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Trace and determinant in Banach algebras

by

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Abstract. We show that the trace and the determinant on a semisimple Banach algebra can be defined in a purely spectral and analytic way and then we obtain many consequences from these new definitions.

1. Introduction. Determinants of infinite matrices were for the first time investigated by the astronomer G. W. Hill in his studies on lunar theory and his ideas were put into a rigorous form by H. Poincaré in 1886. Ten years later H. von Koch refined and generalized Poincaré's results. In 1903, I. Fredholm developed a determinant theory for integral operators. Unlike von Koch, I. Fredholm studied eigenvalues and looked at the analyticity of $\det(I+\lambda M)$. Fredholm's determinant theory is certainly one of the first milestones in the history of functional analysis. In the early fifties A. F. Ruston, T. Leżański and A. Grothendieck almost simultaneously defined determinants for nuclear or integral operators on a Banach space. In the seventies, A. Pietsch developed an axiomatic approach to the determinant of elements of certain operator ideals. In 1978, J. Puhl [16] studied the trace on the socle and nuclear elements of a semisimple Banach algebra, basing his difficult arguments on the standard trace defined for finite-rank linear operators. For more historical information and references on this matter look at [11], Chapters 4 and 5, and [15], 7.5 and 7.6.

The aim of this paper is to show that the trace and determinant on the socle of a Banach algebra can be developed in a purely spectral and analytic way, that is to say internally, without using operators on the algebra. Then we use the analytic properties of the spectrum to prove that the trace and determinant are entire functions and to deduce the basic properties of the trace and determinant in a purely analytic way. The essential ingredient in all these arguments is the fact that the spectrum is an analytic multifunction. So this point of view gives us the possibility of extending almost all the

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results of this paper to more general situations where the spectrum is also analytic, for instance the case of non-associative Jordan–Banach algebras. Using more intricate arguments this was done in [4, 6, 8].

2. Rank and multiplicity. To simplify we shall assume throughout this paper that the Banach algebra A is semisimple with an identity. The *socle* of A, denoted by $\operatorname{Soc} A$, is the sum of all minimal left ideals, or minimal right ideals, of A, if they exist, otherwise it is zero. It is well known that the socle of A is a two-sided ideal of A and that all its elements are algebraic, consequently they have finite spectrum. From Newburgh's theorem ([3], Corollary 3.4.5) it follows in particular that the spectrum function is continuous at every element of the socle. Moreover, this socle is generated by the *minimal projections* of A, that is, the elements $p = p^2$ such that $pAp = \mathbb{C}p$, if they exist. For more general information on the socle see [10], Chapter 4, and [2], Chapter 3, §3. If A is finite-dimensional it coincides with its socle (and conversely by Theorem 5.4.2 of [3]). If A is infinite-dimensional then the socle is a proper two-sided ideal, consequently $0 \in \operatorname{Sp} a$ for every $a \in \operatorname{Soc} A$.

If X is an arbitrary Banach space, then $\mathcal{B}(X)$ contains finite-rank operators, so its socle is non-zero. Another example of Banach algebras with non-zero socle is given by the *scattered algebras*, that is, the algebras for which every element has a finite or countable spectrum (see [3], Theorem 5.7.8). It is not known in general which algebras have a non-zero socle.

The first result of this section was known by the first author since a long time. It first appeared in a slightly different version for Jordan–Banach algebras in [4], Theorem 3.1. It is an improvement of Theorems 2.2 and 3.1 of [14] and Theorem 2.2 of [7]. We denote by # the number of points of a set. The case $m = 0$ of the following theorem includes several characterizations of the radical.

THEOREM 2.1. *Suppose that $a \in A$ and that $m \geq 0$ is an integer. The following properties are equivalent*:

(i) $\#(\operatorname{Sp}(xa) \setminus \{0\}) \leq m$ *for every* $x \in A$,

(ii) $\#(\{t \in \mathbb{C} : 0 \in \operatorname{Sp}(y + ta)\}) \leq m$ *for every y invertible in* A,

(iii) $\bigcap_{t \in F} \operatorname{Sp}(y + ta) \subset \operatorname{Sp} y$ *for every $y \in A$ and every subset F of $\mathbb{C}$ having $m + 1$ non-zero elements.*

Proof. (i)⇒(ii). This is obvious when we notice that $0 \in \operatorname{Sp}(y + ta)$ is equivalent to $-1/t \in \operatorname{Sp}(y^{-1}a)$.

(ii)⇒(iii). Suppose that $\lambda \notin \operatorname{Sp} y$ and λ is in the intersection. This means that $0 \in \operatorname{Sp}(y - \lambda + ta)$ for $t \in F$ and $y - \lambda$ is invertible, so it contradicts (ii) applied to $y - \lambda$.

(iii)⇒(i). This is, conveniently modified, the argument of part (2) (⇐) in the proof of Theorem 2.2 in [7]. ∎

It is easy to see that every element of $\operatorname{Soc} A$ has these three equivalent properties for some integer m. If $a \in \operatorname{Soc} A$, then the sets $A_k = \{x \in A : \#(\operatorname{Sp}(xa) \setminus \{0\}) \leq k\}$ are closed by continuity of the spectrum on the socle. By Baire's theorem there exist a smallest integer m and an open set U in A such that $\#(\operatorname{Sp}(xa) \setminus \{0\}) \leq m$ for $x \in U$. Applying the Scarcity Theorem ([3], Theorem 3.4.25) it is easy to conclude that a has property (i). This argument proves in fact a little more. It implies that if $a \in A$ is such that $\operatorname{Sp}(xa)$ is finite for every $x \in A$, then there exists an integer m such that $\#(\operatorname{Sp}(xa) \setminus \{0\}) \leq m$ for every $x \in A$.

We shall denote by $\mathcal{F}_m$ the set of $a \in A$ satisfying the three equivalent conditions of Theorem 2.1. By property (i) of Theorem 2.1, $\mathcal{F}_0$ is the radical of A, that is, $\{0\}$ when A is supposed to be semisimple. By the previous remark we have $\operatorname{Soc} A \subset \bigcup_{m=0}^{\infty} \mathcal{F}_m$. We shall see later that these two sets coincide.

All of this suggests to define the *rank* of an element a of A to be the smallest integer m such that $a \in \mathcal{F}_m$, if it exists, otherwise the rank is infinite. Consequently,

$$\operatorname{rank}(a) = \sup_{x \in A} \#(\operatorname{Sp}(xa) \setminus \{0\}) \leq \infty.$$

Of course we also have $\operatorname{rank}(a) = \sup_{x \in A} \#(\operatorname{Sp}(ax) \setminus \{0\})$.

Let us give a few elementary properties of the rank:

(a) $\#(\operatorname{Sp} a \setminus \{0\}) \leq \operatorname{rank}(a)$ for $a \in A$,

(b) $\operatorname{rank}(xa) \leq \operatorname{rank}(a)$ and $\operatorname{rank}(ax) \leq \operatorname{rank}(a)$ for $a, x \in A$; moreover, $\operatorname{rank}(ua) = \operatorname{rank}(au) = \operatorname{rank}(a)$ if u is invertible,

(c) let p be a projection of A, then p has rank one if and only if p is minimal,

(d) the rank is lower semicontinuous on $\bigcup_{m=0}^{\infty} \mathcal{F}_m$,

(e) let Φ be an automorphism of A, then $\operatorname{rank}(x) = \operatorname{rank} \Phi(x)$ for every $x \in A$.

The proofs of (a) and (b) are obvious from the definition and the fact that $\operatorname{Sp}(ax)\setminus\{0\} = \operatorname{Sp}(xa)\setminus\{0\}$. We now prove (c). If p is minimal then $pAp = \mathbb{C}p$, consequently $pxp = \lambda p$ for some $\lambda \in \mathbb{C}$, hence $\operatorname{Sp}(xp) = \operatorname{Sp}(pxp) \subset \{0, \lambda\}$. So p has rank one by definition. If p is a projection then $B = pAp$ is a closed semisimple subalgebra of A with identity p. We have $\operatorname{Sp}_B pxp \subset \operatorname{Sp}_A pxp$, because $(pxp - \lambda)q = q(pxp - \lambda) = 1$ implies $(pxp - \lambda p)qp = pq(pxp - \lambda p) = p$ by multiplication by p on the left and right sides. Consequently, if p has rank one we have $\#(\operatorname{Sp}_B pxp \setminus \{0\}) \leq 1$. Associating with $b = pxp$ the element $\chi(b)$ which is the element of $\operatorname{Sp}_B pxp \setminus \{0\}$ if this set is not empty or 0 otherwise, it is possible to prove that B is isomorphic to $\mathbb{C}$, hence p is minimal (see Theorem 2.6 of [4] and [3], Chapter 3, Exercise 21). To prove (d) suppose that there exists a sequence (x_k) of finite-rank elements converging to some finite-rank element x and suppose that $\operatorname{rank}(x) > \underline{\lim}_{n\to\infty} \operatorname{rank}(x_k) = M$. Then $\operatorname{rank}(x) \geq M+1$ and by definition there exists $u \in A$ such that $\#(\operatorname{Sp}(ux) \setminus \{0\}) \geq M+1$. By continuity of the spectrum on the set of finite-rank elements we conclude that $\#(\operatorname{Sp}(ux_k) \setminus \{0\}) \geq M+1$ for k large and this together with (b) leads to a contradiction. Suppose that $a \in \mathcal{F}_m$ and that Φ is an automorphism of A onto itself. Since Φ preserves spectrum we have $\#(\operatorname{Sp} \Phi(x)\Phi(a) \setminus \{0\}) = \#(\operatorname{Sp}(xa) \setminus \{0\}) \leq m$ for all $\Phi(x)$; consequently, $\Phi(a)$ also satisfies property (i) of Theorem 2.1 and (e) is proved.

In the case of $\mathcal{B}(X)$ does this spectral rank coincide with the standard rank? This is clear for the following reasons. Let $m = \dim T(X)$. Then $\dim ST(X) \leq m$, for every $S \in \mathcal{B}(X)$, and $\#(\operatorname{Sp} T \setminus \{0\}) \leq \dim T(X)$ because if $x \neq 0$ there exists a polynomial p of degree $\leq m$ such that $Tp(T)x = 0$, so $0 \in \operatorname{Sp} Tp(T)$. This implies that $\operatorname{Sp} T$ is included in the set of zeroes of $zp(z)$, hence we get the inequality. Using the definition of spectral rank and the preceding remark we obtain $\operatorname{rank}(T) \leq m$. Let $e_1, \ldots, e_m$ be a linear basis of $T(X)$. There exist linearly independent vectors $x_1, \ldots, x_m$ such that $Tx_i = e_i$. By Corollary 4.2.6 of [3] applied to $\mathcal{B}(X)$ there exists $U \in \mathcal{B}(X)$ invertible such that $Ue_i = \frac{1}{i} x_i$ $(i = 1, \ldots, m)$. Consequently, $\operatorname{rank}(T) \geq \#(\operatorname{Sp}(UT) \setminus \{0\}) = \#(\operatorname{Sp}(TU) \setminus \{0\}) \geq m$, because $TUe_i = \frac{1}{i} e_i$ $(i = 1, \ldots, m)$.

If $a \in A$ has finite rank, the set $E(a) = \{x \in A : \#(\operatorname{Sp}(xa) \setminus \{0\}) = \operatorname{rank}(a)\}$ is non-empty by definition of the rank. If $x \in E(a)$ it is clear from properties (a) and (b) that $\operatorname{rank}(a) = \operatorname{rank}(ax) = \#(\operatorname{Sp}(xa) \setminus \{0\})$. This motivates the introduction of *maximal finite-rank elements*, that is, elements $a \in A$ such that $\operatorname{rank}(a) = \#(\operatorname{Sp} a \setminus \{0\})$. We shall see in Theorem 2.8 that they may be written as a linear combination of orthogonal minimal projections.

THEOREM 2.2 (Density of maximal finite-rank elements). *Let $a \in A$ have finite rank. Then $E(a)$ is a dense open subset of A. Consequently, the set of maximal finite-rank elements is dense in the set of finite-rank elements.*

Proof. Let $x_0 \in E(a)$. We set $\alpha_0 = 0$ and denote by $\alpha_1, \ldots, \alpha_m$ the non-zero distinct elements of the spectrum of $\operatorname{Sp}(x_0 a)$, where $m = \operatorname{rank}(a)$. We choose $r > 0$ such that the closed disks $\overline{B}(\alpha_i, r)$ $(i = 0, \ldots, m)$ are disjoint. By upper semicontinuity of the spectrum and Newburgh's theorem ([3], Theorem 3.4.4) there exists $\varepsilon > 0$ such that $\|x - x_0\| < \varepsilon$ implies $\#(\operatorname{Sp}(xa) \cap B(\alpha_i, r)) \geq 1$ $(i = 1, \ldots, m)$. But by hypothesis $\#(\operatorname{Sp}(xa) \setminus \{0\}) \leq \operatorname{rank}(a) = m$, so necessarily $\#(\operatorname{Sp}(xa) \setminus \{0\}) = m$ for $\|x - x_0\| < \varepsilon$. This proves that $E(a)$ is open. If $E(a)$ is not dense in A there exists an open set V containing no point of $E(a)$; consequently, $\#(\operatorname{Sp}(xa) \setminus \{0\}) \leq m - 1$ on V. Then the Scarcity Theorem ([3], Theorem 3.4.25) applied to $\lambda \to \operatorname{Sp}((x_1 + \lambda(x - x_1))a)$ where x_1 is fixed in V and x arbitrary in A implies that $\#(\operatorname{Sp}(xa) \setminus \{0\}) \leq m - 1$ for every $x \in A$, so $\operatorname{rank}(a) \leq m - 1$ and this is absurd. Consequently, $E(a)$ is dense in A. From this, there exists a sequence (x_k) of elements of $E(a)$ converging to the identity. So the elements $x_k a$ are maximal finite-rank elements and they converge to a. ■

This theorem is interesting because if we want to prove some property for finite-rank elements it is sometimes enough to prove it for maximal finite-rank elements (which are very particular by Theorem 2.8) and then to extend it by continuity to all finite-rank elements (see Theorem 2.11 and Theorem 2.14). These maximal finite-rank elements play the rôle of diagonal matrices in the case of matrices.

The next theorem was first given and proved in the situation of Jordan–Banach algebras [4].

THEOREM 2.3 (Scarcity theorem for the rank). *Let f be an analytic function from a domain D of $\mathbb{C}$ into A. Then either the set of λ for which the rank of $f(\lambda)$ is finite has zero capacity or there exist an integer N and a closed discrete subset E of D such that $\operatorname{rank} f(\lambda) = N$ on $D \setminus E$ and $\operatorname{rank} f(\lambda) < N$ on E.*

Proof. It is identical to the proof of Theorem 3.4 of [4] replacing everywhere $U_x(a)$ by xa. ■

THEOREM 2.4. *Let $a \in A$ have finite rank. Let Γ be an oriented regular contour not intersecting $\operatorname{Sp} a$ and denote by Δ_0, Δ_1 respectively its interior and its exterior. By upper semicontinuity of the spectrum there exists a ball U in A, centred at 1, such that $\operatorname{Sp}(xa) \cap \Gamma = \emptyset$ for $x \in U$. Then for $x, y \in U \cap E(a)$ we have*

$$\#(\operatorname{Sp}(xa) \cap \Delta_i) = \#(\operatorname{Sp}(ya) \cap \Delta_i)$$

for $i = 0, 1$.

Proof. By Theorem 2.2, $U \cap E(a)$ is non-empty, so let $x, y \in U \cap E(a)$. Let D be the convex domain of $\mathbb{C}$ containing 0 and 1 such that $\lambda \in D$ is equivalent to $\lambda x + (1 - \lambda)y \in U$. Taking $f(\lambda) = (\lambda x + (1 - \lambda)y)a$, which is also of finite rank by property (b), and applying the Localization Principle ([3], Theorem 7.1.5) to the analytic multifunctions $\lambda \to \operatorname{Sp} f(\lambda) \cap \Delta_i$, by the Scarcity Theorem we conclude that there exist two integers N_0, N_1 and two closed discrete subsets F_0, F_1 of D such that

$$(1)\qquad \begin{cases} \#(\operatorname{Sp} f(\lambda) \cap \Delta_i) = N_i & \text{for } \lambda \in D \setminus F_i, \\ \#(\operatorname{Sp} f(\lambda) \cap \Delta_i) < N_i & \text{for } \lambda \in F_i \ (i = 0, 1). \end{cases}$$

To prove the theorem it is enough to show that $0, 1 \notin F_0 \cup F_1$. Suppose for instance that $0 \in F_0 \cup F_1$. For $\lambda \notin F_0 \cup F_1$ we have

$$\# \operatorname{Sp}(ya) = \#(\operatorname{Sp}(ya) \cap \Delta_0) + \#(\operatorname{Sp}(ya) \cap \Delta_1) < N_0 + N_1 = \# \operatorname{Sp} f(\lambda).$$

If $0 \in \operatorname{Sp}(ya)$ the relation (1) with $y \in E(a)$ implies $m + 1 = \# \operatorname{Sp}(ya) < N_0 + N_1 = \# \operatorname{Sp} f(\lambda) \leq m + 1$, from the definition of the rank, where $m = \operatorname{rank}(a)$, so this is absurd. If $0 \notin \operatorname{Sp}(ya)$ by upper semicontinuity of the spectrum we can choose $\lambda \notin F_0 \cup F_1$ such that $0 \notin \operatorname{Sp} f(\lambda)$. Relation (1) implies

$$m = \# \operatorname{Sp}(ya) < N_0 + N_1 = \# \operatorname{Sp} f(\lambda) = m$$

and this is also absurd. By a similar argument the case $1 \in F_0 \cup F_1$ also gives a contradiction, so the theorem is proved. ∎

The independent number $\#(\operatorname{Sp}(xa) \cap \Delta_0)$ is denoted by $m(\Gamma, a)$ and is called the *multiplicity of a associated with Γ*. It is independent of Γ if the corresponding spectral points of a in Δ_0, Δ_1 do not change. If α is isolated in $\operatorname{Sp} a$ we define $m(\alpha, a)$, the *multiplicity of a at α*, by $m(\Gamma, a)$ where Γ is a small circle centred at α and isolating α from the rest of the spectrum. We have $m(\alpha, a) \geq 1$. It is not difficult to see that

$$m(\Gamma, a) = \sum_{\alpha \in \operatorname{Sp} a \cap \Delta_0} m(\alpha, a).$$

If a is a maximal finite-rank element then necessarily we must have $m(\alpha, a) = 1$ for every $\alpha \neq 0$, $\alpha \in \operatorname{Sp} a$, because otherwise there would be some $x \in U \cap E(a)$ such that $\#(\operatorname{Sp}(xa) \setminus \{0\}) \geq m + 1$ where $m = \operatorname{rank}(a)$ and this would violate the definition of the rank.

If we take for Γ a contour surrounding all the spectrum of a we obtain

$$\sum_{\alpha \in \operatorname{Sp} a} m(\alpha, a) = \begin{cases} 1 + \operatorname{rank}(a) & \text{if } 0 \in \operatorname{Sp} a, \\ \operatorname{rank}(a) & \text{if } 0 \notin \operatorname{Sp} a. \end{cases} \tag{2}$$

In fact, we shall see that the multiplicity of a finite-rank element a at $\alpha \neq 0$ is equal to the rank of the *Riesz projection* associated with a and α.

Let $\alpha \in \mathbb{C}$ and Γ be a small curve isolating α from the rest of the spectrum of a. By definition the Riesz projection is

$$p(\alpha, a) = \frac{1}{2\pi i} \int_\Gamma (\lambda - a)^{-1} \, d\lambda. \tag{3}$$

Obviously it is zero if $\alpha \notin \operatorname{Sp} a$. The Holomorphic Functional Calculus implies that the $p(\alpha, a)$ corresponding to different α have zero product and their sum is one.

The identity $(\lambda - a)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda} a (\lambda - a)^{-1}$ implies that

$$p(\alpha, a) = \frac{a}{2\pi i} \int_\Gamma \frac{1}{\lambda} (\lambda - a)^{-1} \, d\lambda. \tag{4}$$

LEMMA 2.5 (J. Zemánek). *Let p, q be two non-zero projections of A such that $\|p - q\| < 1$. Then there exists an $x \in A$ such that*

$$q = e^{-x} p e^{x}.$$

For the proof, see [21]. For more details and some improvements, see [19]. As a consequence, from property (b), we have $\operatorname{rank}(p) = \operatorname{rank}(q)$.

THEOREM 2.6. *Let $a \in A$ have finite rank and let $\lambda_1, \ldots, \lambda_n$ be non-zero distinct elements of its spectrum. If p denotes the Riesz projection associated with a and $\lambda_1, \ldots, \lambda_n$, that is, $p(\lambda_1, a) + \ldots + p(\lambda_n, a)$, then*

$$\operatorname{rank}(p) = m(\lambda_1, a) + \ldots + m(\lambda_n, a).$$

Proof. By Theorem 2.4 and definition of multiplicity there exists $\varepsilon_0 > 0$ such that

$$m(\lambda_1, a) + \ldots + m(\lambda_n, a) = \max_{\|x-1\|<\varepsilon} \#(\operatorname{Sp}(xa) \cap \Delta_0)$$

for $0 < \varepsilon < \varepsilon_0$, where Δ_0 is the domain limited by a regular contour Γ separating the points $\lambda_1, \ldots, \lambda_n$ from the rest of the spectrum and 0. By [3], Theorem 3.3.4, we have

$$m(\lambda_1, a) + \ldots + m(\lambda_n, a) = \max_{\|x-1\|<\varepsilon} \#(\operatorname{Sp}(xaq) \setminus \{0\}),$$

where q denotes the Riesz projection associated with xa and Γ. Consequently, by property (b) of the rank, we have $m(\lambda_1, a) + \ldots + m(\lambda_n, a) \leq \operatorname{rank}(q)$. If we choose $\varepsilon > 0$ small enough, from the definition of the Riesz projections, it is easy to see that $\|q - p\| < 1$ so, by Lemma 2.5 and the following remark, we have

$$m(\lambda_1, a) + \ldots + m(\lambda_n, a) \leq \operatorname{rank}(q) = \operatorname{rank}(p).$$

We now prove the converse inequality. We choose x arbitrarily near to 1 in such a way that $m(\lambda_1, a) + \ldots + m(\lambda_n, a) = \#(\operatorname{Sp}(xa) \cap \Delta_0)$ and we set $r = \operatorname{rank}(a)$, $m = m(\lambda_1, a) + \ldots + m(\lambda_n, a)$, $b = xa$. The spectrum of b has m points in Δ_0 and $r - m$ non-zero points in Δ_1 (the exterior of Γ). Denote by $q_1, \ldots, q_{r-m}$ the $r - m$ Riesz projections associated with b and these $r - m$ non-zero points. These projections are orthogonal to q by the Holomorphic Functional Calculus. Suppose $\operatorname{rank}(q) > m$. Then there exists $y \in A$ such that $\#(\operatorname{Sp}(yq) \setminus \{0\}) > m$. Using the Holomorphic Functional Calculus we can create $m+1$ orthogonal projections $p_1, \ldots, p_{m+1}$ which are in $Ayq \subset Aq \subset Ab$ by formula (4). Moreover, $p_i q_j = 0$ because q is orthogonal to the q_j. So Ab contains $p_1, \ldots, p_{m+1}, q_1, \ldots, q_{r-m}$; consequently, we can consider

$$zb = p_1 + 2p_2 + \ldots + (m+1)p_{m+1} + (m+2)q_1 + \ldots + (r+1)q_{r-m},$$

whose spectrum contains $r+1$ non-zero points, and this contradicts the definition of the rank of b, because $\operatorname{rank}(b) = \operatorname{rank}(a) = r$, by property (b). Hence $\operatorname{rank}(q) = \operatorname{rank}(p) \le m$. So the theorem is proved. ∎

COROLLARY 2.7. *Let $p \neq 0, 1$ be a finite-rank projection. Then $m(0,p) = 1$ and $\operatorname{rank}(p) = m(1,p)$.*

Proof. We have $1 + \operatorname{rank}(p) = m(0,p) + m(1,p)$ because $\operatorname{Sp} p = \{0,1\}$. Then we apply the previous theorem to $a = p$ and $\lambda_1 = 1$, noticing that the Riesz projection associated with p and 1 is p itself, because by (3),

$$\frac{1}{2\pi i}\int_\Gamma (\lambda - p)^{-1}\,d\lambda = \frac{1}{2\pi i}\int_\Gamma \left(\frac{p}{\lambda-1} + \frac{1-p}{\lambda}\right) d\lambda = p. \ ∎$$

Using an argument very similar to the one used at the end of the proof of Theorem 2.6, we can deduce the precise structure of maximal finite-rank elements.

THEOREM 2.8 (Diagonalization theorem). *Let $a \in A$ be a non-zero maximal finite-rank element. Denote by $\lambda_1, \ldots, \lambda_n$ its non-zero distinct spectral values. Then there exist n orthogonal minimal projections $p_1, \ldots, p_n$ such that $a = \lambda_1 p_1 + \ldots + \lambda_n p_n$.*

Proof. We consider the Riesz projections $p_1, \ldots, p_n$ associated with a and the λ_i. They are orthogonal and commute with a. We now prove that the p_i have rank one, so they are minimal by property (c). By Theorem 2.6 we have $\operatorname{rank}(p_i) = m(\alpha_i, a)$, which is one, by the remark following the definition of multiplicity, because a is maximal. Let $p = p_1 + \ldots + p_n$. We claim that $a = ap$. Suppose $a \neq ap$. Because A is semisimple there exists an $x \in A$ such that $\operatorname{Sp} x(a - ap) \neq \{0\}$. Note that the spectrum of $x(a-ap)$ is finite and let q be the Riesz projection associated with one of its non-zero spectral points. Then $q \in Aa$ because a and p commute and q is of the form $y(1-p)$ so $qp_i = 0$ for $i = 1, \ldots, n$. If we consider the element $c = p_1 + 2p_2 + \ldots + np_n + (n+1)q$ the relations $(c-k)p_k = 0$ for $k = 1, \ldots, n$ and $q(c - (n+1)q) = 0$ imply that $\{1, \ldots, n+1\} \subset \operatorname{Sp} c$, which violates the fact that every element of Aa has at most n non-zero points because $\operatorname{rank}(a) = n$. So $a = ap = p_1 a p_1 + \ldots + p_n a p_n$. By minimality of the p_i we have $p_i a p_i = \alpha_i p_i$ for some α_i. But the spectral mapping theorem implies that $\lambda_i = \alpha_i$. ∎

COROLLARY 2.9. *The socle coincides with the set of finite-rank elements.*

Proof. By the remark following Theorem 2.1 we only have to prove that finite-rank elements are in the socle. So suppose $\operatorname{rank}(a)$ finite. By Theorem 2.2 there exists u invertible such that ua is maximal. By Theorem 2.8, ua is a linear combination of minimal projections, so it is in the socle. Consequently, the same is true for $a = u^{-1}(ua)$. ∎

This last argument also proves that every element of the socle can be written as a sum of rank one elements.

Theorem 2.8 implies easily the known result that every element of the socle is *von Neumann regular.*

COROLLARY 2.10. *If $a \in \operatorname{Soc} A$ there exists $x \in \operatorname{Soc} A$ such that $a = axa$.*

Proof. By Theorem 2.2, let u be invertible such that ua is maximal. We have $ua = \lambda_1 p_1 + \ldots + \lambda_n p_n$, where the p_i are orthogonal minimal projections and the λ_i are non-zero. Then

$$ua = (\lambda_1 p_1 + \ldots + \lambda_n p_a)\left(\frac{p_1}{\lambda_1} + \ldots + \frac{p_n}{\lambda_n}\right)(\lambda_1 p_1 + \ldots + \lambda_n p_n)$$
$$= ua\left(\frac{p_1}{\lambda_1} + \ldots + \frac{p_n}{\lambda_n}\right)ua.$$

So $a = axa$ where $x = (p_1/\lambda_1 + \ldots + p_n/\lambda_n)u \in \operatorname{Soc} A$. ∎

For $x \in A$ denote by $C(x)$ the subalgebra (without identity) generated by x and by $\pi(x)$ the supremum of the number of orthogonal projections in $C(x)$. In a rather forgotten paper [13], H. Kraljević and K. Veselić introduced a concept of rank by taking for $a \in A$, $\sup_{x\in A} \dim C(xa) \le \infty$. In fact, this rank coincides with our spectral rank.

THEOREM 2.11. *For $a \in A$ we have $\operatorname{rank}(a) = \sup_{x\in A} \dim C(xa)$. If $a \in \operatorname{Soc} A$ the preceding number coincides with $\sup_{x\in A} \pi(xa)$.*

Proof. If $\sup_{x\in A} \dim C(xa) \le m < \infty$ then $\#\operatorname{Sp}(xa) \le m$ for all $x \in A$, and consequently $\operatorname{rank}(a) < \infty$. So it is enough to prove the theorem on the socle. If $a \in \operatorname{Soc} A$ we have $\dim C(a) \ge \pi(a) \ge \#(\operatorname{Sp} a \setminus \{0\})$, because the Riesz projections are orthogonal. So it is sufficient to prove that $\sup_{x\in A} \dim C(xa) = \operatorname{rank}(a)$. By Theorem 2.2 there exists a set of invertible elements u such that ua is maximal. By Theorem 2.8 we have $ua = \lambda_1 p_1 + \ldots + \lambda_n p_n$ where $n = \operatorname{rank}(ua) = \operatorname{rank}(a)$ and where the p_i are minimal and the λ_i non-zero. If ϕ is a polynomial without constant coefficient we have $\phi(ua) = \phi(\lambda_1)p_1 + \ldots + \phi(\lambda_n)p_n$. Given $\mu_1, \ldots, \mu_n \in \mathbb{C}$ it is always possible to find, by interpolation, such a polynomial ϕ satisfying $\phi(\lambda_i) = \alpha_i$ $(i = 1, \ldots, n)$; consequently, $C(ua) = \mathbb{C}p_1 + \ldots + \mathbb{C}p_n$, hence $\dim C(ua) = \operatorname{rank}(a)$. Suppose now that for some $x \in A$ we have $\dim C(xa) \ge n+1$. By definition of $C(xa)$, we can find $n+1$ polynomials $\phi_1, \ldots, \phi_{n+1}$ without constant coefficient such that the $\phi_i(xa)$ are linearly independent. By Lemma 2.2.1 of [3] there exists $\varepsilon > 0$ such that $z_1, \ldots, z_{n+1} \in A$ with $\|z_1 - \phi_1(xa)\| < \varepsilon, \ldots, \|z_{n+1} - \phi_{n+1}(xa)\| < \varepsilon$ implies

$z_1, \ldots, z_{n+1}$ linearly independent. By the density of the set $E(xa)$ there exists u near the identity and in $E(xa)$ such that $z_i = \phi_i(uxa)$ satisfy the previous inequalities. But by the previous part of the proof we have $\dim C(uxa) = \operatorname{rank}(xa) \le \operatorname{rank}(a) = n$, and $z_1, \ldots, z_{n+1} \in C(uxa)$ are linearly independent, so we get a contradiction. Hence the theorem is proved. ∎

Several authors, like K. Vala, K. Ylinen, J. C. Alexander, introduced the concept of *finite-dimensional* or *finite-rank element*. This means that the rank of the bounded linear operator $\widehat{a} \in B(A)$ defined by $\widehat{a} : x \to axa$ is finite (for more details see [9], Chapter F). This rank is bad for two reasons: first, it does not coincide with the standard rank, even in the case of matrices, as easily seen with a 2×2 invertible matrix m for which the standard rank is 2 and $\dim \widehat{m}(M_2(\mathbb{C})) = 4$; secondly, it is not subadditive as the standard rank is (see Theorem 2.14). Nevertheless these two ranks are related as seen in Theorem 2.12 which is an improvement of property (c) of Theorem F.2.4 of [9] (which is formulated only in the primitive case) and of a lemma of J. C. Alexander [1], also given as Lemma 4, p. 81, in [2].

THEOREM 2.12. *For $a \in A$ we have*

$$\operatorname{rank}(a) \le \dim aAa \le (\operatorname{rank}(a))^2.$$

Proof. If $\dim aAa$ is finite, then for every $x \in A$ we have $\dim(xa)A(xa) = \dim L_x(aA(xa)) \le \dim L_x(aAa) \le \dim aAa < \infty$, where L_x denotes the left multiplication by x. So xa is algebraic and then $\#\operatorname{Sp}(xa)$ is finite for $x \in A$, hence a has finite rank by Theorem 2.1. Consequently, we can suppose that $a \in \operatorname{Soc} A$, because otherwise the two inequalities are obvious. If $a \in \operatorname{Soc} A$, by Corollary 2.10, we have $C(a) \subset aAa$ so by Theorem 2.11 we have

$$\operatorname{rank}(a) = \sup_{x\in A} \dim C(xa) \le \sup_{x \in A} \dim(xa)A(xa) \le \dim aAa.$$

By Theorem 2.2, there exists u invertible such that $ua = \lambda_1 p_1 + \ldots + \lambda_n p_n$ where the p_i are orthogonal minimal projections and the λ_i are non-zero. But $\dim(ua)A(ua) = \dim L_u(aAa) = \dim aAa$, because L_u is invertible and $Au = A$. Moreover, $(ua)A(ua) = \sum_{i,j=1}^n p_i A p_j$. Because p_i is minimal, $\dim p_i A p_i = 1$. If $i \ne j$ either $p_i A p_j = 0$ or $\dim p_i A p_j = 1$. This can be seen using the second part of the proof of Lemma 4, p. 81 in [2], or the end of the argument of the proof of the theorem in [12] (page 22). Consequently, $\dim aAa \le n^2 = (\operatorname{rank}(ua))^2 = (\operatorname{rank}(a))^2$. ∎

We now arrive at the most important result of this section. This theorem was first proved in the situation of Jordan–Banach algebras ([4], Theorem 3.9), with an intricate proof using a geometrical characterization of algebraic varieties of $\mathbb{C}^2$ [5]. The argument depends on deep considerations on complex analysis in one and two variables and analytic multifunctions. Of course

it could be possible to simplify slightly this proof in the case of Banach algebras, but we now give a truly elementary proof which is only based on Rouché's theorem.

LEMMA 2.13. *Let $a = \lambda_1 p_1 + \ldots + \lambda_n p_n$ where $\lambda_1, \ldots, \lambda_n$ are distinct non-zero numbers and $p_1, \ldots, p_n$ are orthogonal projections. Let b be of rank one. Then there exists an entire function h, depending only on a and b, such that for $\lambda \ne 0$, $z \in \operatorname{Sp}(a + \lambda b) \setminus \{0, \lambda_1, \ldots, \lambda_n\}$ is equivalent to $z \ne 0, \lambda_1, \ldots, \lambda_n$ and $z(z - \lambda_1) \ldots (z - \lambda_n) - \lambda h(z) = 0$. In particular, there exists at most one point in $\operatorname{Sp}(a + \lambda b) \setminus \operatorname{Sp} a$ near zero when λ is small and non-zero.*

Proof. For $z \notin \operatorname{Sp} a$ we have the identity

$$z - (a + \lambda b) = (z - a)(1 - \lambda(z - a)^{-1} b).$$

So for $\lambda \ne 0$, $z \in \operatorname{Sp}(a + \lambda b) \setminus \operatorname{Sp} a$ is equivalent to $z \notin \operatorname{Sp} a$ and $1/\lambda \in \operatorname{Sp}((z - a)^{-1} b)$. But for $z \ne 0, \lambda_1, \ldots, \lambda_n$ we have

$$(z - a)^{-1} b = \frac{p_1 b}{z - \lambda_1} + \ldots + \frac{p_n b}{z - \lambda_n} + \frac{1}{z}(1 - p_1) \ldots (1 - p_n) b.$$

Consequently, for $\lambda \ne 0$, $z \ne 0, \lambda_1, \ldots, \lambda_n$, the property $z \in \operatorname{Sp}(a + \lambda b)$ is equivalent to

$$\frac{z}{\lambda}(z - \lambda_1) \ldots (z - \lambda_n) \in \operatorname{Sp} q(z) b \tag{5}$$

where

$$q(z) = z(z - \lambda_2) \ldots (z - \lambda_n) p_1 + \ldots + z(z - \lambda_1) \ldots (z - \lambda_{n-1}) p_n + (z - \lambda_1) \ldots (z - \lambda_n)(1 - p_1) \ldots (1 - p_n).$$

For $z \in \mathbb{C}$ we have $q(z) b \in \mathcal{F}_1$, so $\#(\operatorname{Sp} q(z) b \setminus \{0\}) \le 1$. By Theorem 3.4.17 of [3] there exists h entire such that $\operatorname{Sp} q(z) b = \{0, h(z)\}$ for every $z \in \mathbb{C}$. Consequently, the first part of the lemma is proved.

Let $\varepsilon > 0$ be such that the closed disks $\overline{B}(0, \varepsilon)$, $\overline{B}(\lambda_i, \varepsilon)$ are disjoint ($i = 1, \ldots, n$). By Newburgh's theorem there exists $\delta_1 > 0$ such that $|\lambda| < \delta_1$ implies

$$\begin{cases} \operatorname{Sp}(a + \lambda b) \subset B(0, \varepsilon) \cup B(\lambda_1, \varepsilon) \cup \ldots \cup B(\lambda_n, \varepsilon), \\ \#(\operatorname{Sp}(a + \lambda b) \cap B(\lambda_i, \varepsilon)) \ge 1 \text{ for } i = 1, \ldots, n. \end{cases} \tag{6}$$

Let Γ be the boundary of $B(0, \varepsilon)$ and let $p(z) = z(z - \lambda_1) \ldots (z - \lambda_n)$. There exists δ such that $0 < \delta < \delta_1$ and

$$|p(z) - (p(z) - \lambda h(z))| < |p(z)|$$

for $z \in \Gamma$ and $|\lambda| < \delta$. By Rouché's theorem (see for instance [17], p. 242) the number of zeros of $p(z) - \lambda h(z)$ on $B(0, \varepsilon)$ is the same as the number of zeros of $p(z)$ on $B(0, \varepsilon)$, which is one. Consequently, for $0 < |\lambda| < \delta$ we have at most one point of $\operatorname{Sp}(a + \lambda b) \setminus \operatorname{Sp} a$ in $B(0, \varepsilon)$ because $\operatorname{Sp} a \subset \{0, \lambda_1, \ldots, \lambda_n\}$. ∎

THEOREM 2.14. *If* $a, b \in A$ *then* $\operatorname{rank}(a+b) \le \operatorname{rank}(a) + \operatorname{rank}(b)$.

Proof. By Corollary 2.9, we can suppose that $a, b \in \operatorname{Soc} A$, because otherwise the inequality is obvious.

First step. We suppose that a is maximal of rank n and that $\operatorname{rank}(b) = 1$. We prove that $\#(\operatorname{Sp}(a+b) \setminus \{0\}) \le \operatorname{rank}(a) + 1$. As in Lemma 2.13, we choose $\varepsilon, \delta > 0$ such that $\operatorname{Sp}(a + \lambda b)$ has at most one non-zero point in $B(0, \varepsilon)$. We consider the diagonalization obtained in Theorem 2.8, and prove that for $|\lambda| < \delta$ we have

$$\#(\operatorname{Sp}(a+\lambda b) \cap B(\lambda_i, \varepsilon)) = 1.$$

Denote by $p_i(\lambda)$ the Riesz projection associated with $a+\lambda b$ and the boundary of $B(\lambda_i, \varepsilon)$. These $p_i(\lambda)$ are analytically connected to p_i, the Riesz projection associated with a and λ_i. Consequently, by Lemma 2.5 and the following remark we have $\operatorname{rank}(p_i(\lambda)) = \operatorname{rank}(p_i) = 1$ $(i = 1, \ldots, n)$. By Theorem 2.6 we have

$$\operatorname{rank}(p_i(\lambda)) = \sum_{\substack{\alpha \in \operatorname{Sp}(a+\lambda b) \\ \alpha \in B(\lambda_i, \varepsilon)}} m(\alpha, a + \lambda b),$$

moreover $m(\alpha, a + \lambda b) \ge 1$, so this proves that there is only one point of multiplicity one in $\operatorname{Sp}(a + \lambda b) \cap B(\lambda_i, \varepsilon)$. Consequently, for $|\lambda| < \delta$ we must have $\# \operatorname{Sp}(a + \lambda b) \setminus \{0\}) \le n+1$. By the Scarcity Theorem ([3], Theorem 3.4.25) we have $\# \operatorname{Sp}(a + \lambda b) \le n + 2$ for all $\lambda \in \mathbb{C}$. The argument given in the proof of Theorem 7.3.5 of [3] implies that the $(n+2)$th symmetric function of the spectral values of $a+\lambda b$, that is, their product counting their multiplicities, is entire. By the Identity Principle either it is identically zero on $\mathbb{C}$ in which case $(\# \operatorname{Sp}(a + \lambda b) \setminus \{0\}) \le n + 1$ on $\mathbb{C}$, or it vanishes on a discrete set so restricting δ if necessary we can suppose that $\# \operatorname{Sp}(a+\lambda b) \le n+1$ on $0 < |\lambda| < \delta$ and then the same is true on all $\mathbb{C}$. Consequently, the assertion is proved taking $\lambda = 1$. (Remark: it is possible to eliminate these last analytic arguments considering the two cases where the dimension of A is finite or infinite.)

Second step. We suppose that $\operatorname{rank}(b) = 1$ and $a \in \operatorname{Soc} A$. We prove that $\#(\operatorname{Sp}(a+b) \setminus \{0\}) \le \operatorname{rank}(a) + 1$. By Theorem 2.2 we know that there exists a sequence (u_n) of invertible elements converging to the identity such that $u_n \in E(a)$. The elements $u_n a$ are maximal of rank equal to $\operatorname{rank}(a)$. So by the first step we have $\#(\operatorname{Sp}(u_n a + b) \setminus \{0\}) \le \operatorname{rank}(a) + 1$. But the spectrum function is continuous on the socle, by Newburgh's theorem ([3], Corollary 3.4.5), and $(u_n a)$ converges to a; consequently, we have proved the assertion.

Third step. We suppose that $\operatorname{rank}(b) = 1$ and $a \in \operatorname{Soc} A$. We prove that $\operatorname{rank}(a+b) \le \operatorname{rank}(a) + 1$. Let $x \in A$ be arbitrary. If $xb = 0$ it is obvious that $\#(\operatorname{Sp}(x(a+b)) \setminus \{0\}) \le \operatorname{rank}(a) \le \operatorname{rank}(a) + 1$. So suppose that $\operatorname{rank}(xb) = 1$. By the second step applied to xa and xb we obtain $\#(\operatorname{Sp}(x(a+b)) \setminus \{0\}) \le \operatorname{rank}(xa) + 1 \le \operatorname{rank}(a) + 1$, so the assertion is proved.

Fourth step. We prove the theorem by induction. By the previous step it is true for $\operatorname{rank}(b) = 1$. So suppose it is true for $\operatorname{rank}(b) = m$ and we prove it at the order $m+1$. Let $b \in \operatorname{Soc} A$ such that $\operatorname{rank}(b) = m + 1$. Multiplying b by some invertible element u we can suppose that ub is maximal of rank $m+1$ so by Theorem 2.8 it can be written as $\alpha_1 p_1 + \ldots + \alpha_{m+1} p_{m+1}$ where the α_i are non-zero and p_i are orthogonal minimal projections. Then $ua + ub = (ua + \alpha_1 p_1 + \ldots + \alpha_m p_m) + \alpha_{m+1} p_{m+1}$. We have $\operatorname{rank}(\alpha_1 p_1 + \ldots + \alpha_m p_m) \le m$ by the third step repeated m times, so by the hypothesis of induction we have $\operatorname{rank}(ua + \alpha_1 p_1 + \ldots + \alpha_m p_m) \le \operatorname{rank}(ua) + m = \operatorname{rank}(a) + m$. Applying again the third step we obtain

$$\operatorname{rank}(a+b) = \operatorname{rank}(ua + ub) \le \operatorname{rank}(a) + m + 1 = \operatorname{rank}(a) + \operatorname{rank}(b).$$

So the theorem is proved. ∎

Applying Theorems 2.6 and 2.14 we can obtain a new property of the rank. The next lemma is part of the folklore.

LEMMA 2.15. *Let* $a, b \in A$ *be such that* $ab = ba = 0$. *Then* $\operatorname{Sp}(a+b) \setminus \{0\} = (\operatorname{Sp} a \setminus \{0\}) \cup (\operatorname{Sp} b \setminus \{0\})$. *Moreover, if* $\lambda_0 \ne 0$ *is isolated in* $\operatorname{Sp}(a+b)$ *then*

$$p(\lambda_0, a+b) = p(\lambda_0, a) + p(\lambda_0, b).$$

Proof. For $\lambda \ne 0$ we have

$$\lambda - (a+b) = \frac{1}{\lambda}(\lambda - a)(\lambda - b) = \frac{1}{\lambda}(\lambda - b)(\lambda - a). \tag{7}$$

So it is easy to see from (7) that $\lambda - (a+b)$ is invertible if and only if both $\lambda - a$ and $\lambda - b$ are invertible.

Let Γ be a circle centred at λ_0 which separates λ_0 from 0 and the rest of the spectrum of $a+b$. If $\lambda \in \Gamma$ by (7) we have $\lambda \ne 0$, $\lambda - a$ and $\lambda - b$ invertible. Because $b = (\lambda - a)\frac{b}{\lambda}$ we have

$$(\lambda - a)^{-1} b = \frac{b}{\lambda} \quad \text{on } \Gamma. \tag{8}$$

Moreover, we have

$$\begin{aligned} (\lambda - (a+b))^{-1} &= \lambda(\lambda - b)^{-1}(\lambda - a)^{-1} \\ &= (\lambda - a)^{-1} + [\lambda(\lambda - b)^{-1} - 1](\lambda - a)^{-1} \\ &= (\lambda - a)^{-1} + b(\lambda - b)^{-1}(\lambda - a)^{-1} \\ &= (\lambda - a)^{-1} + (\lambda - b)^{-1}(\lambda - a)^{-1} b \\ &= (\lambda - a)^{-1} + \frac{b}{\lambda}(\lambda - b)^{-1}. \end{aligned} \tag{9}$$

So integrating this quantity on Γ multiplied by $1/(2\pi i)$ we get

$$(10)\qquad p(\lambda_0, a+b) = p(\lambda_0, a) + \frac{b}{2\pi i}\int_\Gamma \frac{1}{\lambda}(\lambda - b)^{-1}\,d\lambda = p(\lambda_0, a) + p(\lambda_0, b),$$

by formula (4) applied to b. ■

THEOREM 2.16. *Let $p_1, \ldots, p_n$ be orthogonal projections of finite rank. Given non-zero numbers $\alpha_1, \ldots, \alpha_n$ we have*

$$\operatorname{rank}(\alpha_1 p_1 + \ldots + \alpha_n p_n) = \operatorname{rank}(p_1) + \ldots + \operatorname{rank}(p_n).$$

Proof. Without loss of generality we may suppose the projections to be different from zero.

First step. We prove that

$$\operatorname{rank}(\alpha_1 p_1 + \ldots + \alpha_n p_n) = \operatorname{rank}(p_1 + 2p_2 + \ldots + np_n).$$

This comes immediately from the two identities

$$(11)\qquad \begin{cases} p_1 + 2p_2 + \ldots + np_n = (\alpha_1 p_1 + \ldots + \alpha_n p_n) \\ \qquad\qquad \times \left(\dfrac{p_1}{\alpha_1} + \dfrac{2p_2}{\alpha_2} + \ldots + \dfrac{np_n}{\alpha_n}\right) \\ \alpha_1 p_1 + \ldots + \alpha_n p_n = (p_1 + 2p_2 + \ldots + np_n) \\ \qquad\qquad \times \left(\alpha_1 p_1 + \dfrac{\alpha_2}{2} p_2 + \ldots + \dfrac{\alpha_n}{n} p_n\right) \end{cases}$$

and property (b) of the rank.

Second step. Using induction on n it is easy to prove from Lemma 2.15 that if $x_1, \ldots, x_n$ are orthogonal elements then

$$(12)\qquad \begin{cases} \operatorname{Sp}(x_1 + \ldots + x_n) \setminus \{0\} = \bigcup\limits_{i=1}^{n} (\operatorname{Sp} x_i \setminus \{0\}), \\ p(\lambda_0, x_1 + \ldots + x_n) = \sum\limits_{i=1}^{n} p(\lambda_0, x_i), \end{cases}$$

where $\lambda_0 \in \operatorname{Sp}(x_1 + \ldots + x_n)$, $\lambda_0 \neq 0$. Setting $u = p_1 + 2p_2 + \ldots + np_n$ we have $\operatorname{Sp} u \setminus \{0\} = \{1, \ldots, n\}$ and hence

$$\operatorname{rank}(u) \geq m(1, u) + m(2, u) + \ldots + m(n, u) = \operatorname{rank} p(1, u) + \ldots + \operatorname{rank} p(n, u),$$

by Theorem 2.6 applied to u.

Taking $x_k = kp_k$, all the $\operatorname{Sp} x_k \setminus \{0\}$ are disjoint, and consequently by (10) we have

$$p(k, u) = p(k, kp_k) = p_k,$$

because all the terms $p(k, jp_j)$ are 0 for $j \neq k$. So we have

$$\operatorname{rank}(\alpha_1 p_1 + \ldots + \alpha_n p_n) = \operatorname{rank}(u) \geq \operatorname{rank}(p_1) + \ldots + \operatorname{rank}(p_n).$$

The other inequality is a consequence of Theorem 2.14. ■

COROLLARY 2.17. *If $\operatorname{rank}(a) = n = r + s \geq 1$, then there exist $b, c \in \operatorname{Soc} A$ such that $a = b + c$, $\operatorname{rank}(b) = r$ and $\operatorname{rank}(c) = s$.*

Proof. We take u invertible such that ua is maximal of rank n. Then $ua = \lambda_1 p_1 + \ldots + \lambda_n p_n$, where $\lambda_1, \ldots, \lambda_n$ are the non-zero spectral values of ua and $p_1, \ldots, p_n$ the corresponding orthogonal Riesz projections, by Theorem 2.8. We take $b' = \lambda_1 p_1 + \ldots + \lambda_r p_r$ and $c' = \lambda_{r+1} p_{r+1} + \ldots + \lambda_n p_n$. By Theorem 2.16 we have $\operatorname{rank}(b') = r$ and $\operatorname{rank}(c') = s$. Taking $b = u^{-1} b'$ and $c = u^{-1} c'$ and using property (b) of rank we get the result. ■

COROLLARY 2.18 (*). *Let $a \in A$. The rank of a is the smallest integer k such that $a \in I_1 + \ldots + I_k$, where $I_1, \ldots, I_k$ are distinct minimal left ideals.*

Proof. We may suppose that $a \in \operatorname{Soc} A$ because otherwise $k = \infty$. Let $r = \operatorname{rank}(a)$. By Theorems 2.2 and 2.8 there exist u invertible, $\lambda_1, \ldots, \lambda_n$ non-zero and distinct numbers and $p_1, \ldots, p_n$ orthogonal minimal projections such that $\operatorname{rank}(ua) = \operatorname{rank}(a)$ and $ua = \lambda_1 p_1 + \ldots + \lambda_n p_n$. Because $I_1 = Ap_1, \ldots, I_n = Ap_n$ are minimal left ideals we have $a \in I_1 + \ldots + I_n$; consequently, $n = r$ by Theorem 2.16, and so the number k as defined in the statement of the corollary satisfies $k \leq \operatorname{rank}(a)$.

Conversely, if $a \in I_1 + \ldots + I_k$, where the left ideals are minimal, it is well known ([1], Lemma 2, p. 78) that there exist minimal projections $p_1, \ldots, p_k$ such that $I_i = Ap_i$ $(i = 1, \ldots, k)$, so $a = x_i p_1 + \ldots + x_k p_k$ for some $x_1, \ldots, x_k \in A$; consequently, by property (b) and Theorem 2.14 we have

$$\operatorname{rank}(a) \leq \operatorname{rank}(x_1 p_1) + \ldots + \operatorname{rank}(x_k p_k) \leq \operatorname{rank}(p_1) + \ldots + \operatorname{rank}(p_k) = k.$$

So the corollary is proved. ■

We finish this section giving an application of Theorem 2.14 to derivations.

THEOREM 2.19. *Let D be a derivation of A. Then $\operatorname{rank}(Dx) \leq 2\operatorname{rank}(x)$ for every $x \in A$.*

Proof. Since A is semisimple, by the Johnson–Sinclair theorem, D is continuous, so for every $\lambda \in \mathbb{C}$ it defines an automorphism $e^{\lambda D}$ of A. Suppose x is of finite rank. By property (e) of the rank we have $\operatorname{rank}(e^{\lambda D} x) = \operatorname{rank}(x)$.

(*) *Editorial note*: This result has also been obtained by M. Brešar and P. Šemrl (private communication).

Setting

$$f(\lambda)=\begin{cases}\dfrac{e^{\lambda D}-1}{\lambda}x & \text{for } \lambda\neq 0,\\ Dx & \text{for } \lambda=0,\end{cases}$$

which is analytic in λ, by Theorem 2.14 for $\lambda \neq 0$ we have

$$\operatorname{rank} f(\lambda)\le \operatorname{rank}\left(\frac{e^{\lambda D}x}{\lambda}\right)+\operatorname{rank}\left(\frac{x}{\lambda}\right)=2\operatorname{rank}(x).$$

From Theorem 2.3 we conclude that, in particular, $\operatorname{rank} f(0)\le 2\operatorname{rank}(x)$. ■

The constant 2 is the best one. We can verify that for $A=M_2(\mathbb{C})$, $x=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, which is of rank one, and D the inner derivation defined by $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, we have $Dx=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$, which is of rank 2.

3. Trace and determinant. If $a\in\operatorname{Soc}A$ we define the *trace* of a by

$$\operatorname{Tr}(a)=\sum_{\lambda\in\operatorname{Sp}a}\lambda m(\lambda,a),\tag{1}$$

and the *determinant* of $1+a$ by

$$\operatorname{Det}(1+a)=\prod_{\lambda\in\operatorname{Sp}a}(1+\lambda)^{m(\lambda,a)}.\tag{2}$$

It is obvious that $\operatorname{Det}(1+a)\neq 0$ is equivalent to $1+a$ being invertible. From (1) and (2) it is clear that

$$|\operatorname{Tr}(a)|\le \varrho(a)\operatorname{rank}(a),\tag{3}$$

$$|\operatorname{Det}(1+a)|\le \varrho(1+a)^{\operatorname{rank}(a)}\le (1+\varrho(a))^{\operatorname{rank}(a)},\tag{4}$$

$$(1-\varrho(a))^{\operatorname{rank}(a)}\le |\operatorname{Det}(1+a)|\quad \text{for } \varrho(a)<1,\tag{5}$$

where ϱ denotes the spectral radius.

THEOREM 3.1. *Let f be an analytic function from a domain D of $\mathbb{C}$ into the socle of A. Then $\operatorname{Tr}(f(\lambda))$ and $\operatorname{Det}(1+f(\lambda))$ are holomorphic on D.*

Proof. By the Scarcity Theorem there exist a closed discrete subset E of D and an integer n such that $\#\operatorname{Sp} f(\lambda)=n$ for $\lambda\in D\setminus E$ and $\#\operatorname{Sp} f(\lambda)<n$ for $\lambda\in E$.

First we take $\lambda_0\in D\setminus E$. Then $\operatorname{Sp} f(\lambda_0)=\{\alpha_1,\ldots,\alpha_n\}$ and we choose $\varepsilon>0$ such that the $\overline{B}(\alpha_i,\varepsilon)$ are disjoint. By continuity of the spectrum on the socle there exists $\delta>0$ such that $|\lambda-\lambda_0|<\delta$ implies $\operatorname{Sp} f(\lambda)\subset B(\alpha_1,\varepsilon)\cup\ldots\cup B(\alpha_n,\varepsilon)$, $\lambda\in D\setminus E$, and $\#(B(\alpha_i,\varepsilon)\cap \operatorname{Sp} f(\lambda))=1$ for $|\lambda-\lambda_0|<\delta$. Denote by $\alpha_i(\lambda)$ the point in this intersection. We know from [3], Theorem 3.4.25, that these $\alpha_i(\lambda)$ vary locally holomorphically. If δ is chosen small enough, by Lemma 2.5, the Riesz projections $p(\alpha_i(\lambda),f(\lambda))$ and $p(\alpha_i,f(\lambda_0))$ are equivalent, so by Theorem 2.6, we have $m(\alpha_i(\lambda),f(\lambda))=m(\alpha_i,f(\lambda_0))$. From this and the local holomorphy of the $\alpha_i(\lambda)$ we conclude that $\operatorname{Tr}(f(\lambda))$ and $\operatorname{Det}(1+f(\lambda))$ are holomorphic on $D\setminus E$.

Suppose now that $\lambda_0\in E$. Then $\operatorname{Sp} f(\lambda_0)=\{\alpha_1,\ldots,\alpha_m\}$ $(m<n)$. As previously we choose $\varepsilon,\delta>0$ such that the $\overline{B}(\alpha_i,\varepsilon)$ are disjoint and $|\lambda-\lambda_0|<\delta$ implies $\operatorname{Sp} f(\lambda)\subset B(\alpha_1,\varepsilon)\cup\ldots\cup B(\alpha_m,\varepsilon)$. Once again if δ is chosen small enough the projections $p(\partial B(\alpha_i,\varepsilon),\, f(\lambda))$ and $p(\partial B(\alpha_i,\varepsilon)), f(\lambda_0))$ are equivalent for $i=1,\ldots,m$ and consequently we have by Theorem 2.6,

$$m(\alpha_i,f(\lambda_0))=\sum_{\beta\in\operatorname{Sp} f(\lambda)\cap B(\alpha_i,\varepsilon)} m(\beta,f(\lambda)).$$

Continuity of the spectrum on the socle and this relation imply that $\operatorname{Tr}(f(\lambda))$ and $\operatorname{Det}(1+f(\lambda))$ are continuous at every point of E and then at every point of D. By Morera's theorem (see [17], p. 224), $\operatorname{Tr}(f(\lambda))$ and $\operatorname{Det}(1+f(\lambda))$ are holomorphic on all D. ■

The following lemma is implicitly contained in the proof of Theorem 5, p. 29, of [2].

LEMMA 3.2. *Let $f(\lambda,\mu)$ be a complex-valued function of two complex variables which is separately entire in λ,μ and such that $f(\lambda,\mu)\neq 0$ for all λ,μ in $\mathbb{C}$. Suppose moreover that there exist two positive constants A,B such that*

$$|f(\lambda,\mu)|\le e^{A|\lambda|+B|\mu|}.$$

Then there exist three complex constants α,β,γ such that

$$f(\lambda,\mu)=e^{\alpha\lambda+\beta\mu+\gamma}.$$

Proof. Because the complex plane is simply connected, by [17], Theorem 13.11, there exists $\Phi(\lambda,\mu)$ separately entire in λ,μ such that $\exp(\Phi(\lambda,\mu))=f(\lambda,\mu)$. Then we have $\operatorname{Re}\Phi(\lambda,\mu)\le A|\lambda|+B|\mu|$. So if we fix μ and apply Liouville's theorem for the real part we conclude that $\Phi(\lambda,\mu)=\lambda f_1(\mu)+f_2(\mu)$ for every λ. Taking two different values of λ and solving the system of two equations in f_1,f_2 we conclude that f_1,f_2 are entire in μ. Fixing μ again and taking λ real and going to $+\infty$ we conclude that $\operatorname{Re} f_1(\mu)\le A$ for arbitrary μ, so again we conclude that $f_1(\mu)$ is a constant α. A similar argument with λ fixed proves that $\Phi(\lambda,\mu)=\mu g_1(\lambda)+g_2(\lambda)$, where $g_1(\lambda)$ is a constant β. Finally, we have $-\lambda\alpha+g_2(\lambda)=-\beta\mu+f_2(\mu)$ for every λ,μ, hence this quantity must be a constant γ and we get the result. ■

THEOREM 3.3. *Let $x,y\in\operatorname{Soc}A$. Then:*

(i) $\operatorname{Tr}(x+y)=\operatorname{Tr}(x)+\operatorname{Tr}(y)$,
(ii) $\operatorname{Det}(e^{x+y})=\operatorname{Det}(e^xe^y)=\operatorname{Det}(e^x)\operatorname{Det}(e^y)$,
(iii) $\operatorname{Det}(e^x)=e^{\operatorname{Tr}(x)}$,
(iv) $\operatorname{Det}((1+x)(1+y))=\operatorname{Det}(1+x)\operatorname{Det}(1+y)$,
(v) *for every integer n, $\operatorname{Tr}(x)$ and $\operatorname{Det}(1+x)$ are continuous on $\mathcal{F}_n$.*

Proof. (i) By Theorem 3.1, $h(\lambda)=\mathrm{Tr}(x+\lambda y)$ is entire. Furthermore
$$\lim_{|\lambda|\to\infty}\frac{h(\lambda)}{\lambda}=\lim_{|\lambda|\to\infty}\mathrm{Tr}\left(\frac{x}{\lambda}+y\right)=\lim_{\mu\to 0}\mathrm{Tr}(\mu x+y)=\mathrm{Tr}(y)$$
because $\mu\mapsto\mathrm{Tr}(\mu x+y)$ is continuous. So by Liouville's theorem, $h(\lambda)$ is a polynomial of degree one. Identifying the coefficients we get the result.

(ii) We take $f(\lambda,\mu)=\mathrm{Det}(e^{\lambda x+\mu y})$, which is well-defined because $a\in\mathrm{Soc}\,A$ implies $e^a-1\in\mathrm{Soc}\,A$. By Theorem 3.1 this function is separately holomorphic in λ,μ and it is never zero because $e^{\lambda x+\mu y}$ is invertible. Let $N=\max(\mathrm{rank}(x),\mathrm{rank}(y))$. By Theorem 2.14 and property (4) we have
$$f(\lambda,\mu)\le\varrho(e^{\lambda x+\mu y})^{\mathrm{rank}(\lambda x+\mu y)}\le e^{2N(|\lambda|\|x\|+|\mu|\|y\|)},$$
because $\mathrm{rank}(e^a-1)=\mathrm{rank}(a)$ by property (b). So by Lemma 3.2 there exist $\alpha,\beta,\gamma\in\mathbb{C}$ such that $f(\lambda,\mu)=e^{\alpha\lambda+\beta\mu+\gamma}$. Because $f(0,0)=1$ we may supppose that $\gamma=0$. Taking $\lambda=1$, $\mu=0$ we get $e^\alpha=\mathrm{Det}(e^x)$ and a similar argument gives $e^\beta=\mathrm{Det}(e^y)$. Consequently, $\mathrm{Det}(e^{x+y})=f(1,1)=\mathrm{Det}(e^x)\,\mathrm{Det}(e^y)$.

(iii) If $c\in\mathcal{F}_1$ then $e^c-1=c\big(1+\frac{c}{2}+\ldots\big)\in\mathcal{F}_1$. If $\mathrm{Sp}\,c=\{0,\alpha\}$ then $\mathrm{Sp}\,e^c=\{1,e^\alpha\}$ and consequently
$$e^{\mathrm{Tr}(c)}=e^\alpha=\mathrm{Det}(e^c).$$
By the remark following Corollary 2.9, every $x\in\mathrm{Soc}\,A$ can be written as $x=c_1+\ldots+c_n$, where $c_1,\ldots,c_n\in\mathcal{F}_1$. So applying (i) and (ii) we have
$$\mathrm{Det}(e^x)=\mathrm{Det}(e^{c_1})\times\ldots\times\mathrm{Det}(e^{c_n})=e^{\mathrm{Tr}(c_1)+\ldots+\mathrm{Tr}(c_n)}=e^{\mathrm{Tr}(x)}.$$

(iv) We take $\lambda\in\mathbb{C}$ such that $|\lambda|\|x\|,|\lambda|\|y\|<1$. By the Holomorphic Functional Calculus there exist $u,v\in\mathrm{Soc}\,A$ such that $1-\lambda x=e^u$ and $1-\lambda y=e^v$. Hence by (iii), we have
$$\begin{aligned}\mathrm{Det}((1-\lambda x)(1-\lambda y))&=\mathrm{Det}(e^ue^v)=\mathrm{Det}(e^u)\,\mathrm{Det}(e^v)\\&=\mathrm{Det}(1-\lambda x)\,\mathrm{Det}(1-\lambda y).\end{aligned}$$
Because the functions $\mathrm{Det}((1-\lambda x)(1-\lambda y))$ and $\mathrm{Det}(1-\lambda x)$, $\mathrm{Det}(1-\lambda y)$ are entire by Theorem 3.1, by the Identity Principle we conclude that the same property is true for every λ, so (iv) is proved.

(v) If $x,y\in\mathcal{F}_n$ then $x-y\in\mathcal{F}_{2n}$ by Theorem 2.14, so by (i) and property (3) we have $|\mathrm{Tr}(x)-\mathrm{Tr}(y)|=|\mathrm{Tr}(x-y)|\le\varrho(x-y)\,\mathrm{rank}(x-y)\le 2n\|x-y\|$.

Suppose first that $1+x$ is invertible. Then by (iv) we have
$$\frac{\mathrm{Det}(1+y)}{\mathrm{Det}(1+x)}=\mathrm{Det}(1+(1+x)^{-1}(y-x)).$$
Taking $\|(1-x)^{-1}(y-x)\|<1$, by properties (4) and (5) we get
$$(1-\|(1+x)^{-1}\|\cdot\|x-y\|)^{2n}\le\left|\frac{\mathrm{Det}(1+y)}{\mathrm{Det}(1+x)}\right|\le(1+\|(1+x)^{-1}\|\cdot\|x-y\|)^{2n},$$

which proves continuity of $\mathrm{Det}(1+y)$ at $y=x$. If $D(1+x)=0$ we may suppose that $\mathrm{Sp}\,x=\{\alpha_1=-1,\alpha_2,\ldots,\alpha_m\}$. We take $0<\varepsilon<1$ such that the balls $\overline{B}(\alpha_i,\varepsilon)$ are disjoint and, by continuity of the spectrum on $\mathrm{Soc}\,A$, we take $\delta>0$ such that $\|y-x\|<\delta$ implies $\mathrm{Sp}\,y\subset B(\alpha_1,\varepsilon)\cup\ldots\cup B(\alpha_m,\varepsilon)$ and $\mathrm{Sp}\,y\cap B(\alpha_i,\varepsilon)\ne\emptyset$ $(i=1,\ldots,m)$. Then
$$|\mathrm{Det}(1+y)|=\prod_{\lambda\in\mathrm{Sp}\,y}|1+\lambda|^{m(\lambda,y)}\le(1+\|x\|+\varepsilon)^n\varepsilon,$$
because $B(-1,\varepsilon)$ contains at least one point of $\mathrm{Sp}\,y$. Consequently, $\mathrm{Det}(1+y)$ goes to zero as y goes to x in $\mathcal{F}_n$. ∎

Let H be a Hilbert space and $A=\mathcal{B}(H)$. In this case the socle of A consists of the finite-rank linear operators on H. We consider a sequence (p_n) of orthogonal projections onto subspaces of dimension n, when n varies from one to infinity. Then $\|p_n/n\|=1/n$, $\mathrm{Tr}(p_n/n)=1$, so this shows that the trace, and also the determinant, cannot be continuous in general on all the socle.

From Theorem 3.3 it is possible to give another proof of Theorem 2.16 avoiding Lemma 2.15. Because
$$(6)\qquad\begin{cases}p_1+\ldots+p_n=(\alpha_1p_1+\ldots+\alpha_np_n)\left(\dfrac{p_1}{\alpha_1}+\ldots+\dfrac{p_n}{\alpha_n}\right),\\ \alpha_1p_1+\ldots+\alpha_np_n=(p_1+\ldots+p_n)(\alpha_1p_1+\ldots+\alpha_np_n),\end{cases}$$
we have $\mathrm{rank}(\alpha_1p_1+\ldots+\alpha_np_n)=\mathrm{rank}(p_1+\ldots+p_n)$. Also $p=p_1+\ldots+p_n$ is a non-zero projection. By Corollary 2.7 we have $\mathrm{rank}(p)=m(1,p)=\mathrm{Tr}(p)=\mathrm{Tr}(p_1)+\ldots+\mathrm{Tr}(p_n)=\mathrm{rank}(p_1)+\ldots+\mathrm{rank}(p_n)$.

We finish this article proving that there exists a connection between the spectral radius and the trace for elements of the socle. To prove Theorem 3.5 we essentially follow the very simple and beautiful argument due to H. K. Wimmer [20] in the case of matrices, modifying it slightly with the help of Lemma 3.4 in the general case of elements of the socle.

LEMMA 3.4. *Let $a\in\mathrm{Soc}\,A$, λ_1 be a non-zero spectral value of a and let $1/z\notin\mathrm{Sp}\,a$. The Riesz projection associated with $\mu_1=\frac{\lambda_1}{1-z\lambda_1}$ and $a(1-za)^{-1}$ is equal to the Riesz projection associated with λ_1 and a. Consequently, μ_1 and λ_1 have the same multiplicity in the corresponding spectra.*

Proof. Let Γ be a small curve surrounding μ_1. Because the transformation $\mu=\lambda/(1-z\lambda)$ is injective we have
$$\begin{aligned}p(\mu_1,a(1-za)^{-1})&=\frac{1}{2\pi i}\int_\Gamma(\mu-a(1-za)^{-1})^{-1}\,d\mu\\&=\frac{1}{2\pi i}\int_\gamma\left(\frac{\lambda}{1-z\lambda}-a(1-za)^{-1}\right)^{-1}\frac{d\lambda}{(1-z\lambda)^2}\end{aligned}$$

$$= \frac{1}{2\pi i} \int_\gamma (1 - za)(\lambda - a)^{-1} \frac{d\lambda}{1 - z\lambda}$$
$$= \frac{1}{2\pi i} \int_\gamma (1 - z\lambda + z(\lambda - a))(\lambda - a)^{-1} \frac{d\lambda}{1 - z\lambda}$$
$$= \frac{1}{2\pi i} \int_\gamma \frac{z\,d\lambda}{1 - z\lambda} + p(\lambda_1, a)$$

where γ, the preimage of Γ, is a small curve surrounding λ_1. But the first integral is zero because γ does not surround the pole $\frac{1}{z}$ of $\lambda \mapsto \frac{1}{1-z\lambda}$ if Γ is taken small enough. So the first part is proved. The last part is a consequence of Theorem 2.6. ■

THEOREM 3.5. *If $a \in \operatorname{Soc} A$ then*

$$\varrho(a) = \overline{\lim_{k\to\infty}} |\operatorname{Tr}(a^k)|^{1/k}.$$

Moreover, if $\varrho(a) \neq 0$, the number of spectral values of a with modulus $\varrho(a)$ is given by

$$\overline{\lim_{k\to\infty}} \frac{|\operatorname{Tr}(a^k)|}{\varrho(a)^k}.$$

Proof. We have in general $|\operatorname{Tr}(a^k)| \le \operatorname{rank}(a)\varrho(a)^k$ by property (b) of the rank and formula (2) of Section 2, so the theorem is true for $\varrho(a) = 0$. Suppose now $\varrho(a) \neq 0$ and denote by $\lambda_1, \ldots, \lambda_n$ the non-zero spectral values of a and by $m_1, \ldots, m_n$ their multiplicities. We consider the series $\sum_{k=1}^\infty \operatorname{Tr}(a^k) z^k$ whose radius of convergence R satisfies

$$\frac{1}{R} = \overline{\lim_{k\to\infty}} |\operatorname{Tr}(a^k)|^{1/k} \le \varrho(a),$$

by the inequality at the beginning. Let $|z| < 1/\varrho(a) \le R$. Then the series converges and we have

$$\sum_{k=1}^\infty \operatorname{Tr}(a^k) z^k = \sum_{k=1}^\infty \operatorname{Tr}((za)^k) = \operatorname{Tr}\Big(\sum_{k=1}^\infty (za)^k\Big) = \operatorname{Tr}(a(1 - za)^{-1})),$$

because the trace is additive on the socle and continuous on a set of bounded rank elements. We just have to notice that the elements $\sum_{k=1}^\ell (za)^k$ have rank less than or equal to $\operatorname{rank}(a)$ for every $\ell \le \infty$. Now by Lemma 3.4 we have

$$\operatorname{Tr}(a(1 - za)^{-1}) = \frac{m_1\lambda_1}{1 - z\lambda_1} + \ldots + \frac{m_n\lambda_n}{1 - z\lambda_n},$$

which is a rational function with poles $1/\lambda_1, \ldots, 1/\lambda_n$. Consequently, $R = \min(1/|\lambda_1|, \ldots, 1/|\lambda_n|)$. Hence $1/R = \varrho(a)$. The second assertion comes from Kronecker's theorem, exactly as in [18]. ■

The similar result, where the superior limit is replaced by the limit, is not true. To see this take

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{C}).$$

Then $\operatorname{Tr}(a^{2k}) = 2$, $\operatorname{Tr}(a^{2k+1}) = 0$.

COROLLARY 3.6. *Let $a \in A$. Suppose that $\operatorname{Tr}(ax) = 0$ for all $x \in \operatorname{Soc} A$. Then $a \operatorname{Soc} A = 0$. If moreover $a \in \operatorname{Soc} A$ then $a = 0$.*

Proof. Let $x \in \operatorname{Soc} A$ and $y \in A$. Then $\operatorname{Tr}((axy)^k) = 0$ for $k \ge 1$, so by the previous theorem we have $\varrho((ax)y) = 0$ for all $y \in A$. Hence $ax \in \operatorname{Rad} A = \{0\}$, that is, $a \operatorname{Soc} A = \{0\}$. If $a \in \operatorname{Soc} A$ then $(ay) \operatorname{Soc} A = 0$, hence $(ay)^2 = 0$. Again we have $a \in \operatorname{Rad} A = \{0\}$. So $a = 0$. ■

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Complexité de la famille des ensembles de synthèse d'un groupe abélien localement compact

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Résumé. On montre que si $\mathbf{G}$ est un groupe abélien localement compact non discret à base dénombrable d'ouverts, alors la famille des fermés de synthèse pour l'algèbre de Fourier $A(\mathbf{G})$ est une partie coanalytique non borélienne de $\mathcal{F}(\mathbf{G})$, l'ensemble des fermés de $\mathbf{G}$ muni de la structure borélienne d'Effros. On généralise ainsi un résultat connu dans le cas du groupe $\mathbb{T}$.

Introduction. Depuis une dizaine d'années, les relations entre l'Analyse Harmonique et la Théorie Descriptive des Ensembles ont été largement explorées (voir par exemple [KL1] ou [KL2]). L'étude des propriétés descriptives de certaines familles de fermés issues de l'Analyse Harmonique s'est révélée très profitable aux deux disciplines.

Cet article est consacré à la famille des fermés de synthèse d'un groupe abélien localement compact.

Dans [KMG2], Katznelson et McGehee construisent dans le groupe $\mathbb{T}^{\mathbb{N}}$ des ensembles de synthèse de "rang" arbitrairement grand, et indiquent très brièvement qu'une construction analogue est réalisable dans $\mathbb{T}$. Ce résultat est utilisé par Kechris et Solovay pour montrer que les ensembles de synthèse du groupe $\mathbb{T}$ forment une partie coanalytique non borélienne de $\mathcal{K}(\mathbb{T})$, l'ensemble des compacts de $\mathbb{T}$ muni de la topologie de Hausdorff. Dans cet article, on obtient la même conclusion pour tous les groupes abéliens localement compacts (non discrets) à base dénombrable d'ouverts.

Dans toute la suite, la lettre $\mathbf{G}$ désigne un groupe abélien localement compact non discret à base dénombrable d'ouverts, et on note $\mathbf{\Gamma}$ le groupe dual de $\mathbf{G}$.

Par transformation de Fourier, on identifie $L^1(\mathbf{\Gamma})$ à une sous-algèbre de $C_0(\mathbf{G})$, que l'on note $A(\mathbf{G})$. Le dual de $A(\mathbf{G})$ est l'espace $PM(\mathbf{G})$ des *pseudomesures* sur $\mathbf{G}$ (qui s'identifie à $L^\infty(\mathbf{\Gamma})$). L'espace $M(\mathbf{G})$ des

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