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Upper and lower solutions satisfying the inverse inequality

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Abstract. We consider multipoint and two-point BVPs for second order ordinary differential equations with a Carathéodory right hand side. We prove the existence of solutions provided there exist upper and lower solutions of the BVP and the upper solution is less than the lower one.

1. Introduction. In this paper we consider the four-point boundary value problem

$$x'' = f(t, x, x'), \tag{1}$$

$$x(a) = x(c), \quad x(d) = x(b), \tag{2}$$

where $a, b, c, d \in \mathbb{R}$, $a < c \le d < b$, $J = [a, b]$ and $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a function satisfying the Carathéodory conditions. We prove an existence result for (1), (2) under the assumption that there exist upper and lower solutions of (1), (2) which fulfil the inverse inequality, i.e. the upper solution is less than the lower one (Theorem 1).

Since the four-point conditions (2) can be considered as an approximation of the Neumann conditions

$$x'(a) = 0, \quad x'(b) = 0, \tag{3}$$

our result is valid for problem (1), (3) as well. Moreover, one of the sign conditions for f may be omitted (Theorem 5). The same approach can be used for the periodic conditions

$$x(a) = x(b), \quad x'(a) = x'(b) \tag{4}$$

(Theorem 5).

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Let us recall that $\sigma_1, \sigma_2 \in \mathbf{AC}^1(J)$ (i.e. with absolutely continuous first derivatives on J) are *lower* and *upper solutions* of (1), (k) if for a.e. $t \in J$,

$$\sigma_1''(t) \geq f(t, \sigma_1, \sigma_1'), \quad \sigma_2''(t) \leq f(t, \sigma_2, \sigma_2'), \tag{5}$$

and for $k = 2$,

$$\sigma_1(c) \geq \sigma_1(a), \quad \sigma_1(b) \leq \sigma_1(d),$$
$$\sigma_2(c) \leq \sigma_2(a), \quad \sigma_2(b) \geq \sigma_2(d);$$

for $k = 3$,

$$\sigma_1'(a) \geq 0, \quad \sigma_1'(b) \leq 0,$$
$$\sigma_2'(a) \leq 0, \quad \sigma_2'(b) \geq 0;$$

for $k = 4$,

$$\sigma_1(b) = \sigma_1(a), \quad \sigma_1'(b) \leq \sigma_1'(a),$$
$$\sigma_2(b) = \sigma_2(a), \quad \sigma_2'(b) \geq \sigma_2'(a).$$

Under the classical assumption that

$$\sigma_1(t) \leq \sigma_2(t) \quad \text{for all } t \in J \tag{6}$$

the existence of solutions of various second order boundary value problems has been proved by many authors. We can refer for example to [1], [3], [5]–[7], [17]. For the periodic problem with a Carathéodory right hand side f and a generalized Nagumo condition see [4] or [12].

Here, we investigate the case where σ_1, σ_2 satisfy the opposite ordering condition

$$\sigma_2(t) \leq \sigma_1(t) \quad \text{for all } t \in J. \tag{7}$$

Conditions (5) and (7) are satisfied for example if

$$\limsup_{|x|\to\infty} f(t, x, 0)/x < 0, \tag{8}$$

uniformly for a.e. $t \in J$. We can see that (8) yields the existence of constants $r_1 > 0$ and $r_2 < 0$ such that $f(t, r_1, 0) \leq 0$, $f(t, r_2, 0) \geq 0$ for a.e. $t \in J$. So, we put $\sigma_1(t) \equiv r_1$, $\sigma_2(t) \equiv r_2$ and get lower and upper solutions for problem (1), (k), $k \in \{2, 3, 4\}$. On the other hand, condition (8) expresses the fact that for large $|x|$ the nonlinearity $f(t, x, 0)/x$ lies on the side of the spectrum of the linear differential operator $\mathbf{L} : x \mapsto x''$ acting on the space of functions belonging to $\mathbb{A}C^1(J)$ and satisfying (k), $k \in \{2, 3, 4\}$. This means that the nonlinearity $f(t, x, 0)/x$ could interact with higher eigenvalues of $\mathbb{L}$. This situation was considered e.g. in [2], [9]–[11], [16], [18], mainly for periodic or Dirichlet boundary conditions and for special differential equations like the Liénard or Rayleigh equations.

Our approach is quite different and we present other types of conditions which guarantee the existence of solutions of (1), (k), $k \in \{2, 3, 4\}$, provided

(7) is valid. We do not impose growth restrictions on f with respect to x or x' but we need some sign conditions for f.

2. Main results

THEOREM 1. *Let σ_1 be a lower solution and σ_2 an upper solution of* (1), (2), *$\sigma_1'', \sigma_2'' \in \mathbf{L}_\infty(J)$ and let* (7) *be satisfied. Suppose that there exist real numbers R_1, R_2, R_3, R_4 such that $R_1 \neq R_3$, $R_2 \neq R_4$, $R_1 \le \sigma_i'(t) \le R_2$, $R_3 \le \sigma_i'(t) \le R_4$ for each $t \in J$, $i = 1, 2$, and for a.e. $t \in J$,*

$$f(t, x, \sigma_2') \ge \sigma_2'' \quad \text{for all } x \in [c_1, \sigma_2(t)], \tag{9}$$

$$f(t, x, \sigma_1') \le \sigma_1'' \quad \text{for all } x \in [\sigma_1(t), c_2], \tag{10}$$

and for all $x \in [c_1, c_2]$,

$$f(t, x, R_1) \le 0, \quad f(t, x, R_2) \ge 0 \quad \text{for a.e. } t \in J, \tag{11}$$

$$f(t, x, R_3) \ge 0, \quad f(t, x, R_4) \le 0 \quad \text{for a.e. } t \in [d, b], \tag{12}$$

where $c_1 = (b-a)L_1 + \min\{\sigma_2(t) : t \in J\}$, $c_2 = (b-a)L_2 + \max\{\sigma_1(t) : t \in J\}$, $L_1 = \min\{R_1, R_3\}$ and $L_2 = \max\{R_2, R_4\}$. Then problem (1), (2) *has at least one solution u with*

$$\sigma_2(t_u) \le u(t_u) \le \sigma_1(t_u) \quad \text{for some } t_u \in J. \tag{13}$$

COROLLARY 2. *Let f be nonincreasing in the second variable x. Then, in Theorem* 1, *conditions* (9), (10) *can be omitted and* (11), (12) *can be replaced by*

$$f(t, c_1, R_1) \le 0, \quad f(t, c_2, R_2) \ge 0 \quad \text{for a.e. } t \in J, \tag{14}$$

$$f(t, c_2, R_3) \ge 0, \quad f(t, c_1, R_4) \le 0 \quad \text{for a.e. } t \in [d, b]. \tag{15}$$

EXEMPLE 3. The function $f(t,x,y) = x - x^3 + \cos 2\pi t + \alpha y(y^2-1)$ satisfies the conditions of Theorem 1 for sufficiently large α. If we choose $J = [0,1]$, we can put $\sigma_1 = 1.35$, $\sigma_2 = -1.35$, $R_1 = -0.4$, $R_2 = 0.4$, $R_3 = -1.1$, $R_4 = 1.1$ and $|\alpha| \ge 60$.

EXEMPLE 4. The function $f(t,x,y) = -x + \sin t + \alpha \sin y$ satisfies the conditions of Corollary 2 for sufficiently large α. If we choose $J = [a,b]$, we can put $\sigma_1 = 1$, $\sigma_2 = -1$, $R_1 = -\pi/2$, $R_2 = \pi/2$, $R_3 = -3\pi/2$, $R_4 = -3\pi/2$, $|\alpha| > 3\pi(b-a)/2 + 1$.

THEOREM 5. *Let σ_1 be a lower solution and σ_2 an upper solution of* (1), (k), $k \in \{3, 4\}$, *with $\sigma_1'', \sigma_2'' \in \mathbf{L}_\infty(J)$, and let* (7) *be satisfied. Suppose that there exist real numbers R_1, R_2 such that $R_1 \le \sigma_i'(t) \le R_2$ for each $t \in J$, $i = 1, 2$, conditions* (9), (10) *are satisfied for a.e. $t \in J$ and condition* (11) *is satisfied for all $x \in [c_1, c_2]$, where $c_1 = (b-a)R_1 + \min\{\sigma_2(t) : t \in J\}$ and $c_2 = (b-a)R_2 + \max\{\sigma_1(t) : t \in J\}$. Then problem* (1), (k), $k \in \{3, 4\}$, *has at least one solution satisfying* (13).

Corollary 6. *Let f be nonincreasing in the second variable x. Then, in Theorem* 5, *conditions* (9), (10) *can be omitted and* (11) *can be replaced by* (14).

Example 7. If we choose f, J, σ_1, σ_2, R_1, R_2 as in Example 3, then the conditions of Theorem 5 are satisfied for $|\alpha| > 1$.

Example 8. If we choose f, J, σ_1, σ_2, R_1, R_2 as in Example 4, then the conditions of Corollary 6 are satisfied for $|\alpha| > \pi(b-a)/2 + 1$.

Remark. Let f be bounded (sublinear or linear with appropriately small coefficients) in x and y. Further, suppose that there exist upper and lower solutions of problem (1), (k), $k \in \{2,3,4\}$, with σ_1'', $\sigma_2'' \in \mathbf{L}_\infty(J)$, and for a.e. $t \in J$,

$$\begin{aligned} &f(t,x,\sigma_2') \ge \sigma_2'' \quad \text{for all } x \le \sigma_2(t),\\ &f(t,x,\sigma_1') \le \sigma_1'' \quad \text{for all } x \ge \sigma_1(t). \end{aligned} \tag{16}$$

Then condition (7) is sufficient for the solvability of (1), (k), $k \in \{2,3,4\}$. See Lemma 11 for f bounded and [15] for sublinear and linear cases.

3. Proofs. We will consider a one-parameter system of equations

$$x'' = \lambda f^*(t,x,x',\lambda), \qquad \lambda \in [0,1], \tag{17}$$

where $f^* : J \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e.

- $f(\cdot,x,y,\lambda) : J \to \mathbb{R}$ is measurable for all $(x,y,\lambda) \in \mathbb{R}^2 \times [0,1]$,
- $f(t,\cdot,\cdot,\cdot) : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ is continuous for a.e. $t \in J$,
- $\sup\{|f(\cdot,x,y,\lambda)| : |x|+|y| < \varrho,\ \lambda \in [0,1]\} \in \mathbf{L}(J)$ for any $\varrho \in \mathbb{R}_+$.

Let f^* be chosen such that $f^*(t,x,y,1) = f(t,x,y)$ on $J \times \mathbb{R}^2$. Further, put

$$f_{0k}(x) = \frac{1}{b-a}\int_a^b f^*(t,x,0,0)\,dt \qquad \text{for } k \in \{3,4\}$$

and

$$f_{02}(x) = \frac{1}{c_0}\left[\frac{1}{b-d}\int_d^b\int_a^s f^*(\tau,x,0,0)\,d\tau\,ds - \frac{1}{c-a}\int_a^c\int_a^s f^*(\tau,x,0,0)\,d\tau\,ds\right]$$

where

$$c_0 = \frac{b+d}{2} - \frac{c+a}{2}.$$

In our proofs we exploit the following lemma:

Lemma 9. *Let $k \in \{2,3,4\}$ and suppose there exists an open bounded set $\Omega \subset \mathbf{C}^1(J)$ such that*: (a) *for any $\lambda \in (0,1)$, every solution u of problem* (17), (k) *satisfies $u \notin \partial\Omega$*; (b) *for any root $x_0 \in \mathbb{R}$ of the equation $f_{0k}(x) = 0$, the condition $x_0 \notin \partial\Omega$ holds, where x_0 is considered as a constant function* on J; (c) *the Brouwer degree* $d[f_{0k}, D, 0]$ *is not zero, where* $D \subset \mathbb{R}$ *is the set of constants* c *such that the constant functions* $u(t) \equiv c$ *belong to* Ω. *Then problem* (17), (k) *has at least one solution in* $\overline{\Omega}$.

Proof. For $k = 2$ see [13] or [14], for $k \in \{3, 4\}$ see [3] or [8]. ■

First we prove the existence for problem (1), (k), $k \in \{2, 3, 4\}$, provided f is bounded and upper and lower solutions are constants.

LEMMA 10. *Suppose that there exist* $r_1, r_2 \in \mathbb{R}$ *and* $K \in (0, \infty)$ *such that* $r_1 \geq r_2$ *and for a.e.* $t \in J$,

$$f(t, x, 0) \geq 0 \quad \text{for all } x \leq r_2,$$
$$f(t, x, 0) \leq 0 \quad \text{for all } x \geq r_1$$

and

$$\int_a^b |f(t, x, y)|\, dt \leq K \quad \text{for all } x, y \in \mathbb{R}.$$

Then problem (1), (k), $k \in \{2, 3, 4\}$, *has at least one solution* u *with*

$$r_2 \leq u(t_u) \leq r_1,$$

where t_u *is a point from* J.

Proof. For every $m \in \mathbb{N}$, $m \geq 2$, and $(t, x, y) \in J \times \mathbb{R}^2$, set

$$f_m(t, x, y) = \begin{cases} f(t, x, y) & \text{for } |y| > 2/m, \\ f(t, x, y) + [f(t, x, 0) - f(t, x, y)]m(2/m - |y|) & \\ & \text{for } 1/m < |y| \leq 2/m, \\ f(t, x, 0) & \text{for } |y| \leq 1/m, \end{cases}$$

and consider system (17) where

$$f^*(t, x, y, \lambda) = \lambda f_m(t, x, y) + (1 - \lambda)\frac{r_1 - x}{|r_1| + |x|}.$$

Define

$$\Omega = \{x \in \mathbf{C}^1(J) : \|x\|_\infty < r,\ \|x'\|_\infty < K + 2(b - a)\},$$

where $r = \max\{|r_1|, |r_2|\} + 1 + (b - a)K + 2(b - a)^2$. Now we use Lemma 9 and first prove that for any $\lambda \in (0, 1)$ no solution of (17), (k), $k \in \{2, 3, 4\}$, belongs to $\partial\Omega$. Suppose that u is a solution of (17), (k), $k \in \{2, 3, 4\}$, for some $\lambda \in (0, 1)$. Put $v(t) = u(t) - r_1 - 1/m$ and suppose

$$\min\{v(t) : t \in J\} = v(t_0) > 0.$$

Then there exists an interval $[\alpha, \beta] \subset J$ containing t_0 with $v(t) \geq 0$ and $|v'(t)| \leq 1/m$ for all $t \in [\alpha, \beta]$, $v'(\alpha) \leq 0$, $v'(\beta) \geq 0$. Then for a.e. $t \in (\alpha, \beta)$ we get

$$v''(t) = u''(t) = \lambda f_m(t, u, u') + (1 - \lambda)\frac{r_1 - u}{|r_1| + |u|} < \lambda f(t, u, 0) \leq 0.$$

On the other hand, $\int_\alpha^\beta v''(t)\,dt = v'(\beta) - v'(\alpha) \ge 0$, a contradiction. Analogously we can get a contradiction if

$$\max\{u(t) : t \in J\} < r_2 - 1/m.$$

Thus there exists $t_u \in J$ such that

$$r_2 - 1/m \le u(t_u) \le r_1 + 1/m.$$

Now, by integrating (1) from $\bar{t}$ to t, where $\bar{t}$ is a zero of u', we get $|u'(t)| \le K + 2(b-a)$ for all $t \in J$, and integrating the last inequality from t_u to t we get $|u(t)| \le \max\{|r_1|, |r_2|\} + 1/m + 2(b-a)^2 + K(b-a)$ for all $t \in J$. Thus $u \notin \partial\Omega$.

Now, consider the function f_{0k} from Lemma 9 which has for all three cases of $k = 2, 3, 4$ the same form $f_{0k}(x) = (r_1 - x)/(|r_1| + |x|)$. Since the unique root of the equation $f_{0k}(x) = 0$ is $x_0 = r_1$ and the constant function $x_0 \notin \partial\Omega$, condition (b) of Lemma 9 is satisfied. Finally, we compute the Brouwer degree $d[f_{0k}, (-r, r), 0]$. It is equal to ± 1, because $f_{0k}(-r) < 0$ and $f_{0k}(r) > 0$. Therefore problem (17), (k), $k \in \{2, 3, 4\}$, has for $\lambda = 1$, $m \ge 2$, at least one solution $u_m \in \overline{\Omega}$. Following these considerations for each $m \in \mathbb{N}$, $m \ge 2$, we get a sequence $(u_m)_{m=2}^\infty$ of solutions of the problems

$$x'' = f_m(t, x, x'),\ (k), \qquad k \in \{2, 3, 4\}.$$

This sequence is equi-continuous and bounded in $\mathbf{C}^1(J)$ and therefore, by the Arzelà–Ascoli Theorem, we can choose a subsequence converging in $\mathbf{C}^1(J)$ to u_0 which is a solution of (17), (k), $k \in \{2, 3, 4\}$. Moreover, there exists a sequence $(t_{u_m})_{m=2}^\infty$ of points with the property

$$r_2 - 1/m \le u_m(t_{u_m}) \le r_1 + 1/m,$$

which implies the existence of $\tau \in J$ such that $r_2 \le u_0(\tau) \le r_1$. ∎

The second step consists in the change of constant upper and lower solutions to functions depending on t.

Lemma 11. *Let σ_1, σ_2 be lower and upper solutions of* (1), (k), $k \in \{2, 3, 4\}$, *with $\sigma_1'', \sigma_2'' \in \mathbf{L}_\infty(J)$. Suppose that conditions* (7) *and* (16) *are fulfilled. Further, suppose there exists $K \in (0, \infty)$ such that*

$$\int_a^b |f(t, x, y)|\,dt \le K \quad \text{for all } x, y \in \mathbb{R}.$$

Then problem (1), (k), $k \in \{2, 3, 4\}$, *has at least one solution u satisfying* (13).

Proof. For $n \in \mathbb{N}$ and $(t,x,y) \in J \times \mathbb{R}^2$ set

$$f_n(t,x,y) = \begin{cases} f(t,x,y) + (-1)^i/n & \text{for } |y - \sigma_i'(t)| > 2/n, \\ f(t,x,y) + (-1)^i/n + \kappa_{i,n} & \text{for } 1/n < |y - \sigma_i'(t)| \le 2/n, \\ f(t,x,\sigma_i'(t)) + (-1)^i/n & \text{for } |y - \sigma_i'(t)| \le 1/n, \end{cases}$$

where

$$\kappa_{i,n} = [f(t,x,\sigma_i'(t)) - f(t,x,y)]n(2/n - |y - \sigma_i'(t)|), \qquad i = 1,2,$$

and

$$g_n(t,x,y) = \begin{cases} f_n(t,x,\sigma_1') + \frac{A+1/n-x}{x-A}\|\sigma_1''\|_\infty & \text{for } x \ge A + 1/n, \\ f_n(t,x,y) + w_{1,n} & \text{for } A < x < A + 1/n, \\ f_n(t,x,y) & \text{for } -A \le x \le A, \\ f_n(t,x,y) - w_{2,n} & \text{for } -A - 1/n < x < -A, \\ f_n(t,x,\sigma_2') + \frac{A+1/n+x}{A+x}\|\sigma_2''\|_\infty & \text{for } x \le -A - 1/n, \end{cases}$$

where

$$w_{i,n} = [f_n(t,x,\sigma_i'(t)) - f_n(t,x,y)]n(x + (-1)^i A), \qquad i = 1,2,$$
$$A = (b-a)L + \frac{1}{n} + \max\{\|\sigma_1\|_\infty, \|\sigma_2\|_\infty\},$$
$$L = K + \frac{b-a}{n} + 2(b-a)\max\{\|\sigma_1''\|_\infty, \|\sigma_2''\|_\infty\}.$$

We can easily see that $w_{i,n} \equiv 0$ for $|y - \sigma_i'(t)| \le 1/n$. Now consider the problem

$$x'' = g_n(t,x,x'),\ (k), \qquad k \in \{2,3,4\}. \tag{18}$$

We can check that for $r_1 = 2A + 1/n$ and $r_2 = -2A - 1/n$ the function g_n satisfies all assumptions of Lemma 10, so problem (18) has a solution u_n for each $n \in \mathbb{N}$. Now, using similar arguments to the proof of Lemma 10, we can prove that there exist $t_n = t_n(u_n) \in J$ such that

$$-1/n + \sigma_2(t_n) \le u_n(t_n) \le \sigma_1(t_n) + 1/n. \tag{19}$$

Indeed, putting $v(t) = u_n(t) - \sigma_1(t) - 1/n$ (or $v(t) = \sigma_2(t) - u_n(t) - 1/n$) and supposing $\min\{v(t) : t \in J\} > 0$ (or $\max\{v(t) : t \in J\} < 0$), we get a contradiction. We can also easily check that

$$\int_a^b |g_n(t,x,y)|\,dt \le L \qquad \text{for all } x,y \in \mathbb{R}.$$

From the latter inequality we get $\|u_n'\|_\infty \le L$ and $\|u_n\|_\infty \le A$, so u_n is a solution of the problem

$$x'' = f_n(t,x,x'),\ (k), \qquad k \in \{2,3,4\}.$$

Using the Arzelà–Ascoli Theorem for the sequence $(u_n)_{n=1}^{\infty}$ in the space $\mathbf{C}^1(J)$, by a limiting process we obtain a solution u of problem (1), (k), $k \in \{2,3,4\}$, satisfying (13). ■

P r o o f o f T h e o r e m 1. Suppose that $R_1 < R_3$ and $R_2 > R_4$. Then for sufficiently large $n_0 \in \mathbb{N}$ we have $R_4 + 2/n_0 < R_2$ and $R_3 - 2/n_0 > R_1$. Suppose that $n \in \mathbb{N}$, $n \ge n_0$ and put $r_1 = \max\{\sigma_1(t) : t \in J\}$, $r_2 = \min\{\sigma_2(t) : t \in J\}$,

$$g(t,x,y) = \begin{cases} f(t, r_1 + R_2(b-a), y) & \text{for } x > r_1 + R_2(b-a), \\ f(t,x,y) & \text{for } r_2 + R_1(b-a) \le x \le r_1 + R_2(b-a), \\ f(t, r_2 + R_1(b-a), y) & \text{for } x < r_2 + R_1(b-a), \end{cases}$$

$$h_n(t,x,y) = \begin{cases} g(t,x,R_2) & \text{for } y \ge R_2, \\ g(t,x,y) & \text{for } R_4 + 2/n \le y < R_2, \\ g(t,x,R_4 + 2/n) + w_4 & \text{for } R_4 + 1/n < y < R_4 + 2/n, \\ g(t,x,R_4) & \text{for } R_4 < y \le R_4 + 1/n, \\ g(t,x,y) & \text{for } R_3 \le y \le R_4, \\ g(t,x,R_3) & \text{for } R_3 - 1/n \le y < R_3, \\ g(t,x,R_3 - 2/n) - w_3 & \text{for } R_3 - 2/n < y < R_3 - 1/n, \\ g(t,x,y) & \text{for } R_1 < y \le R_3 - 2/n\ , \\ g(t,x,R_1) & \text{for } R_1 \ge y, \end{cases}$$

where

$$w_3 = [g(t,x,R_3 - 2/n) - g(t,x,R_3)]n(y - R_3 + 2/n),$$
$$w_4 = [g(t,x,R_4 + 2/n) - g(t,x,R_4)]n(y - R_4 - 2/n).$$

Then for all $x, y \in \mathbb{R}$,

$$\int_a^b |h_n(t,x,y)|\,dt \le \int_a^b h(t)\,dt = K,$$

where $h(t) = \sup\{|h_n(t,x,y)| : x \in [r_2 + R_1(b-a), r_1 + R_2(b-a)],\ y \in [R_1, R_2]\}$. We can see that h_n satisfies the conditions of Lemma 11 and so the problem

$$x'' = h_n(t,x,x'),\ (2)$$

has a solution u_n satisfying (13).

Let us prove a priori estimates for u_n'. It follows from (2) that there exist $a_0 \in (a,c)$ and $b_0 \in (d,b)$ with $u_n'(a_0) = u_n'(b_0) = 0$. Suppose that $\max\{u_n'(t) : t \in [a,b_0]\} = u_n'(\gamma) > R_2 + 1/n$. Then $\gamma \ne b_0$ and there exists $(\alpha,\beta) \subset (a,b_0)$ such that $u_n'(\beta) = R_2$, $u_n'(\alpha) = R_2 + 1/n$ and $R_2 \le u_n'(t) \le R_2 + 1/n$ for all $t \in (\alpha,\beta)$. Thus $\int_\alpha^\beta u_n''(t)\,dt = -1/n < 0$. On the other hand, by (11), (12) and the construction of h_n,

$$\int_\alpha^\beta u_n''(t)\,dt = \int_\alpha^\beta g(t,u_n,R_2)\,dt \ge 0,$$

a contradiction. Similarly, supposing $\min\{u_n'(t) : t\in[a,b_0]\} < R_1 - 1/n$, we get a contradiction. Therefore

$$R_1 - 1 \le R_1 - 1/n \le u_n'(t) \le R_2 + 1/n \le R_2 + 1 \tag{21}$$

for all $t \in [a,b_0]$.

Now, suppose that $\max\{u_n'(t) : t\in[b_0,b]\} = u_n'(\gamma) > R_4 + 1/n$. Then $\gamma\in(b_0,b]$ and there exists $(\alpha,\beta)\subset(b_0,b)$ such that $u_n'(\alpha) = R_4$, $u_n'(\beta) = R_4 + 1/n$, and $R_4 \le u_n'(t) \le R_4 + 1/n$ for all $t\in(\alpha,\beta)$. Thus $\int_\alpha^\beta u_n''(t)\,dt = 1/n > 0$. On the other hand, by (11), (12) and the construction of h_n,

$$\int_\alpha^\beta u_n''(t)\,dt = \int_\alpha^\beta g(t,u_n,R_4)\,dt \le 0,$$

a contradiction. Similarly, supposing $\min\{u_n'(t) : t\in[b_0,b]\} < R_3 - 1/n$, we get a contradiction. Thus, by (21),

$$R_1 - 1/n \le u_n'(t) \le R_2 + 1/n \tag{22}$$

for all $t\in J$. Integrating (22) from t_u to t and using (13), we get

$$r_2 + R_1(b-a) - \frac{b-a}{n} \le u_n(t) \le r_1 + R_2(b-a) + \frac{b-a}{n} \tag{23}$$

for all $t\in J$.

For each $n\in\mathbb{N}$, $n\ge n_0$, we have a solution u_n satisfying estimates (22) and (23). Since the sequence $(u_n)_{n=n_0}^\infty$ is bounded and equi-continuous in $\mathbf{C}^1(J)$, we can use the Arzelà–Ascoli Theorem and get a solution u of the problem

$$x'' = g(t,x,x'),\ (2),$$

with $r_2 + R_1(b-a) \le u(t) \le r_1 + R_2(b-a)$ and $R_1 \le u'(t) \le R_2$ for all $t\in J$. Hence u is a solution of (1), (2) as well. ∎

Proof of Theorem 5. We can follow the previous proof and make some simplifications. E.g. instead of h_n we use

$$h(t,x,y) = \begin{cases} g(t,x,R_2) & \text{for } y > R_2,\\ g(t,x,y) & \text{for } R_1 \le y \le R_2,\\ g(t,x,R_1) & \text{for } y < R_1,\end{cases}$$

and for a priori estimates of u' we only need condition (11), because in the case of problem (1), (3), u' has zero values at the end points, and if we consider periodic problem (1), (4), it is sufficient to prove the estimate of u' only at one of the end points. ∎

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