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J. BARTOSZEWICZ (Wrocław)

TAIL ORDERINGS AND THE TOTAL TIME ON TEST TRANSFORM

Abstract. The paper presents some connections between two tail orderings of distributions and the total time on test transform. The procedure for testing the pure-tail ordering is proposed.

1. Introduction. The concept of tail-heaviness of a distribution function F permeats both the theory and practice of statistics. Among other applications it is very important in study of the efficiency and robustness of estimators as well as in the theory of extreme value statistics. Many authors have studied this concept and stochastic orderings related to them, e.g. van Zwet [21], Doksum [11], Barlow and Proschan [4], Lawrance [13], Parzen [16], Shaked [20], Loh [15], Lehmann [14], Rojo [17]. Most of the literature on tail orderings concerns orderings on the whole distribution, with the resulting ordering strongly affected by the behavior at the center. We recall some of them.

Let random variables X and Y have distribution functions F and G respectively with F^{-1} and G^{-1} the corresponding left-continuous inverses. We say that F is less than G in *dispersive ordering* ($F <^{\mathrm{disp}} G$) if and only if $F^{-1}(\beta)-F^{-1}(\alpha) \le G^{-1}(\beta)-G^{-1}(\alpha)$ whenever $0<\alpha<\beta<1$ (see [20]).

Let $F(0) = G(0) = 0$ and let the supports S_F and S_G of F and G respectively be intervals. We say that F is less than G in *convex ordering* ($F <^{\mathrm{c}} G$) if and only if $G^{-1}F$ is convex on S_F. We say that F is less than G in *star ordering* ($F <^{*} G$) if and only if $G^{-1}F$ is starshaped on S_F, i.e. $G^{-1}F(x)/x$ is increasing on S_F (see [4]).

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Recently Rojo [17] has studied a pure-tail ordering, the so called *q-ordering*.

We say that F is not more than G in *q-ordering* ($F \leq_{\mathrm{q}} G$) if

$$\limsup_{u\to 1} \frac{F^{-1}(u)}{G^{-1}(u)} < \infty.$$

Define similarly

$$F <_{\mathrm{q}} G \quad \text{if} \quad F \leq_{\mathrm{q}} G \text{ but } G \not\leq_{\mathrm{q}} F,$$

and

$$F \sim_{\mathrm{q}} G \quad \text{if} \quad F \leq_{\mathrm{q}} G \text{ and } G \leq_{\mathrm{q}} F.$$

Under very mild conditions it follows easily that q-ordering is location and scale invariant. It is obvious that if $F^{-1}(1) := \lim_{u\to 1} F^{-1}(u) < \infty$ and $G^{-1}(1) < \infty$, then $F \sim_{\mathrm{q}} G$. Also if $F \leq_{\mathrm{q}} G$ and $G^{-1}(1) < \infty$, then $F^{-1}(1) < \infty$.

Let f and g be densities of F and G respectively. If $G^{-1}(u) \to \infty$ as $u \to 1$ and $\lim_{u\to 1}[gG^{-1}(u)/fF^{-1}(u)]$ exists, then by L'Hôpital's rule

$$\lim_{u\to\infty} \frac{F^{-1}(u)}{G^{-1}(u)} = \lim_{u\to 1} \frac{gG^{-1}(u)}{fF^{-1}(u)},$$

so the q-ordering in some subclasses of distributions may be studied in terms of so-called *density-quantile functions* fF^{-1} and gG^{-1}.

Parzen [16] (see also [18]) has classified probability distributions according to the limiting behavior of $fF^{-1}(u)$ as $u \to 1$ or 0. This concept has been developed and systematized by Alzaid and Al-Osh [1].

Let f be differentiable. Parzen's approach begins with the observation that the density-quantile function has the representation

$$fF^{-1}(u) = L_F(u)(1-u)^{\alpha_F}, \qquad u \in (0,1),$$

where $L_F(u)$ is a slowly varying function as $u \to 1$ and α_F, called *the right tail exponent*, is defined by

$$\alpha_F = \lim_{u\to 1}(1-u)J_F(u)/fF^{-1}(u),$$

with

$$J_F(u) = -f'F^{-1}(u)/fF^{-1}(u).$$

(The function J_F is the well known score-function of the distribution F, which is frequently used in nonparametric statistics (see [12])). Therefore the probability distribution can be classified according to the value of its tail exponent α_F. The ranges $\alpha_F < 1$, $\alpha_F = 1$ and $\alpha_F > 1$ correspond, respectively, to short tail or limited type, medium tail or exponential type, long tail or Cauchy type. In the same manner one can define the left tail exponents and classify the distributions according to them. Our considerations will be restricted to the right tail.

Alzaid and Al-Osh [1] have considered the function

$$\alpha_F(1-u) = \frac{(1-u)J_F(u)}{fF^{-1}(u)}, \qquad u \in (0,1).$$

Then

$$\alpha_F = \lim_{u\to 1} \alpha_F(1-u).$$

For the distribution G define $\alpha_G(1-u)$ and α_G similarly. We say that F has *a shorter tail* than G in *Parzen sense* (written $F <^{\mathrm{P}} G$) if $\alpha_F < \alpha_G$. If $\alpha_F = \alpha_G$, we say F has a *similar tail* to that of G (written $F =^{\mathrm{P}} G$). It is easy to prove that the Parzen ordering is invariant under location and scale transformations.

Bartoszewicz [10] has remarked that many stochastic orderings, among them the q-ordering, are preserved by the total time on test transform, used in reliability theory. The present paper develops this observation. First we recall some definitions.

Let $\mathcal{F}$ be the class of absolutely continuous distribution functions F with positive and right (or left) continuous density f on the interval, where $0 < F < 1$. We take $F^{-1}(0)$ and $F^{-1}(1)$ to be equal to the left and right endpoints of the support of F (possibly $-\infty$ and $+\infty$). Let G be a fixed distribution from $\mathcal{F}$ with density g. Barlow and Doksum [3] have introduced the *generalized total time on test transform* (TTT transform) of the distribution F by

$$H_F^{-1}(t) = \int_{F^{-1}(0)}^{F^{-1}(t)} g[G^{-1}F(x)]\,dx, \qquad t \in (0,1), \tag{1}$$

providing the integral exists and is finite for all $t \in (0,1)$. It is obvious that H_F^{-1} depends on G, but for simplicity we will use this notation as in [3] and [7]. Since H_F^{-1} is nondecreasing, H_F is a distribution function and $H_G^{-1}(t) = t$, $t \in (0,1)$, it follows that H_G is the uniform distribution $U(0,1)$. It is worthwhile to notice that the density h_F of H_F satisfies

$$h_F H_F^{-1}(u) = \frac{fF^{-1}(u)}{gG^{-1}(u)} = r_F(F^{-1}(u)), \qquad u \in (0,1),$$

where

$$r_F(x) = \frac{f(x)}{g[G^{-1}F(x)]}$$

is the so-called *generalized failure rate function* (see [2]).

Denote by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ order statistics of a sample from the distribution F. Define $X_{0:n} = \sup\{x : F(x) = 0\}$ if it is finite. The random variables $V_{i:n} = X_{i:n} - X_{i-1:n}$, $i = 1, \ldots, n$, are called *spacings* from the distribution F. Let F_n be the empirical distribution function based on a sample of size n from the distribution $F \in \mathcal{F}$. Assuming that H_F^{-1} exists, we can estimate it by

$$H_{F_n}^{-1}(t) := \int\limits_{F_n^{-1}(0)}^{F_n^{-1}(t)} g[G^{-1}F_n(u)]\,du$$

and hence

$$H_{F_n}^{-1}(i/n) = \int\limits_{X_{0:n}}^{X_{i:n}} g[G^{-1}F_n(u)]\,du = \sum_{j=1}^{i} gG^{-1}\left(\frac{j-1}{n}\right)V_{j:n}, \qquad i = 1,\ldots,n,$$

if $X_{0:n}$ is finite while

$$H_{F_n}^{-1}(i/n) = \int\limits_{X_{1:n}}^{X_{i:n}} g[G^{-1}F_n(u)]\,du = \sum_{j=2}^{i} gG^{-1}\left(\frac{j-1}{n}\right)V_{j:n}, \qquad i = 2,\ldots,n,$$

otherwise.

Let $G \in \mathcal{F}$ be a fixed distribution and $F, K \in \mathcal{F}$ be distributions for which the respective TTT transforms H_F^{-1} and H_K^{-1}, defined by (1), exist. Since

$$h_F H_F^{-1}(u) = \frac{fF^{-1}(u)}{gG^{-1}(u)} \quad \text{and} \quad h_K H_K^{-1}(u) = \frac{kK^{-1}(u)}{gG^{-1}(u)},$$

it follows that

$$\frac{h_K H_K^{-1}(u)}{h_F H_F^{-1}(u)} = \frac{kK^{-1}(u)}{fF^{-1}(u)}. \tag{2}$$

2. Results. Let $\mathcal{F}_0 \subset \mathcal{F}$ be a class of distributions F such that $F^{-1}(u) \to \infty$ as $u \to 1$ and for any $F, K \in \mathcal{F}_0$ the limit $\lim_{u\to 1}[kK^{-1}(u)/fF^{-1}(u)]$ exists. Thus the q-ordering in the class $\mathcal{F}_0$ is defined as follows:

$$F \le_{\mathrm{q}} K \quad \text{if} \quad \lim_{u\to 1} \frac{kK^{-1}(u)}{fF^{-1}(u)} < \infty.$$

We can prove the following theorem.

THEOREM 1. *Fix $G \in \mathcal{F}_0$ and let $F, K \in \mathcal{F}_0$ be distributions for which H_F^{-1} and H_K^{-1}, defined by* (1), *exist.*

(a) *If $F \le_{\mathrm{q}} G$, then $\lim_{u\to 1} H_F^{-1}(u) < \infty$.*
(b) *If $F \le_{\mathrm{q}} K \le_{\mathrm{q}} G$, then $H_F \sim_{\mathrm{q}} H_K \sim_{\mathrm{q}} H_G \equiv U(0,1)$.*
(c) *If $G <_{\mathrm{q}} F \le_{\mathrm{q}} K$ and $\lim_{u\to 1} H_F^{-1}(u) = \infty$, then $H_G <_{\mathrm{q}} H_F \le_{\mathrm{q}} H_K$.*
(d) *If $H_G <_{\mathrm{q}} H_F \le_{\mathrm{q}} H_K$, then $G <_{\mathrm{q}} F \le_{\mathrm{q}} K$.*

P r o o f. (a) *A contrario*: suppose that $\lim_{u\to 1} H_F^{-1}(u) = \infty$. Since $H_G^{-1}(u) = u$, by L'Hôpital's rule we have

$$\lim_{u\to 1}\frac{G^{-1}(u)}{F^{-1}(u)} = \lim_{u\to 1}\frac{fF^{-1}(u)}{gG^{-1}(u)} = \lim_{u\to 1} h_F H_F^{-1}(u)$$
$$= \lim_{u\to 1}\frac{h_F H_F^{-1}(u)}{h_G H_G^{-1}(u)} = \lim_{u\to 1}\frac{H_G^{-1}(u)}{H_F^{-1}(u)} = 0,$$

and hence $\lim_{u\to 1}[F^{-1}(u)/G^{-1}(u)] = \infty$. Thus $G <_{\mathrm{q}} F$, contrary to the assumption.

(b) It follows from (a) that H_F and H_K have finite supports and hence the assertion holds.

(c) The relation $H_G <_{\mathrm{q}} H_F$ is obvious, since $H_G = U(0,1)$. Since $\lim_{u\to 1} H_F^{-1}(u) = \infty$ and $F \le_{\mathrm{q}} K$, we also have $\lim_{u\to 1} H_K^{-1}(u) = \infty$. Thus by L'Hôpital's rule and (2) we have

$$\lim_{u\to 1}\frac{H_F^{-1}(u)}{H_K^{-1}(u)} = \lim_{u\to 1}\frac{h_K H_K^{-1}(u)}{h_F H_F^{-1}(u)} = \lim_{u\to 1}\frac{kK^{-1}(u)}{fF^{-1}(u)} = \lim_{u\to 1}\frac{F^{-1}(u)}{K^{-1}(u)} < \infty,$$

i.e. $H_F \le_{\mathrm{q}} H_K$.

(d) From $H_G <_{\mathrm{q}} H_F \le_{\mathrm{q}} H_K$ it follows that $\lim_{u\to 1} H_F^{-1}(u) = \infty$ and also $\lim_{u\to 1} H_K^{-1}(u) = \infty$. Thus the statement easily follows by L'Hôpital's rule and (2), similarly to the proofs of (a) and (c).

R e m a r k 1. Theorem 1(a) gives a general condition for the finiteness of $H_F^{-1}(1) := \lim_{u\to 1} H_F^{-1}(u)$ in the class $\mathcal{F}_0$. It is well known that under some conditions each of the orderings $F <^{\mathrm{c}} G$, $F <^{*} G$ and $F <^{\mathrm{disp}} G$ implies $H_F^{-1}(1) < \infty$ (see [5], [7] and [9]). However, it is easy to notice that all these orderings imply the q-ordering.

Let now $\mathcal{F}_1 \subset \mathcal{F}$ be the class of distributions F with a differentiable density f.

L E M M A 1. *Fix $G \in \mathcal{F}_1$ and let $F \in \mathcal{F}_1$ be a distribution for which H_F^{-1} exists. Then*

$$\alpha_{H_F}(1-u) = \alpha_F(1-u) - \alpha_G(1-u), \qquad u \in (0,1),$$

and $\alpha_{H_F} = \alpha_F - \alpha_G$.

P r o o f. By the definition we have

$$\alpha_{H_F}(1-u) = \frac{(1-u)J_{H_F}(u)}{h_F H_F^{-1}(u)} = -(1-u)\frac{h'_F H_F^{-1}(u)}{[h_F H_F^{-1}(u)]^2}.$$

Since

$$h_F(x) = \frac{fF^{-1}(H_F(x))}{gG^{-1}(H_F(x))},$$

we have

$$h'_F(x) = \frac{h_F(x)}{[gG^{-1}(H_F(x))]^2} \times \left[\frac{f'F^{-1}(H_F(x))}{fF^{-1}(H_F(x))} gG^{-1}(H_F(x)) - \frac{g'G^{-1}(H_F(x))}{gG^{-1}(H_F(x))} fF^{-1}(H_F(x))\right]$$

and hence

$$h'_F(H_F^{-1}(u)) = \frac{f'F^{-1}(u) - g'G^{-1}(u)[h_F H_F^{-1}(u)]^2}{[gG^{-1}(u)]^2}.$$

Therefore

$$\begin{aligned}\alpha_{H_F}(1-u) &= -(1-u)\frac{f'F^{-1}(u) - g'G^{-1}(u)[h_F H_F^{-1}(u)]^2}{[h_F H_F^{-1}(u) gG^{-1}(u)]^2} \\ &\quad - (1-u)\left\{\frac{f'F^{-1}(u)}{[fF^{-1}(u)]^2} - \frac{g'G^{-1}(u)}{[gG^{-1}(u)]^2}\right\} \\ &= \alpha_F(1-u) - \alpha_G(1-u).\end{aligned}$$

Lemma 1 shows that the TTT transform reduces the tail exponent. The following result concerns the preservation of the Parzen ordering by the TTT transform.

THEOREM 2. *Fix $G \in \mathcal{F}_1$ and let $F, K \in \mathcal{F}_1$ be distributions for which H_F^{-1} and H_K^{-1} exist. Then $F <^{\mathrm{P}} K$ if and only if $H_F <^{\mathrm{P}} H_K$ and also $F =^{\mathrm{P}} K$ if and only if $H_F =^{\mathrm{P}} H_K$.*

Proof. The theorem follows immediately from Lemma 1 and the definition of the Parzen ordering by noticing that

$$\alpha_{H_F}(1-u) - \alpha_{H_K}(1-u) = \alpha_F(1-u) - \alpha_K(1-u), \quad u \in [0,1].$$

Hence $\alpha_{H_F} - \alpha_{H_K} = \alpha_F - \alpha_K$.

Remark 2. Theorem 2 may also be proved in another way using the Parzen representation of the density-quantile function. Let $fF^{-1}(u) = L_F(u)(1-u)^{\alpha_F}$ and $gG^{-1}(u) = L_G(u)(1-u)^{\alpha_G}$, $u \in (0,1)$. Then from (2) we have

$$h_F H_F^{-1}(u) = \frac{L_F(u)}{L_G(u)}(1-u)^{\alpha_F - \alpha_G} = L_{H_F}(u)(1-u)^{\alpha_{H_F}},$$

where $L_{H_F}(u) = L_F(u)/L_G(u)$ is also a slowly varying function (see [19]) and $\alpha_{H_F} = \alpha_F - \alpha_G$. Similarly we obtain

$$h_K H_K^{-1}(u) = L_{H_K}(u)(1-u)^{\alpha_{H_K}},$$

where $L_{H_K}(u) = L_K(u)/L_G(u)$ and $\alpha_{H_K} = \alpha_K - \alpha_G$. Thus $\alpha_{H_F} - \alpha_{H_K} = \alpha_F - \alpha_K$.

Remark 3. From Theorem 3 of Alzaid and Al-Osh [1] and Lemma 1 we have a new proof that the TTT transform preserves the convex ordering.

3. Application: testing the pure tail ordering. Let $F, G \in \mathcal{F}_0$. Assume that G is known and F is a distribution for which H_F^{-1}, defined by (1), exists. Consider the problem of testing the goodness-of-fit hypothesis

$$\mathcal{H}_0 : F(\cdot) = G\left(\frac{\cdot - \mu}{\sigma}\right), \qquad \mu \in \mathbb{R},\ \sigma > 0, \tag{3}$$

against the hypothesis

$$\mathcal{H}_1 : G <_{\mathrm{q}} F \tag{4}$$

given a random sample $\mathbf{X} = (X_1, \ldots, X_n)$ from F. Similarly to Barlow and Doksum [3], who have considered tests for convex ordering, and to Bartoszewicz [7], [8] (see also [10]), who has studied tests for dispersive ordering, we shall use the TTT transformation for the problem (3)–(4). From Theorem 1 it follows that this problem may be replaced by that of testing

$$\mathcal{H}_0' : H_F = U(0, \sigma), \qquad \sigma > 0, \tag{5}$$

against

$$\mathcal{H}_1' : H_F^{-1}(1) = \infty \qquad (\text{i.e. } H_G <_{\mathrm{q}} H_F). \tag{6}$$

The rejection of $\mathcal{H}_0'$ in favour of $\mathcal{H}_1'$ implies the rejection of $\mathcal{H}_0$ in favour of $\mathcal{H}_1$.

Notice that the testing problem (3)–(4) is location and scale invariant, while the problem (5)–(6) is scale invariant. If $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ is a sample from H_F, then the vector of order statistics $\mathbf{Z}^* = (Z_{1:n}, Z_{2:n}, \ldots, Z_{n:n})$ is a sufficient statistic for H_F and the vector

$$\widetilde{\mathbf{Z}} = \left(\frac{Z_{1:n}}{Z_{n:n}}, \frac{Z_{2:n}}{Z_{n:n}}, \ldots, \frac{Z_{n-1:n}}{Z_{n:n}}\right)$$

is a maximal invariant with respect to the group of scale transformations. Hence an invariant test of $\mathcal{H}_0'$ against $\mathcal{H}_1'$ is a function of the vector $\widetilde{\mathbf{Z}}$.

One can propose many tests for the problem (5)–(6). In particular, one can use tests which are usually applied to detect outliers. The excellent review of these procedures has been given by Barnett and Lewis [6]. It seems that some of the so-called *Dixon statistics* which are of the form

$$\frac{Z_{s:n} - Z_{r:n}}{Z_{q:n} - Z_{p:n}},$$

where $0 \le p \le r < q \le s \le n$ and $p < r$ if $s = q$, may be applied to the test problem (5)–(6). For example, consider the level α test of the form

$$\phi(\widetilde{\mathbf{z}}) = \begin{cases} 1 & \text{if } T(\widetilde{\mathbf{z}}) > c_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$T(\widetilde{\mathbf{Z}}) = Z_{n:n}/Z_{n-1:n}.$$

Under the hypothesis $\mathcal{H}_0'$ the distribution of $T(\widetilde{\mathbf{Z}})$ does not depend on σ, i.e. it is the same as in the case $H_F = U(0,1)$. One can easily show that under $\mathcal{H}_0'$ the distribution function P of $T(\widetilde{\mathbf{Z}})$ is of the form

$$P(t) = \begin{cases} 0 & \text{if } t < 1, \\ 1 - t^{-n+1} & \text{if } t \geq 1. \end{cases}$$

However, since F is unknown, $Z_{i:n}$, $i = 1, \ldots, n$, are unobservable except in the case $G(x) = x$, $0 \leq x \leq 1$. Therefore we have to replace these variables by some observable ones, uniformly close to $Z_{i:n}$, $i = 1, \ldots, n$. It is easy to notice that if $U_{1:n}, \ldots, U_{n:n}$ are order statistics from the uniform distribution $U(0,1)$, then

$$\begin{aligned} Z_{i:n} &=^{\mathrm{st}} H_F^{-1}(U_{i:n}) := \int\limits_{F^{-1}(0)}^{F^{-1}(U_{i:n})} g[G^{-1}F(x)]\,dx \\ &=^{\mathrm{st}} \int\limits_{F^{-1}(0)}^{X_{i:n}} g[G^{-1}F(x)]\,dx = H_F^{-1}(F(X_{i:n})). \end{aligned}$$

The following result, a modification of Theorems 2.1 and 2.2 of Barlow and van Zwet [5], allows us to replace $Z_{i:n}$ by the observable variables $H_F^{-1}(i/n)$, $i = 1, \ldots, n$.

THEOREM 3. *Fix $G \in \mathcal{F}$ and let $F \in \mathcal{F}$ be a distribution for which H_F^{-1} exists. Let $E(X^+) = \int_0^\infty x\,dF(x) < \infty$ and let gG^{-1} be uniformly continuous on $(0,1)$. Assume that either*

(i) *$F^{-1}(1) < \infty$, or*

(ii) *$gG^{-1}(u)/(1-u)$ is bounded on (0,1), or*

(iii) *$F, G \in \mathcal{F}_0$, $F \leq_{\mathrm{q}} G$ and there exists a number $0 < \eta < 1$ such that $gG^{-1}(u)$ is nonincreasing and $gG^{-1}(u)/(1-u)$ is nondecreasing in $u \in [\eta, 1)$.*

Then, as $n \to \infty$,

$$\max_{i \leq n} |H_{F_n}^{-1}(i/n) - Z_{i:n}| \to 0 \quad \textit{almost surely.}$$

P r o o f. Under the assumptions (i) and (ii) the theorem follows immediately from Theorem 2.1 of Barlow and van Zwet [5]. Under the assumption (iii) the proof may be copied from the proof of Theorem 2.2 of Barlow and van Zwet [5] with some modifications in those parts where properties of the convex ordering are used. It suffices to observe that $G^{-1}(1) = \infty$ and $\lim_{u\to 1}[gG^{-1}(u)/fF^{-1}(u)] < \infty$ imply that either $r_F(x) = f(x)/g[G^{-1}F(x)] \to \infty$ or $r_F(x) \to C > 0$ as $x \to \infty$.

Thus under the assumptions of Theorem 3 the statistics $H_{F_n}^{-1}(i/n)$, $i = 1, \ldots, n$, behave asymptotically like order statistics from H_F. This suggests the following invariant test of $\mathcal{H}_0$ against $\mathcal{H}_1$, which has approximately the level α:

$$\psi(\mathbf{x}) = \begin{cases} 1 & \text{if } T^*(\mathbf{x}) > c'_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$T^*(\mathbf{X}) = \frac{gG^{-1}\left(\frac{n-1}{n}\right)(X_{n:n} - X_{n-1:n})}{\sum_{j=1}^{n-1} gG^{-1}\left(\frac{j-1}{n}\right)(X_{j:n} - X_{j-1:n})}$$

and $c'_\alpha = 1/\sqrt[n-1]{\alpha}$.

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Jarosław Bartoszewicz
Mathematical Institute
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: jarbar@math.uni.wroc.pl

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