3) On peut définir de manière évidente le rang d'un ensemble de synthèse pour $V(\mathbf{2}^\omega)$, et montrer que ce rang est un $\mathbf{\Pi}_1^1$-rang. Il résulte alors de 2) et du théorème de la borne qu'il existe, dans $\mathbf{2}^\omega \times \mathbf{2}^\omega$, des ensembles de synthèse pour $V(\mathbf{2}^\omega)$ de rang arbitrairement grand.

Références

[GMG] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Grundlehren Math. Wiss. 238, Springer, New York, 1979.

[Ka] J. P. Kahane, *Séries de Fourier absolument convergentes*, Springer, Berlin, 1970.

[KMG1] Y. Katznelson and O. C. McGehee, *Measures and pseudomeasures on compact subsets of the line*, Math. Scand. 23 (1968), 57–68.

[KMG2] —, —, *Some sets obeying Harmonic Synthesis*, Israel J. Math. 23 (1976), 88–93.

[KR] Y. Katznelson and W. Rudin, *The Stone–Weierstrass property in Banach algebras*, Pacific J. Math. 11 (1961), 253–265.

[Ke] A. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1995.

[KL1] A. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Math. Soc. Lecture Note Ser. 128, Cambridge University Press, 1987.

[KL2] —, —, *Descriptive set theory and harmonic analysis*, J. Symbolic Logic 57 (1992), 413–441.

[L] A. Louveau, *The Descriptive Theory of Borel Sets*, livre en préparation.

[M] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North-Holland, Amsterdam, 1972.

[R] W. Rudin, *Fourier Analysis on Groups*, Interscience Tract 12, Wiley, New York, 1962.

[Va] N. T. Varopoulos, *Tensor algebras and harmonic analysis*, Acta Math. 119 (1967), 51–112.

Equipe d'Analyse
Université Paris VI
Boîte 186
4, Place Jussieu
75252, Paris Cedex 05, France

Received July 7, 1995 (3512)

STUDIA MATHEMATICA 121 (2) (1996)

A Fourier analytical characterization of the Hausdorff dimension of a closed set and of related Lebesgue spaces

by

HANS TRIEBEL and HEIKE WINKELVOSS (Jena)

Abstract. Let Γ be a closed set in $\mathbb{R}^n$ with Lebesgue measure $|\Gamma| = 0$. The first aim of the paper is to give a Fourier analytical characterization of the Hausdorff dimension of Γ.

Let $0 < d < n$. If there exist a Borel measure μ with $\operatorname{supp}\mu \subset \Gamma$ and constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 r^d \le \mu(B(x,r)) \le c_2 r^d$ for all $0 < r < 1$ and all $x \in \Gamma$, where $B(x,r)$ is a ball with centre x and radius r, then Γ is called a *d-set*. The second aim of the paper is to provide a link between the related Lebesgue spaces $L_p(\Gamma)$, $0 < p \le \infty$, with respect to that measure μ on the one hand and the Fourier analytically defined Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ ($s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$) on the other hand.

1. Introduction. Let $0 < \sigma < 1$. Then $\mathcal{C}^\sigma(\mathbb{R}^n)$ stands for the usual Hölder space on $\mathbb{R}^n$, that is, the collection of all complex-valued continuous functions f on $\mathbb{R}^n$ such that

$$\|f \mid \mathcal{C}^\sigma(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\sigma} < \infty. \tag{1}$$

Let $s \in \mathbb{R}$ and $s = \varrho + \sigma$ with $0 < \sigma < 1$. Then the Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$ are the lifted Hölder spaces,

$$\mathcal{C}^s(\mathbb{R}^n) = (\mathrm{id} - \Delta)^{-\varrho/2}\mathcal{C}^\sigma(\mathbb{R}^n), \tag{2}$$

where Δ is the Laplacian and everything must be interpreted in the framework of tempered distributions $S'(\mathbb{R}^n)$. Of course $\mathcal{C}^s(\mathbb{R}^n)$ in (2) does not depend on the choice of ϱ and σ, and the Hölder spaces are included in the Zygmund scale. Assume Γ to be a closed subset of $\mathbb{R}^n$ with Lebesgue measure $|\Gamma| = 0$. Let

$$\mathcal{C}^{s,\Gamma}(\mathbb{R}^n) = \{f \in \mathcal{C}^s(\mathbb{R}^n) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^n) \text{ and } \varphi|\Gamma = 0\}, \tag{3}$$

where $f(\varphi)$ is the usual pairing for the Schwartz space $S(\mathbb{R}^n)$ and its dual $S'(\mathbb{R}^n)$. Furthermore, $\varphi|\Gamma$ is the restriction of $\varphi \in S(\mathbb{R}^n)$ to Γ. In particular,

1991 *Mathematics Subject Classification*: 46E35, 28A78, 28A80.

Key words and phrases: Hausdorff dimension, Hausdorff measure, function spaces.

$\operatorname{supp} f \subset \Gamma$ if $f \in \mathcal{C}^{s,\Gamma}(\mathbb{R}^n)$. Only spaces $\mathcal{C}^{s,\Gamma}(\mathbb{R}^n)$ with $s \le 0$ are of interest since $\mathcal{C}^{s,\Gamma}(\mathbb{R}^n) = \{0\}$ if $s > 0$. On the other hand, if $-\infty < s_1 \le s_2 \le 0$, then $\mathcal{C}^{s_2,\Gamma}(\mathbb{R}^n) \subset \mathcal{C}^{s_1,\Gamma}(\mathbb{R}^n)$. Hence it makes sense to ask for the largest value s such that $\mathcal{C}^{s,\Gamma}(\mathbb{R}^n)$ is non-trivial. More precisely: The *distributional dimension* $\dim_{\mathrm{D}} \Gamma$ is, by definition,

$$\dim_{\mathrm{D}} \Gamma = \sup\{d : \mathcal{C}^{-n+d,\Gamma}(\mathbb{R}^n) \text{ is non-trivial}\}. \tag{4}$$

Of course "non-trivial" means $\mathcal{C}^{-n+d,\Gamma}(\mathbb{R}^n) \ne \{0\}$. The Hausdorff dimension of Γ is denoted by $\dim_{\mathrm{H}} \Gamma$. The first aim of our paper is to prove the following assertion.

THEOREM 1. *Let Γ be a closed subset of $\mathbb{R}^n$ with Lebesgue measure 0. Then*

$$\dim_{\mathrm{H}} \Gamma = \dim_{\mathrm{D}} \Gamma. \tag{5}$$

To describe the second main result we recall the notion of a d-set. Let $0 < d < n$. Then a closed set Γ in $\mathbb{R}^n$ is called a *d-set* if there exist a Borel measure μ with $\operatorname{supp} \mu \subset \Gamma$ and positive numbers c_1 and c_2 such that

$$c_1 r^d \le \mu(B(x,r)) \le c_2 r^d \quad \text{for all } 0 < r < 1 \text{ and } x \in \Gamma, \tag{6}$$

where $B(x,r)$ stands for the ball centred at x and of radius r. The measure μ can be identified with the Hausdorff measure $\mathcal{H}^d$ restricted to Γ, and we have $\dim_{\mathrm{H}} \Gamma = d$. Self-similar fractals are typical examples of d-sets. Let $L_p(\Gamma)$ with $0 < p \le \infty$ be the related Lebesgue spaces with respect to that measure μ, which is equivalent to $\mathcal{H}^d|\Gamma$, the restriction of $\mathcal{H}^d$ to Γ. Any $f^\Gamma \in L_p(\Gamma)$, $1 \le p \le \infty$, can be interpreted as a tempered distribution $f \in S'(\mathbb{R}^n)$ given by

$$f(\varphi) = \int_\Gamma f^\Gamma(\gamma)(\varphi|\Gamma)(\gamma)\,\mu(d\gamma), \quad \varphi \in S(\mathbb{R}^n). \tag{7}$$

A Fourier analytical characterization of the distributions f obtained in that way will be given in the framework of the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. These spaces can be introduced parallel to the above spaces $\mathcal{C}^s(\mathbb{R}^n)$. In particular, $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$. The subspaces $B^{s,\Gamma}_{p,q}(\mathbb{R}^n)$ are defined similarly to (3). A second possibility of characterizing the space $L_p(\Gamma)$ is to find out whether it can be identified with the trace space of suitable spaces $B^s_{p,q}(\mathbb{R}^n)$. Here the trace $\operatorname{tr}_\Gamma f$ of $f \in B^s_{p,q}(\mathbb{R}^n)$ is defined pointwise if f is smooth and the rest is a matter of completion (if the necessary norm inequality holds). The second aim of our paper is to prove the following assertion.

THEOREM 2. *Let Γ be a closed d-set in $\mathbb{R}^n$ with $0 < d < n$.*

(i) *Let $1 < p \le \infty$ and $1/p + 1/p' = 1$. Then*

$$L_p(\Gamma) = B^{-(n-d)/p',\Gamma}_{p,\infty}(\mathbb{R}^n) \tag{8}$$

in the sense of (7).

(ii) *Let $1 \le p < \infty$. Then*

$$\operatorname{tr}_\Gamma B^{(n-d)/p}_{p,1}(\mathbb{R}^n) = L_p(\Gamma). \tag{9}$$

The two parts of this theorem provide a perfect link between L_p spaces on Γ and the Fourier analytically defined spaces $B^s_{p,q}(\mathbb{R}^n)$. If one steps from Γ into $\mathbb{R}^n$ according to (7) and (8), one loses $(n-d)/p'$ smoothness, returning via (9) one loses $(n-d)/p$, i.e. together $n-d$, which is rather natural. Although these two theorems seem to be of interest for their own sake, they also pave the way to a spectral theory of fractal pseudodifferential operators. We shift this task to later papers. But to provide a feeling what can be expected on that basis we outline a simple but typical example:

Let Ω be a bounded, say, smooth domain in $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a compact d-set with $n-2 < d < n$. Let $(-\Delta)^{-1}$ be the inverse of the Dirichlet Laplacian in Ω. Let $W^1_2(\Omega)$ be the classical Sobolev space and $\mathring{W}^1_2(\Omega) = \{g \in W^1_2(\Omega) : \operatorname{tr}_{\partial\Omega} g = 0\}$. By (7)–(9) with $p = 2$ the operator tr^μ,

$$(\operatorname{tr}^\mu f)(\varphi) = \int_\Gamma (\operatorname{tr}_\Gamma f)(\gamma)(\varphi|\Gamma)(\gamma)\,\mu(d\gamma), \quad \varphi \in D(\Omega), \tag{10}$$

makes sense as a mapping from $\mathring{W}^1_2(\Omega)$ in $D'(\Omega)$, and it comes out that the so defined fractal differential operator

$$B = (-\Delta)^{-1} \circ \operatorname{tr}^\mu \tag{11}$$

generates a compact, self-adjoint, non-negative operator on $\mathring{W}^1_2(\Omega)$. For its positive eigenvalues λ_k, $k \in \mathbb{N}$, there exist two positive numbers c_1 and c_2 such that

$$c_1 \lambda_k \le k^{-(2-n+d)/d} \le c_2 \lambda_k, \quad k \in \mathbb{N}. \tag{12}$$

Compared with the classical Weyl exponent $2/n$ one has the natural replacement of n by d and a fractal defect $n-d$, which comes from both parts of Theorem 2. We return to this subject later on in greater detail and greater generality.

The plan of the present paper is the following. In Section 2 we collect some definitions and references. We are very brief. Special properties of function spaces will be mentioned later on when they are needed. The proofs of the two theorems above are given in Section 3. Finally, Section 4 contains comments and further results. In particular, in Theorem 3 we extend (9) to $p < 1$ and to some spaces $F^s_{p,q}(\mathbb{R}^n)$.

2. Definitions and preliminaries

2.1. *Function spaces.* Let $\mathbb{R}^n$ be the euclidean n-space. The Schwartz space $S(\mathbb{R}^n)$ and its dual space $S'(\mathbb{R}^n)$ of all complex-valued tempered distributions have the usual meaning here. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\| \cdot \mid L_p(\mathbb{R}^n)\|$. Let $\varphi \in S(\mathbb{R}^n)$ be such that

$$(13)\qquad \operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \text{ if } |x| \le 1;$$

let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for each $j \in \mathbb{N}$ (natural numbers) and put $\varphi_0 = \varphi$. Then, since $1 = \sum_{j=0}^{\infty} \varphi_j(x)$ for all $x \in \mathbb{R}^n$, the φ_j form a dyadic resolution of unity. Let $\hat{f}$ and $\check{f}$ be the Fourier transform and its inverse, respectively, of $f \in S'(\mathbb{R}^n)$. Then for any $f \in S'(\mathbb{R}^n)$, $(\varphi_j \hat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$.

DEFINITION 1. Let $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$. Then $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$(14)\qquad \|f \mid B^s_{p,q}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q}$$

(with the usual modification if $q = \infty$) is finite.

Remark 1. A systematic treatment of the theory of these spaces may be found in [10] and [11]. In particular, $B^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space, which is independent of the function $\varphi \in S(\mathbb{R}^n)$ satisfying (13), in the sense of equivalent quasi-norms. This justifies our omission of the subscript φ in (14) in what follows. If $p \ge 1$ and $q \ge 1$, then $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space.

Remark 2. We have

$$(15)\qquad \mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n), \quad s \in \mathbb{R},$$

in the sense of (1) and (2).

2.2. *Closed sets.* Let $0 \le d \le n$ and let Γ be a non-empty closed set in $\mathbb{R}^n$. Furthermore, $\mathcal{H}^d$ stands for the d-dimensional Hausdorff measure on $\mathbb{R}^n$, and $\mathcal{H}^d(\Gamma)$ is the corresponding Hausdorff measure of Γ. Moreover, $\dim_{\mathrm{H}} \Gamma$ denotes the Hausdorff dimension of Γ. We assume that the reader is familiar with the basic notions of fractal geometry (see [1] and [2]). Our restriction to closed sets comes from our aim described in the introduction. The Lebesgue measure of Γ is denoted by $|\Gamma|$. For the definition of a d-set we refer to the introduction. We mention that our notation is slightly different from that used in fractal geometry (see [1]), but it coincides with that of [8]. In particular, any measure μ on Γ with (6) is equivalent to the Hausdorff measure $\mathcal{H}^d$ restricted to Γ (see [8, pp. 28–32]). We also refer to [6], [7] and [9], where the concept of a d-set is used in connection with function spaces preferably of $B^s_{p,q}$-type.

DEFINITION 2. Let Γ be a non-empty closed subset of $\mathbb{R}^n$ with $|\Gamma| = 0$. Suppose that $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$. Then

$$(16)\qquad B^{s,\Gamma}_{p,q}(\mathbb{R}^n) = \{f \in B^s_{p,q}(\mathbb{R}^n) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^n),\ \varphi|\Gamma = 0\}.$$

Remark 3. By (15), the space $B^{s,\Gamma}_{\infty,\infty}(\mathbb{R}^n)$ coincides with $\mathcal{C}^{s,\Gamma}(\mathbb{R}^n)$ given by (3).

Remark 4. Let $0 < p \le \infty$, $0 < q \le \infty$ and

$$(17)\qquad s > \sigma_p = n(1/p - 1)_+.$$

Then $B^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and, hence,

$$(18)\qquad B^{s,\Gamma}_{p,q}(\mathbb{R}^n) = \{0\},$$

i.e. is trivial. In other words, only values $s \le \sigma_p$ (in particular $s \le 0$ if $1 \le p \le \infty$) are of interest. Recall $\operatorname{supp} f \subset \Gamma$ if $f \in B^{s,\Gamma}_{p,q}(\mathbb{R}^n)$ in any case.

2.3. *Explanations of the theorems*

The distributional dimension. Let again Γ be a non-empty closed subset of $\mathbb{R}^n$ with $|\Gamma| = 0$. Then the distributional dimension $\dim_{\mathrm{D}} \Gamma$ is given by (4). By (15), (17) and (18) on the one hand and $\delta \in \mathcal{C}^{-n}(\mathbb{R}^n)$ (δ-distribution) on the other hand it follows immediately that

$$(19)\qquad 0 \le \dim_{\mathrm{D}} \Gamma \le n$$

(as it should be). We introduced the concept of the distributional dimension in [13]. But we must confess that our claim at the very end of that paper (hastily written down during proof reading) that (5) holds for any Borel set (or even Suslin set) is not correct: Assume Γ to be an arbitrary (Borel or Suslin) set in $\mathbb{R}^n$ with $|\Gamma| = 0$. Then (3) and (4) make sense and obviously $\dim_{\mathrm{D}} \Gamma = \dim_{\mathrm{D}} \overline{\Gamma}$, where $\overline{\Gamma}$ is the closure of Γ. But a corresponding assertion for the Hausdorff dimension is not true.

Traces. Let $\varphi \in S(\mathbb{R}^n)$. Then $\operatorname{tr}_\Gamma \varphi = \varphi|\Gamma$ makes sense pointwise. Let Γ be a closed d-set in $\mathbb{R}^n$ with $0 < d < n$. Then (9) must be understood as follows: There exists a positive number c such that

$$(20)\qquad \|\operatorname{tr}_\Gamma \varphi \mid L_p(\Gamma)\| \le c\|\varphi \mid B^{(n-d)/p}_{p,1}(\mathbb{R}^n)\|$$

for any $\varphi \in S(\mathbb{R}^n)$. Since $S(\mathbb{R}^n)$ is dense in $B^{(n-d)/p}_{p,1}(\mathbb{R}^n)$ (see [10, p. 48]), this inequality can be extended by completion to any $f \in B^{(n-d)/p}_{p,1}(\mathbb{R}^n)$ and, hence,

$$(21)\qquad \operatorname{tr}_\Gamma B^{(n-d)/p}_{p,1}(\mathbb{R}^n) \subset L_p(\Gamma).$$

The equality (9) means that any $f^\Gamma \in L_p(\Gamma)$ is the trace of a suitable $g \in B^{(n-d)/p}_{p,1}(\mathbb{R}^n)$ on Γ and $\|f^\Gamma \mid L_p(\Gamma)\|$ is equivalent to

$$\inf\{\|g \mid B^{(n-d)/p}_{p,1}(\mathbb{R}^n)\| : \mathrm{tr}_\Gamma\, g = f^\Gamma\}. \tag{22}$$

Hyperplanes. To illustrate the two theorems we take a look at $\Gamma = \mathbb{R}^d$ with $d \in \mathbb{N}$, $d < n$, interpreted in the usual way as a hyperplane in $\mathbb{R}^n$. That is, $\mathbb{R}^d$ will be identified with

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_{d+1} = \ldots = x_n = 0\}. \tag{23}$$

Let δ^{n-d} be the δ-distribution in $\mathbb{R}^{n-d}$. Then

$$f \in S'(\mathbb{R}^n), \quad f(\varphi) = 0 \quad \text{for all } \varphi \in S(\mathbb{R}^n) \text{ with } \varphi|\Gamma = 0 \tag{24}$$

if and only if

$$f = f_d \otimes \delta^{n-d} \qquad \text{with } f_d \in S'(\mathbb{R}^d). \tag{25}$$

On that basis we proved

$$\dim_{\mathrm{D}} \Gamma = \dim_{\mathrm{H}} \Gamma = d \tag{26}$$

in [13, p. 127] by direct reasoning. Here the latter equality is obvious. Hence, (26) illustrates (5).

Traces of function spaces on hyperplanes (and on smooth surfaces) have been studied in great detail (see [10, 2.7.2] and [11, 4.4]). For assertions of type (9) (also for $p < 1$) we refer to [4, §11] and [11, p. 220]. Hence, (9) is their fractal extension, whereas (8) is their dual assertion; see also Remark 7 below. A part of Theorem 2(ii) may also be found in [5].

3. Proofs

3.1. *Some preliminaries.* The proofs of the theorems mainly rely on the atomic characterization of $B^s_{p,q}(\mathbb{R}^n)$ and on local means. Thus we briefly recall the main facts.

Atomic decomposition. Let $\nu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}^n$. Then $Q_{\nu k}$ denotes the cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-\nu}k$ and with side-length $2^{-\nu}$. If $r > 0$ and Q is a cube in $\mathbb{R}^n$, then rQ is the cube concentric with Q and with side-length r times that of Q. Let $C(\mathbb{R}^n)$ be the usual space of all complex-valued bounded continuous functions on $\mathbb{R}^n$ equipped with the L_∞-norm.

Definition 3. (i) Let $\sigma \ge 0$ with $\sigma \notin \mathbb{N}$. Then a function a is called a *1-atom* (or more precisely 1_σ*-atom*) if

$$\operatorname{supp} a \subset 5Q_{0k} \quad \text{for some } k \in \mathbb{Z}^n \tag{27}$$

and

$$a \in \mathcal{C}^\sigma(\mathbb{R}^n), \qquad \|a \mid \mathcal{C}^\sigma(\mathbb{R}^n)\| \le 1, \tag{28}$$

with $C(\mathbb{R}^n)$ in place of $\mathcal{C}^0(\mathbb{R}^n)$ if $\sigma = 0$.

(ii) Let $s \in \mathbb{R}$, $0 < p \le \infty$, $\sigma \ge 0$ with $\sigma \notin \mathbb{N}$ and $L + 1 \in \mathbb{N}_0$. Then a function a is called an (s,p)*-atom* (or more precisely $(s,p)_{\sigma,L}$*-atom*) if

$$\operatorname{supp} a \subset 5Q_{\nu k} \quad \text{for some } \nu \in \mathbb{N} \text{ and some } k \in \mathbb{Z}^n, \tag{29}$$

$$a \in \mathcal{C}^\sigma(\mathbb{R}^n), \qquad \|a(2^{-\nu}\cdot) \mid \mathcal{C}^\sigma(\mathbb{R}^n)\| \le 2^{-\nu(s-n/p)} \tag{30}$$

with $C(\mathbb{R}^n)$ in place of $\mathcal{C}^0(\mathbb{R}^n)$ if $\sigma = 0$ and

$$\int x^\beta\, a(x)\, dx = 0 \quad \text{for } |\beta| \le L. \tag{31}$$

Remark 5. We adopt the custom that the condition (31) is empty if $L = -1$. If the atom a is supported by $5Q_{\nu k}$, we write it $a_{\nu k}$, i.e.

$$\operatorname{supp} a_{\nu k} \subset 5Q_{\nu k}. \tag{32}$$

Let $0 < p \le \infty$ and $0 < q \le \infty$. We introduce the sequence space

$$b_{p,q} = \Big\{\lambda = \{\lambda_{\nu k}\}_{\nu \in \mathbb{N}_0, k \in \mathbb{Z}^n} \subset \mathbb{C} : \|\lambda \mid b_{p,q}\| = \Big(\sum_{\nu=0}^\infty \Big(\sum_{k \in \mathbb{Z}^n} |\lambda_{\nu k}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Proposition 1. *Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $\sigma \ge 0$, $\sigma > s$ with $\sigma \notin \mathbb{N}$, and $L + 1 \in \mathbb{N}_0$ with $L \ge \max\{[\sigma_p - s], -1\}$, where σ_p is given by* (17). *Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n)$ if and only if it can be represented as*

$$f = \sum_{\nu=0}^\infty \sum_{k \in \mathbb{Z}^n} \lambda_{\nu k} a_{\nu k} \quad (\text{convergence in } S'(\mathbb{R}^n)), \tag{33}$$

where $a_{\nu k}$ are 1_σ-atoms ($\nu = 0$) or $(s,p)_{\sigma,L}$-atoms ($\nu \in \mathbb{N}$) in the sense of Definition 3(i) *and* (ii), *respectively, with* (32), *and $\lambda = \{\lambda_{\nu k}\}_{\nu \in \mathbb{N}_0,\, k \in \mathbb{Z}^n} \in b_{p,q}$. Furthermore,* $\inf \|\lambda \mid b_{p,q}\|$, *where the infimum is taken over all admissible representations* (33), *is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$.*

For the proof we refer to [3].

Local means. A function $K \in S(\mathbb{R}^n)$ is an *admissible kernel* if it can be represented as $K = \Delta^N K^0$, where $N \in \mathbb{N}_0$ is sufficiently large, Δ is the Laplacian, $K^0 \in S(\mathbb{R}^n)$ has a compact support and $\widehat{K^0}(0) \ne 0$. If $f \in S'(\mathbb{R}^n)$, then the *local means* are defined as

$$K(t,f)(x) = \int_{\mathbb{R}^n} K(y) f(x + ty)\, dy, \qquad t > 0,\ x \in \mathbb{R}^n \tag{34}$$

(appropriately interpreted). Suppose that $K_0 \in S(\mathbb{R}^n)$ also has a compact support and that $\widehat{K}_0(0) \ne 0$. Let $K_0(1,f)$ be defined in a similar way to (34).

PROPOSITION 2. *Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ and $N \in \mathbb{N}$ with $2N > \max\{s, \sigma_p\}$, where σ_p is given by* (17). *Then*

$$(35)\qquad \|K_0(1,f) \mid L_p(\mathbb{R}^n)\| + \Big(\int_0^1 t^{-sq}\|K(t,f) \mid L_p(\mathbb{R}^n)\|^q \frac{dt}{t}\Big)^{1/q}$$

(*modification if $q = \infty$*) *is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$* (see [11, 2.5.3]).

Remark 6. If $s < 0$, then $N = 0$ is admissible in the above proposition (see [14, 3.1]). In particular, one can drop the first summand in (35).

3.2. *Proof of Theorem* 1. Step 1. Due to the local nature of $\dim_{\mathrm{D}}$ we have

$$\dim_{\mathrm{D}} \Gamma = \sup_{k \in \mathbb{Z}^n} \dim_{\mathrm{D}}(\Gamma \cap 2Q_{0k})$$

with $\dim_{\mathrm{D}}(\Gamma \cap 2Q_{0k}) = 0$ if the intersection is empty. The same is true for $\dim_{\mathrm{H}}$. Hence we may assume that Γ is compact.

Step 2. We prove $\dim_{\mathrm{D}} \Gamma \ge \dim_{\mathrm{H}} \Gamma$. This is clear if $\dim_{\mathrm{H}} \Gamma = 0$ (see (19)). If $\dim_{\mathrm{H}} \Gamma > 0$, then we have $\mathcal{H}^\gamma(\Gamma) = \infty$ for every γ with

$$(36)\qquad 0 \le \gamma < \dim_{\mathrm{H}} \Gamma.$$

By [1, Theorem 5.6] there exist a compact set $C \subset \Gamma$ and a constant $c > 0$ such that

$$(37)\qquad \mathcal{H}^\gamma(C) > 0$$

and

$$(38)\qquad \mathcal{H}^\gamma(C \cap B(x,r)) \le cr^\gamma, \quad x \in \mathbb{R}^n,\ 0 < r \le 1.$$

We define $f \in S'(\mathbb{R}^n)$ by

$$(39)\qquad f(\varphi) = \int_C \varphi(x)\,\mathcal{H}^\gamma(dx), \quad \varphi \in S(\mathbb{R}^n).$$

Obviously, $f(\varphi) = 0$ if $\varphi|\Gamma = 0$. Let K be an admissible kernel in the sense of 3.1 with support in the unit ball, say. Then Proposition 2 and Remark 6 yield

$$(40)\qquad \|f \mid \mathcal{C}^{-n+\gamma}(\mathbb{R}^n)\| \sim \sup_{0<t\le 1} t^{n-\gamma}\|K(t,f) \mid L_\infty(\mathbb{R}^n)\|,$$

where "$\sim$" as usual means that there exist positive numbers c_1 and c_2 independent of f such that the left-hand side of (40) can be estimated from below by c_1 times the right-hand side and from above by c_2 times the right-hand side. Now, (34), (38) and (39) imply

$$|K(t,f)(x)| \le t^{-n} \int_C \Big|K\Big(\frac{y-x}{t}\Big)\Big| \mathcal{H}^\gamma(dy) \le c_1 t^{-n}\mathcal{H}^\gamma(C \cap B(x,t)) \le c_2 t^{-n+\gamma}$$

for every $x \in \mathbb{R}^n$. Together with (40) and (39) this gives $f \in \mathcal{C}^{-n+\gamma,\Gamma}(\mathbb{R}^n)$. Moreover, by (37), $f \ne 0$. Consequently, $\dim_{\mathrm{D}} \Gamma \ge \gamma$ and the choice (36) of γ implies $\dim_{\mathrm{D}} \Gamma \ge \dim_{\mathrm{H}} \Gamma$.

Step 3. To prove the reverse inequality we need a preparation:

LEMMA. *Let $s \le 0$ and assume $\Gamma \ne \emptyset$ to be a closed subset of $\mathbb{R}^n$ with $|\Gamma| = 0$. Let $B^s_{1,\Gamma}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ be the closure of $\{\varphi \in S(\mathbb{R}^n) : \varphi|\Gamma = 0\}$ in $B^s_{1,1}(\mathbb{R}^n)$. Then*

$$\mathcal{C}^{s,\Gamma}(\mathbb{R}^n) \ne \{0\} \quad \textit{if and only if} \quad B^{-s}_{1,\Gamma}(\mathbb{R}^n) \ne B^{-s}_{1,1}(\mathbb{R}^n).$$

Proof. Suppose that $B^{-s}_{1,\Gamma}(\mathbb{R}^n) \ne B^{-s}_{1,1}(\mathbb{R}^n)$. Then $B^{-s}_{1,\Gamma}(\mathbb{R}^n)$ is a closed proper subspace of $B^{-s}_{1,1}(\mathbb{R}^n)$. By the Hahn–Banach theorem there exists a non-trivial element in $(B^{-s}_{1,1}(\mathbb{R}^n))' = \mathcal{C}^s(\mathbb{R}^n)$, which vanishes on $B^{-s}_{1,\Gamma}(\mathbb{R}^n)$. Hence $\mathcal{C}^{s,\Gamma}(\mathbb{R}^n)$ is non-trivial.

The reverse assertion can easily be seen by contraposition: If $B^{-s}_{1,\Gamma}(\mathbb{R}^n) = B^{-s}_{1,1}(\mathbb{R}^n)$ and $f \in \mathcal{C}^{s,\Gamma}(\mathbb{R}^n)$, then f vanishes on a dense subset of $B^{-s}_{1,1}(\mathbb{R}^n)$. From this it follows that $\mathcal{C}^{s,\Gamma}(\mathbb{R}^n)$ is trivial.

Step 4. We now handle $\dim_{\mathrm{D}} \Gamma \le \dim_{\mathrm{H}} \Gamma$. Without loss of generality we may assume $\dim_{\mathrm{H}} \Gamma < n$ (see (19)). If

$$(41)\qquad \dim_{\mathrm{H}} \Gamma < \gamma < n,$$

then $\mathcal{H}^\gamma(\Gamma) = 0$. We intend to prove that $B^{n-\gamma}_{1,\Gamma}(\mathbb{R}^n) = B^{n-\gamma}_{1,1}(\mathbb{R}^n)$ and to use the Lemma. By the definition of $\mathcal{H}^\gamma$ and the compactness of Γ, for every $\varrho > 0$ there exist a $\delta > 0$ and a finite covering of Γ by open balls B_j centred at Γ and with diameters less than δ such that

$$(42)\qquad \sum_{j=1}^N (\operatorname{diam} B_j)^\gamma < \varrho,$$

where $\operatorname{diam} B_j$ is the diameter of B_j. The union $\bigcup_{j=1}^N B_j$ also covers the closure $\overline{\Gamma}_\varepsilon$ of some neighbourhood $\Gamma_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Gamma) < \varepsilon\}$ of Γ, where ε depends on ϱ. Let now $\{\varphi_j\}_{j=1}^N$ be a smooth resolution of unity on $\overline{\Gamma}_\varepsilon$ subordinate to $\{B_j\}_{j=1}^N$, i.e.

$$(43)\qquad \varphi(x) = \sum_{j=1}^N (\operatorname{diam} B_j)^\gamma (\operatorname{diam} B_j)^{-\gamma}\varphi_j(x) = 1 \quad \text{if } x \in \overline{\Gamma}_\varepsilon$$

and $\operatorname{supp}\varphi_j \subset B_j$. The functions $(\operatorname{diam} B_j)^{-\gamma}\varphi_j$ may be assumed to be $(n-\gamma, 1)$-atoms (up to constants) (see Definition 3). Hence, by (43), Proposition 1 and (42),

$$\|\varphi \mid B^{n-\gamma}_{1,1}(\mathbb{R}^n)\| \le c\sum_{j=1}^{N}(\operatorname{diam} B_j)^\gamma < c\varrho. \tag{44}$$

Let $\psi \in S(\mathbb{R}^n)$ have a compact support and choose some $\eta \in S(\mathbb{R}^n)$ with $\eta(x) = 1$ if $x \in \operatorname{supp}\psi \cup \operatorname{supp}\varphi$, where φ is given by (43). Then

$$\psi(x)(\eta(x) - \varphi(x)) = 0 \quad \text{if } x \in \Gamma \tag{45}$$

and

$$\|\psi - \psi(\eta-\varphi) \mid B^{n-\gamma}_{1,1}(\mathbb{R}^n)\| = \|\psi\varphi \mid B^{n-\gamma}_{1,1}(\mathbb{R}^n)\| \le c\|\varphi \mid B^{n-\gamma}_{1,1}(\mathbb{R}^n)\|,$$

where we used the fact that ψ is a pointwise multiplier for $B^{n-\gamma}_{1,1}(\mathbb{R}^n)$, (see [10, 2.8]). Since the set of all compactly supported functions in $S(\mathbb{R}^n)$ is dense in $B^{n-\gamma}_{1,1}(\mathbb{R}^n)$, the last inequality together with (44) and (45) and the Lemma lead to $\mathcal{C}^{-n+\gamma,\Gamma}(\mathbb{R}^n) = \{0\}$. Thus, $\dim_{\mathrm{D}}\Gamma \le \gamma$ and the choice (41) of γ implies $\dim_{\mathrm{D}}\Gamma \le \dim_{\mathrm{H}}\Gamma$.

3.3. *Proof of Theorem* 2(i). Step 1. Let $f^\Gamma \in L_p(\Gamma)$ and let f be given by (7). We prove $f \in B^{-(n-d)/p'}_{p,\infty}(\mathbb{R}^n)$ and

$$\|f \mid B^{-(n-d)/p'}_{p,\infty}(\mathbb{R}^n)\| \le c\|f^\Gamma \mid L_p(\Gamma)\| \tag{46}$$

for some $c > 0$ independent of f^Γ. We apply Proposition 2. Let K be an admissible kernel in the sense of 3.1 with support in the unit ball, say. Suppose that $p < \infty$. The modifications of the following estimates are obvious if $p = \infty$. The definition of f, (34) and Hölder's inequality imply

$$\begin{aligned}|K(t,f)(x)| &\le t^{-n}\Big(\int_\Gamma |f^\Gamma(y)|^p\Big|K\Big(\frac{y-x}{t}\Big)\Big|\mathcal{H}^d(dy)\Big)^{1/p}\Big(\int_\Gamma\Big|K\Big(\frac{y-x}{t}\Big)\Big|\mathcal{H}^d(dy)\Big)^{1/p'}\\ &\le ct^{-n}\Big(\int_\Gamma |f^\Gamma(y)|^p\Big|K\Big(\frac{y-x}{t}\Big)\Big|\mathcal{H}^d(dy)\Big)^{1/p}(\mathcal{H}^d(\Gamma\cap B(x,t)))^{1/p'}.\end{aligned}$$

By the right-hand inequality of (6), we obtain

$$\begin{aligned}&\|K(t,f) \mid L_p(\mathbb{R}^n)\|\\ &\quad\le c_1 t^{-n}t^{d/p'}\Big(\int_\Gamma |f^\Gamma(y)|^p\Big(\int_{\mathbb{R}^n}\Big|K\Big(\frac{y-x}{t}\Big)\Big|\,dx\Big)\mathcal{H}^d(dy)\Big)^{1/p}\\ &\quad\le c_2 t^{-n}t^{d/p'}t^{n/p}\|K \mid L_1(\mathbb{R}^n)\|\cdot\|f^\Gamma \mid L_p(\Gamma)\|,\end{aligned}$$

where we also used Fubini's theorem. This inequality, Proposition 2 and Remark 6 now imply (46). By (7) we have $f \in B^{-(n-d)/p',\Gamma}_{p,\infty}(\mathbb{R}^n)$.

Step 2. To prove the reverse assertion we need some elementary preparations. Let $0 < q < \infty$ and suppose that Γ is a closed d-set in $\mathbb{R}^n$ with $0 < d < n$. Furthermore, $L_q(\Gamma)$ has the above meaning with respect to the restriction of the d-dimensional Hausdorff measure to Γ. Assume that φ satisfies

$$\varphi \in S(\mathbb{R}^n), \quad \operatorname{supp}\varphi \subset 2Q_{00}, \quad \sum_{k\in\mathbb{Z}^n}\varphi(x-k) = 1 \quad \text{if } x \in \mathbb{R}^n, \tag{47}$$

where, by the above notations, Q_{00} is the unit cube centred at the origin. Let $\varrho > 0$ be a fixed number, which will be chosen sufficiently large later on. For every $\nu \in \mathbb{N}_0$ we determine a maximal $\varrho 2^{-\nu}$-distant set $\{z^{\nu,l}\}_l$ on Γ, i.e. a set of points $z^{\nu,l} \in \Gamma$ with the properties

$$|z^{\nu,l} - z^{\nu,m}| > \varrho 2^{-\nu} \quad \text{if } l \ne m \quad \text{and} \quad \Gamma \subset \bigcup_l \overline{B(z^{\nu,l}, \varrho 2^{-\nu})}.$$

Now we fix a reference point $y_{\nu k} \in \{z^{\nu,l}\}_l$ for every cube $Q_{\nu k}$ with $2Q_{\nu k}\cap\Gamma \ne \emptyset$, which minimizes the distance between $2^{-\nu}k$ (the centre of $Q_{\nu k}$) and the set $\{z^{\nu,l}\}_l$:

$$|y_{\nu k} - 2^{-\nu}k| = \min_l |z^{\nu,l} - 2^{-\nu}k|.$$

We assume ϱ to be so large that

$$y_{\nu k} = z^{\nu,l} \quad \text{if } 2Q_{\nu k}\cap B(z^{\nu,l}, 2^{-\nu}) \ne \emptyset. \tag{48}$$

Let g be a continuous function on Γ with compact support (equivalently, the restriction of a compactly supported continuous function in $\mathbb{R}^n$ to Γ). By the above assumptions it follows that

$$g_\nu(x) = \sum_k g(y_{\nu k})\varphi(2^\nu x - k)|\Gamma \to g(x) \quad \text{in } L_q(\Gamma) \tag{49}$$

as $\nu \to \infty$ and

$$\|g_\nu \mid L_q(\Gamma)\| \sim 2^{-\nu d/q}\Big(\sum_k |g(y_{\nu k})|^q\Big)^{1/q} \tag{50}$$

if ν is sufficiently large. We used (6) and the fact that $g_\nu(x) = g(y_{\nu k})$ near $y_{\nu k}$ by the above construction (see (47)–(49)). Of course the summation over k in (49) and (50) may be restricted to those $k \in \mathbb{Z}^n$ with $2Q_{\nu k}\cap\Gamma \ne \emptyset$.

Step 3. We prove the reverse assertion to Step 1. Let f be a function in $B^{-(n-d)/p',\Gamma}_{p,\infty}(\mathbb{R}^n)$. Let $g \in S(\mathbb{R}^n)$ with compact support and

$$\begin{aligned}g^\nu(x) &= \sum_k g(x)\varphi(2^\nu x - k)\\ &= \sum_k g(y_{\nu k})\varphi(2^\nu x - k) + \sum_k (g(x) - g(y_{\nu k}))\varphi(2^\nu x - k)\\ &= g_0^\nu(x) + g_1^\nu(x), \quad \nu \in \mathbb{N},\end{aligned} \tag{51}$$

where φ has the same meaning as in (47), the summation in (51) is restricted to those $k \in \mathbb{Z}^n$ with $2Q_{\nu k} \cap \Gamma \neq \emptyset$ and $y_{\nu k}$ is given by (48). By Proposition 1, $2^{(\nu d)/p'}\varphi(2^\nu x - k)$ are $((n-d)/p', p')$-atoms (up to constants), we have

$$(52) \qquad \|g_0^\nu \mid B^{(n-d)/p'}_{p',1}(\mathbb{R}^n)\| \le c_1 2^{-\nu d/p'}\Big(\sum_k |g(y_{\nu k})|^{p'}\Big)^{1/p'} \le c_2\|g_0^\nu | L_{p'}(\Gamma)\| \le c_3\|g | L_{p'}(\Gamma)\|,$$

where we used (49) and (50) for large ν. Representing the differences $g(x) - g(y_{\nu k})$ in (51) as integral remainder terms of the corresponding Taylor series, one may interpret the second sum as an atomic decomposition of g_1^ν. The application of Proposition 1 then leads to a similar estimate for g_1^ν, where now an additional factor $2^{-\nu}$ comes out. Assuming $g|\Gamma \neq 0$, we choose ν large (in dependence on g) and estimate the left-hand side of (52) with g_1^ν in place of g_0^ν by the right-hand side of (52). Hence we have a corresponding estimate for g^ν itself. By the assumptions about f and duality (see [10, 2.11]), we obtain

$$(53) \quad |f(g)| = |f(g^\nu)| \le \|f \mid B^{-(n-d)/p'}_{p,\infty}(\mathbb{R}^n)\| \cdot \|g^\nu \mid B^{(n-d)/p'}_{p',1}(\mathbb{R}^n)\| \le c\|f \mid B^{-(n-d)/p'}_{p,\infty}(\mathbb{R}^n)\| \cdot \|g \mid L_{p'}(\Gamma)\|.$$

Basic properties of the Hausdorff measure (see [1, 1.2]) imply that the continuous functions on Γ with compact support are dense in $L_{p'}(\Gamma)$. By Step 2 this applies also to the restrictions to Γ of $g \in S(\mathbb{R}^n)$ with compact support. Now (53) shows that $g \mapsto f(g)$ is a bounded linear functional on this dense subset of $L_{p'}(\Gamma)$. The representation theorem for linear continuous functionals on $L_{p'}(\Gamma)$ implies the existence of a uniquely determined $f^\Gamma \in L_p(\Gamma)$ such that f is given by (7) and

$$(54) \qquad \|f^\Gamma \mid L_p(\Gamma)\| = \|f \mid L_{p'}(\Gamma) \to \mathbb{C}\| \le c\|f \mid B^{-(n-d)/p'}_{p,\infty}(\mathbb{R}^n)\|.$$

3.4. *Proof of Theorem* 2(ii). As announced at the end of the introduction, we intend to generalize (9) in Theorem 3 below (see 4.4). Having this in mind we now prove a little more than stated in (9) covering those parts of Theorem 3 for which no additional considerations are needed.

Step 1. Let $0 < p < \infty$ and let $q \le \min\{1, p\}$. We prove

$$(55) \qquad \operatorname{tr}_\Gamma B^{(n-d)/p}_{p,q}(\mathbb{R}^n) \subset L_p(\Gamma);$$

the necessary explanation of how to understand (55) may be found in 2.3 (see in particular (20)–(22)). We rely again on atomic representations. Let $\varphi \in S(\mathbb{R}^n)$ be represented in $B^{(n-d)/p}_{p,q}(\mathbb{R}^n)$ as

$$(56) \quad \varphi = \sum_{\nu=0}^\infty \sum_{k\in\mathbb{Z}^n} \lambda_{\nu k} a_{\nu k}, \qquad \|\{\lambda_{\nu k}\}_{\nu\in\mathbb{N}_0, k\in\mathbb{Z}^n} \mid b_{p,q}\| \le c\|\varphi \mid B^{(n-d)/p}_{p,q}(\mathbb{R}^n)\|$$

(see Proposition 1). By $|a_{\nu k}(x)| \le 2^{-\nu((n-d)/p)-n/p)}$ and the fact that Γ is a d-set it follows that (56) restricted to Γ converges in $L_p(\Gamma)$ and

$$\|\varphi|\Gamma \mid L_p(\Gamma)\| \le c\|\varphi \mid B^{(n-d)/p}_{p,q}(\mathbb{R}^n)\|$$

(see also [7, Lemma 1.2/1]). The rest is a matter of completion according to 2.3, (20)–(22).

Step 2. Let $d/n < p < \infty$ and $0 < q \le \min\{p, 1\}$. We prove that under these circumstances tr_Γ in (55) is an onto map and that the related (quasi-) norms are equivalent. As mentioned in 3.3/Step 3 the compactly supported continuous functions on Γ are dense in $L_p(\Gamma)$. By 3.3/Step 2 the even functions of type (49) are dense in $L_p(\Gamma)$. Hence any $g \in L_p(\Gamma)$ can be approximated by a sequence $\{g_j|\Gamma\}_{j\in\mathbb{N}_0}$ of functions of type (49) with

$$(57) \qquad g_j(x) = \sum_k \lambda_{\nu_j k}\varphi(2^{\nu_j}x - k), \qquad j \in \mathbb{N}_0,$$

where $\nu_j \in \mathbb{N}_0$, such that

$$(58) \qquad \Big\|g - \sum_{j=0}^J g_j \;\Big|\; L_p(\Gamma)\Big\| \le c2^{-J}\|g \mid L_p(\Gamma)\|, \qquad J \in \mathbb{N}_0.$$

Again the summation over k in (57) and in what follows is restricted to those $k \in \mathbb{Z}^n$ with $2Q_{\nu k} \cap \Gamma \neq \emptyset$. Furthermore, whether functions defined on $\mathbb{R}^n$ are considered on $\mathbb{R}^n$ or on Γ will be clear from the context. Now, (58) implies

$$(59) \qquad \|g_j \mid L_p(\Gamma)\| \le c\,2^{-j}\|g \mid L_p(\Gamma)\|,$$

and under the above assumption, (50) with p in place of q yields

$$(60) \qquad \|g_j \mid L_p(\Gamma)\| \sim 2^{-\nu_j d/p}\Big(\sum_k |\lambda_{\nu_j k}|^p\Big)^{1/p}.$$

Moreover, (57) is an atomic decomposition of g_j in $B^{(n-d)/p}_{p,q}(\mathbb{R}^n)$ with the atoms $2^{\nu_j d/p}\varphi(2^{\nu_j}x - k)$ and the corresponding coefficients $2^{-\nu_j d/p}\lambda_{\nu_j k}$. Here we use $d/n < p$, since in that case we have $(n-d)/p > \sigma_p$ (see (17)) and no moment conditions for the atoms are required. Thus, Proposition 1 implies

$$(61) \qquad \|g_j \mid B^{(n-d)/p}_{p,q}(\mathbb{R}^n)\| \le c\,2^{-\nu_j d/p}\Big(\sum_k |\lambda_{\nu_j k}|^p\Big)^{1/p}.$$

Applying the t-triangle inequality with $t = \min\{1, p, q\}$ to

$$\Big\|\sum_{j=0}^J g_j \;\Big|\; B^{(n-d)/p}_{p,q}(\mathbb{R}^n)\Big\|$$

and taking (59)–(61) into account we obtain the convergence of the sequence $\{\sum_{j=0}^{J} g_j\}_{J\in\mathbb{N}_0}$ in $B^{(n-d)/p}_{p,q}(\mathbb{R}^n)$. Its limit is denoted by $\operatorname{ext} g$, i.e.

$$\operatorname{ext} g = \sum_{j=0}^{\infty}\sum_{k} 2^{-\nu_j d/p}\lambda_{\nu_j k}\, 2^{\nu_j d/p}\varphi(2^{\nu_j}x-k). \tag{62}$$

Again (62) is an atomic decomposition in $B^{(n-d)/p}_{p,q}(\mathbb{R}^n)$, and Proposition 1, (59) and (60) yield

$$\|\operatorname{ext} g \mid B^{(n-d)/p}_{p,q}(\mathbb{R}^n)\| \le c\|g \mid L_p(\Gamma)\|. \tag{63}$$

The definition of tr_Γ in Step 1 (see also 2.3/(20)–(22)) finally implies

$$\operatorname{tr}_\Gamma(\operatorname{ext} g) = g. \tag{64}$$

Remark 7. We mentioned that the two parts of Theorem 2 are dual to each other. To underline what is meant by this, we give a new proof of (55) restricted to $1<p<\infty$ and $q\le 1$ by dualizing (8). Let $\varphi\in S(\mathbb{R}^n)$. Then $\operatorname{tr}_\Gamma\varphi = \varphi|\Gamma$ and the $L_p(\Gamma)$-$L_{p'}(\Gamma)$ duality implies

$$\begin{aligned}&\|\operatorname{tr}_\Gamma\varphi \mid L_p(\Gamma)\|\\ &\quad= \sup\Big\{\Big|\int_\Gamma \operatorname{tr}_\Gamma\varphi(x) f^\Gamma(x)\,\mathcal{H}^d(dx)\Big| : f^\Gamma\in L_{p'}(\Gamma),\ \|f^\Gamma \mid L_{p'}(\Gamma)\|\le 1\Big\}.\end{aligned}$$

Now we define f by (7) and apply (8) to obtain

$$\begin{aligned}&\|\operatorname{tr}_\Gamma\varphi \mid L_p(\Gamma)\|\\ &\quad\le c_1\sup\{|f(\varphi)| : f\in B^{-(n-d)/p,\Gamma}_{p',\infty}(\mathbb{R}^n),\ \|f \mid B^{-(n-d)/p}_{p',\infty}(\mathbb{R}^n)\|\le c_2\}\\ &\quad\le c_3\|\varphi \mid B^{(n-d)/p}_{p,q}(\mathbb{R}^n)\|,\end{aligned}$$

making also use of the $B^{(n-d)/p}_{p,1}(\mathbb{R}^n)$-$B^{-(n-d)/p}_{p',\infty}(\mathbb{R}^n)$ duality and the elementary embedding $B^{(n-d)/p}_{p,q}(\mathbb{R}^n)\subset B^{(n-d)/p}_{p,1}(\mathbb{R}^n)$. The rest is again a matter of completion.

4. Comments and complements

4.1. *An independence assertion.* Let again Γ be a closed d-set in $\mathbb{R}^n$ with $0<d<n$. Let $m\in\mathbb{N}$ with $m>n$ and $\mathbb{R}^n\subset\mathbb{R}^m$ be interpreted as a hyperplane in the usual way. Of course we also have $\Gamma\subset\mathbb{R}^m$ and one can replace $B^{(n-d)/p}_{p,1}(\mathbb{R}^n)$ in (9) by $B^{(m-d)/p}_{p,1}(\mathbb{R}^m)$. For our later purposes it will be helpful to have a closer look at this independence, in particular in connection with values $p<1$. Let

$$d/n<p<\infty. \tag{65}$$

Then we have $(n-d)/p > n(1/p-1)_+$ and the corresponding assertions with $n+1,\ldots,m$ in place of n. By [10, 2.7.2] or, better, [11, p. 219], it follows that

$$\operatorname{tr}_{\mathbb{R}^n} B^{(m-d)/p}_{p,q}(\mathbb{R}^m) = B^{(n-d)/p}_{p,q}(\mathbb{R}^n),\quad 0<q\le\infty, \tag{66}$$

which again sheds light on (9).

4.2. *An open set condition.* The extension of (9) to $p<1$ causes some trouble if $p\le d/n$. For that purpose we introduce the following notation.

DEFINITION 4. Let $\Gamma\subset\mathbb{R}^n$ be a non-empty Borel set. Γ is said to satisfy the *open set condition* if there exists a $\mu\in\mathbb{N}$ such that for every cube $2Q_{\nu k}$, $\nu\in\mathbb{N}_0$, $k\in\mathbb{Z}^n$, with $2Q_{\nu k}\cap\Gamma\ne\emptyset$ one of the $2^{\mu n}$ congruent subcubes with side-length $2^{-(\nu+\mu-1)}$ does not intersect Γ.

Remark 8. Without loss of generality we may assume that the proportional cubes in the definition for $\nu\in\mathbb{N}_0$ and $k\in\mathbb{Z}^n$ are mutually disjoint. This can be achieved by enlarging the number μ if necessary.

4.3. *The spaces $F^s_{p,q}(\mathbb{R}^n)$.* We intend to extend (9) also to the spaces $F^s_{p,q}(\mathbb{R}^n)$. For that purpose we briefly recall the definition of these spaces. Let $\varphi\in S(\mathbb{R}^n)$ satisfy (13), let $\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x)$ for $j\in\mathbb{N}$ and $\varphi_0=\varphi$.

DEFINITION 5. Let $0<p<\infty$, $0<q\le\infty$ and $s\in\mathbb{R}$. Then $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f\in S'(\mathbb{R}^n)$ such that

$$\|f \mid F^s_{p,q}(\mathbb{R}^n)\|_\varphi = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j\widehat{f})^\vee(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\| \tag{67}$$

(with the usual modification if $q=\infty$) is finite.

Remark 9. As in the case of $B^s_{p,q}(\mathbb{R}^n)$, the space $F^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space (Banach space if $p\ge 1$ and $q\ge 1$) and does not depend on the function φ. Hence we omit the subscript φ in (67) in the sequel. A systematic treatment of the theory of these spaces may be found in [10] and [11].

Let $0<p\le\infty$, $0<q\le\infty$ and $\chi^{(p)}_{\nu k}(x)=2^{\nu n/p}\chi_{\nu k}(x)$, where $\chi_{\nu k}$ is the characteristic function of the cube $Q_{\nu k}$. We introduce the sequence space

$$\begin{aligned}f_{p,q} = \Big\{&\lambda=\{\lambda_{\nu k}\}_{\nu\in\mathbb{N}_0,k\in\mathbb{Z}^n}\subset\mathbb{C} :\\ &\|\lambda \mid f_{p,q}\| = \Big\|\Big(\sum_{\nu=0}^{\infty}\sum_{k\in\mathbb{Z}^n}|\lambda_{\nu k}\chi^{(p)}_{\nu k}(\cdot)|^q\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\|<\infty\Big\}\end{aligned}$$

(with the usual modification if $q=\infty$).

PROPOSITION 3. *Let $0<p<\infty$, $0<q\le\infty$ and $s\in\mathbb{R}$. Let $\sigma\ge 0$, $\sigma>s$ with $\sigma\notin\mathbb{N}$ and $L+1\in\mathbb{N}_0$ with*

$$L\ge\max\Big\{\Big[n\Big(\frac{1}{\min\{p,q\}}-1\Big)_+ - s\Big], -1\Big\}.$$

Then $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{p,q}(\mathbb{R}^n)$ *if and only if it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \lambda_{\nu k} a_{\nu k} \quad (\textit{convergence in } S'(\mathbb{R}^n)), \tag{68}$$

where $a_{\nu k}$ *are* 1_σ*-atoms* ($\nu = 0$) *or* $(s,p)_{\sigma,L}$*-atoms* ($\nu \in \mathbb{N}$) *in the sense of Definition* 3(i) *and* (ii), *respectively, with* (32), *and* $\lambda = \{\lambda_{\nu k}\}_{\nu\in\mathbb{N}_0,\, k\in\mathbb{Z}^n} \in f_{p,q}$. *Furthermore,* $\inf \|\lambda \mid f_{p,q}\|$, *where the infimum is taken over all admissible representations* (68), *is an equivalent quasi-norm in* $F^s_{p,q}(\mathbb{R}^n)$.

A proof of this assertion may be found in [4].

4.4. *An extension of Theorem* 2(ii).

THEOREM 3. *Let* Γ *be a closed* d*-set in* $\mathbb{R}^n$ *with* $0 < d < n$.

(i) *Let* $d/n < p < \infty$ *and* $0 < q \le \min\{1,p\}$. *Then*

$$\operatorname{tr}_\Gamma B^{(n-d)/p}_{p,q}(\mathbb{R}^n) = L_p(\Gamma). \tag{69}$$

(ii) *Let* Γ, *in addition, satisfy the open set condition. Then* (69) *holds for all* $0 < p < \infty$ *and* $0 < q \le \min\{1,p\}$.

(iii) *Let* $0 < p \le 1$, $0 < q \le \infty$ *and assume* Γ *to satisfy the open set condition. Then*

$$\operatorname{tr}_\Gamma F^{(n-d)/p}_{p,q}(\mathbb{R}^n) = L_p(\Gamma).$$

4.5. *Proof of Theorem* 3. Step 1. Part (i) of the theorem is covered by Steps 1 and 2 of 3.4.

Step 2. We prove (ii). The embedding $\operatorname{tr}_\Gamma B^{(n-d)/p}_{p,q}(\mathbb{R}^n) \subset L_p(\Gamma)$ is clear by 3.4/Step 1. To prove the reverse assertion we apply a similar construction to 3.4/Step 2. Since $p \le d/n$ is not excluded, we are now forced to construct atoms with vanishing moments up to a certain order $L \ge 0$ (see Definition 3(ii)).

Let $\varphi \in S(\mathbb{R}^n)$ satisfy (47). For $j \in \mathbb{N}_0$ we determine coefficients $\{\lambda_{\nu_j k}\}$ such that (57) and (58) are satisfied. If we now furnish $2^{\nu_j d/p}\varphi(2^{\nu_j}x - k)$ with vanishing moments up to order L for a fixed L with $L \ge \max\{[\sigma_p - (n-d)/p], -1\}$, where σ_p is given by (17), then it becomes an $((n-d)/p, p)$-atom. For that purpose we use a similar construction to one in the proof of [12, Theorem 3.6], i.e. we put $\varphi_{\nu_j k}(x) = \varphi(2^{\nu_j}x - k) - \psi_{\nu_j k}(x)$, where the function $\psi_{\nu_j k} \in S(\mathbb{R}^n)$ has a support in a proportional cube related to $2Q_{\nu_j k}$ via the open set condition and has the property that

$$\int_{\mathbb{R}^n} x^\beta \varphi_{\nu_j k}(x)\, dx = 0, \quad |\beta| \le L.$$

Since the proportional cubes do not intersect Γ, we have $\sum_k \varphi_{\nu_j k} \equiv 1$ on Γ. Moreover, the construction in [12, 3.6] guarantees that the sequence $\{\varphi_{\nu_j k}\}_{j\in\mathbb{N}_0,\, k\in\mathbb{Z}^n}$ is uniformly bounded. Consequently, we may proceed as in 3.4/Step 2.

Step 3. In order to prove (iii) we show that $\operatorname{tr}_\Gamma F^{(n-d)/p}_{p,q}(\mathbb{R}^n)$ is independent of the parameter q and use $F^{(n-d)/p}_{p,p}(\mathbb{R}^n) = B^{(n-d)/p}_{p,p}(\mathbb{R}^n)$. Let $0 < q < r \le \infty$. The elementary embedding $F^{(n-d)/p}_{p,q}(\mathbb{R}^n) \subset F^{(n-d)/p}_{p,r}(\mathbb{R}^n)$ implies the inclusion $\operatorname{tr}_\Gamma F^{(n-d)/p}_{p,q}(\mathbb{R}^n) \subset \operatorname{tr}_\Gamma F^{(n-d)/p}_{p,r}(\mathbb{R}^n)$. To show the reverse inclusion let $f \in F^{(n-d)/p}_{p,r}(\mathbb{R}^n)$ and choose an atomic decomposition

$$f = \sum_{\nu=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \lambda_{\nu k} a_{\nu k} \quad \text{with} \quad \|\lambda \mid f_{p,r}\| \sim \|f \mid F^{(n-d)/p}_{p,r}(\mathbb{R}^n)\|,$$

according to Proposition 3. Then we have $\operatorname{tr}_\Gamma f = \sum_{\nu=0}^\infty \sum_{k\in\mathbb{Z}^n} \lambda_{\nu k} a_{\nu k}|\Gamma$. We define a sequence $\widetilde{\lambda} = \{\widetilde{\lambda}_{\nu k}\}_{\nu\in\mathbb{N}_0,\, k\in\mathbb{Z}^n}$ with

$$\widetilde{\lambda}_{\nu k} = \begin{cases} \lambda_{\nu k} & \text{if } 2Q_{\nu k} \cap \Gamma \ne \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

(where we tacitly assumed that the atom $a_{\nu k}$ has a support in $2Q_{\nu k}$). Since Γ satisfies the open set condition, a similar construction to [12, 3.6] enables us to extend $a_{\nu k}|\Gamma$ to an atom $\widetilde{a}_{\nu k}$ on $\mathbb{R}^n$ with sufficiently many vanishing moments, i.e., in particular, to an atom for $F^{(n-d)/p}_{p,q}(\mathbb{R}^n)$. As in Step 2 of the proof of [12, Theorem 3.5] we verify

$$\|\widetilde{\lambda} \mid f_{p,q}\| \sim \Big\| \Big(\sum_{\nu=0}^{\infty}\sum_{k\in\mathbb{Z}^n} |\widetilde{\lambda}_{\nu k}\chi^{(p)}_{\nu k,\mathrm{i}}(\cdot)|^q\Big)^{1/q} \Bigm| L_p(\mathbb{R}^n)\Big\|,$$

where $\chi_{\nu k,\mathrm{i}}$ denotes the characteristic function of a proportional cube related to $2Q_{\nu k}$ via Definition 4 and $\chi^{(p)}_{\nu k,\mathrm{i}} = 2^{\nu n/p}\chi_{\nu k,\mathrm{i}}$. This is a consequence of the vector-valued Hardy–Littlewood maximal inequality. Recall now that the proportional cubes may be chosen mutually disjoint (see Remark 8). Hence the parameter q on the right-hand side of the last expression cancels and may be replaced by r. So we finally obtain

$$\|\widetilde{\lambda} \mid f_{p,q}\| \le c\|\widetilde{\lambda} \mid f_{p,r}\| \le c\|\lambda \mid f_{p,r}\|.$$

This shows that

$$g = \sum_{\nu=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \widetilde{\lambda}_{\nu k}\widetilde{a}_{\nu k}$$

belongs to $F^{(n-d)/p}_{p,q}(\mathbb{R}^n)$, and, by the construction of $\widetilde{\lambda}$ and $\widetilde{a}_{\nu k}$, it satisfies $\operatorname{tr}_\Gamma g = \operatorname{tr}_\Gamma f$. This is the reverse inclusion.

References

[1] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985.
[2] —, *Fractal Geometry*, Wiley, 1990.
[3] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 34 (1985), 777–799.
[4] —, —, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.
[5] A. B. Gulisashvili, *Traces of differentiable functions on subsets of the euclidean space*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 149 (1986), 52–66 (in Russian).
[6] A. Jonsson, *Atomic decomposition of Besov spaces on closed sets*, in: Function Spaces, Differential Operators and Non-Linear Analysis, Teubner-Texte Math. 133, Teubner, Leipzig, 1993, 285–289.
[7] A. Jonsson, *Besov spaces on closed sets by means of atomic decompositions*, preprint, Umeå, 1993.
[8] A. Jonsson and H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$, Math. Reports 2 Part 1, Harwood Acad. Publ., London, 1984.
[9] —, —, *The dual of Besov spaces on fractals*, Studia Math. 112 (1995), 285–300.
[10] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[11] —, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.
[12] H. Triebel and H. Winkelvoß, *Intrinsic atomic characterizations of function spaces on domains*, Math. Z. 221 (1996), 647–673.
[13] —, —, *The dimension of a closed subset of* $\mathbb{R}^n$ *and related function spaces*, Acta Math. Hungar. 68 (1995), 117–133.
[14] H. Winkelvoß, *Function spaces related to fractals. Intrinsic atomic characterizations of function spaces on domains*, Thesis, Jena, 1995.

Mathematisches Institut
Friedrich-Schiller-Universität Jena
D-07740 Jena, Germany
E-mail: triebel@minet.uni-jena.de

Received September 26, 1995
Revised version June 10, 1996 (3530)

STUDIA MATHEMATICA 121 (2) (1996)

A quantitative asymptotic theorem for contraction semigroups with countable unitary spectrum

by

CHARLES J. K. BATTY (Oxford), ZDZISŁAW BRZEŹNIAK (Hull) and DAVID A. GREENFIELD (Benfleet)

Abstract. Let T be a semigroup of linear contractions on a Banach space X, and let $X_s(T) = \{x \in X : \lim_{s\to\infty} \|T(s)x\| = 0\}$. Then $X_s(T)$ is the annihilator of the bounded trajectories of T^*. If the unitary spectrum of T is countable, then $X_s(T)$ is the annihilator of the unitary eigenvectors of T^*, and $\lim_s \|T(s)x\| = \inf\{\|x-y\| : y \in X_s(T)\}$ for each x in X.

1. Introduction. Let T be a semigroup of linear contractions on a Banach space X, and suppose that the unitary spectrum of T is countable. Let

$$X_s(T) = \{x \in X : \lim_{s\to\infty} \|T(s)x\| = 0\}.$$

The ABLP Theorem [2], [19], [6], [25] shows that $X_s(T) = X$ if the adjoint semigroup T^* has no unitary eigenvalues.

A variant of the ABLP Theorem [20], [21], [6] shows that $X = X_s(T) \oplus X_b(T)$, where T acts as a group of isometries on $X_b(T)$, provided that T satisfies a suitable ergodic spectral condition (which is automatic if X is reflexive). It follows easily that, for any x in X,

$$(*) \qquad \lim_s \|T(s)x\| = \inf\{\|x-y\| : y \in X_s(T)\}.$$

There are several instances where T is a C_0-semigroup generated by a differential operator A, and results have been obtained which identify the space $X_s(T)$ or which evaluate $\lim_s \|T(s)x\|$, without the conditions above being satisfied [4], [7], [10], [32]. Typically, $X = L^1(\mathbb{R}^n)$ and A^* has finitely many independent unitary eigenvectors in $L^\infty(\mathbb{R}^n)$. These results can usually be obtained by means of some more or less explicit estimates

1991 *Mathematics Subject Classification*: 47D03, 35B40.
Key words and phrases: contraction semigroup, unitary spectrum, unitary eigenvector trajectory, asymptotic stability, trivially asymptotically stable, countable, spectral synthesis.