof. Let $w \in G_1^c$ with $|w| = r_1$. Then $\Phi(w/(w - z))$ is locally equal to $g(z) := \sum_{n=0}^{\infty} z^{\varphi(n)}/w^n$. Since $g(z)$ is holomorphic on G_2 it has convergence radius at least r_2. Let x with $1 < x < r_2$ be arbitrary. Then there exists $n_0 \in \mathbb{N}$ with $x^{\varphi(n)}/|w|^n \le 1$ for all $n \ge n_0$. It follows that $\varphi(n)/n \le \log|w|/\log x$ for all $n \ge n_0$ and $1 \le \limsup \varphi(n)/n \le \log|w|/\log x$. We infer that $\log x \le \log r_1$ and therefore $r_2 \le r_1$. By symmetry we have $r_1 = r_2$ and now it is easy to see that $\varphi(n)/n$ converges to 1. ∎

Except for the case $G = \mathbb{D}_r$ we have not been able to prove that Hadamard-isomorphic domains are equal if $1 \in G$.

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