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A generalized periodic boundary value problem for the one-dimensional p-Laplacian

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Abstract. The generalized periodic boundary value problem $-[g(u')]' = f(t,u,u')$, $a < t < b$, with $u(a) = \xi u(b) + c$ and $u'(b) = \eta u'(a)$ is studied by using the generalized method of upper and lower solutions, where $\xi, \eta \geq 0$, a, b, c are given real numbers, $g(s) = |s|^{p-2}s$, $p > 1$, and f is a Carathéodory function satisfying a Nagumo condition. The problem has a solution if and only if there exists a lower solution α and an upper solution β with $\alpha(t) \leq \beta(t)$ for $a \leq t \leq b$.

1. Introduction. The present paper is a continuation of the papers [1] and [2].

In this paper, we study the following generalized periodic boundary value problem for the one-dimensional p-Laplacian:

$$\begin{cases} -[g(u')]' = f(t,u,u'), \quad t \in I := [a,b], \\ u(a) = \xi u(b) + c, \quad u'(b) = \eta u'(a), \end{cases} \tag{1.1}$$

by using the generalized method of upper and lower solutions. Here $\xi, \eta \geq 0$, a, b, c are given real numbers, $g(s) = |s|^{p-2}s$, $p > 1$, and $f(t,u,v)$ is a Carathéodory function satisfying a Nagumo condition.

We name the problem a *generalized periodic boundary value problem* since the periodic boundary value problem is its particular case.

We call a function $\alpha : I \to \mathbb{R}$ a *lower solution* to problem (1.1) if $\alpha \in C^1(I)$, $g(\alpha') \in AC(I)$, and

$$\begin{cases} -[g(\alpha'(t))]' \leq f(t,\alpha(t),\alpha'(t)) \quad \text{for a.e. } t \in I, \\ \alpha(a) = \xi\alpha(b) + c, \quad \alpha'(b) \leq \eta\alpha'(a), \end{cases}$$

where $AC(I)$ is the set of all absolutely continuous functions defined on I.

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Similarly, a function $\beta : I \to \mathbb{R}$ is called an *upper solution* to (1.1) if $\beta \in C^1(I)$, $g(\beta') \in AC(I)$, and

$$\begin{cases} -[g(\beta'(t))]' \geq f(t, \beta(t), \beta'(t)) & \text{for a.e. } t \in I, \\ \beta(a) = \xi\beta(b) + c, \quad \beta'(b) \geq \eta\beta'(a). \end{cases}$$

A function $u : I \to \mathbb{R}$ is said to be a *solution* to (1.1) if it is both a lower solution and an upper solution to (1.1).

We call a function $f : I \times \mathbb{R}^2 \to \mathbb{R}$ a *Carathéodory function* if the following two conditions are satisfied:

(1) for almost all $t \in I$, the function $(u, v) \to f(t, u, v)$ is continuous on $\mathbb{R}^2$, and

(2) for every $(u, v) \in \mathbb{R}^2$, the function $t \to f(t, u, v)$ is measurable on I.

The function f is said to satisfy a *Nagumo condition* on the set

$$D := \{(t, u, v) : t \in I,\ \alpha(t) \leq u \leq \beta(t),\ v \in \mathbb{R}\}$$

for given $\alpha, \beta \in C(I)$ with $\alpha(t) \leq \beta(t)$ on I if there exists a positive measurable function $k \in L_\sigma(I)$, $1 \leq \sigma \leq \infty$, and a positive continuous function $H \in C(\mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$ such that

$$|f(t, u, v)| \leq k(t)H(|v|) \quad \text{a.e. on } D \tag{1.2}$$

and

$$\int_{g(A)}^{\infty} \frac{|G(s)|^{(\sigma-1)/\sigma}}{H(|G(s)|)}\, ds > B^{(\sigma-1)/\sigma}\|k\|_\sigma, \tag{1.3}$$

where G is the function inverse to g,

$$A := \max\{|\beta(a) - \alpha(b)|, |\beta(b) - \alpha(a)|\}/(b - a), \tag{1.4}$$

$$B := \max\{\beta(t) : t \in I\} - \min\{\alpha(t) : t \in I\} \tag{1.5}$$

and

$$\|k\|_\sigma := \begin{cases} \Big(\int_a^b |k(s)|^\sigma\, ds\Big)^{1/\sigma} & \text{if } \sigma \in [1, \infty), \\ \operatorname{ess\,sup}\{|k(t)| : t \in I\} & \text{if } \sigma = \infty. \end{cases}$$

Here we set $B^0 := 1$ and $|G(s)|^0 := 1$.

The main result of this paper is as follows.

Theorem 1. *Assume that f is a Carathéodory function satisfying a Nagumo condition. Then a necessary and sufficient condition for the problem* (1.1) *to have a solution u is that there exists a lower solution α and an upper solution β with $\alpha(t) \leq \beta(t)$ on I. Moreover*,

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \textit{and} \quad |u'(t)| \leq N \quad \textit{on } I,$$

where N is a constant depending only on α, β, g, H and k.

Obviously, Theorem 1 extends and improves Theorem 1 of [1] and Theorem 2.4 of [2].

2. Proof of Theorem 1. The necessity part is obvious. We prove the sufficiency.

Now assume that α and β are lower and upper solutions to problem (1.1), respectively, and $\alpha(t) \le \beta(t)$ on I. To prove the existence of solutions (1.1), we consider the modified problem

$$\left\{\begin{aligned} &-[g(u')]' + Mg(u') \\ &\qquad = f^*\Big(t, q(t,u), \frac{d}{dt}q(t,u)\Big) + Mg\Big(\frac{d}{dt}q(t,u)\Big), \quad t \in I, \\ &u(a) = \xi q(b, u(b)) + c, \quad u'(b) = \eta u'(a), \end{aligned}\right. \tag{2.1}$$

where M is a positive number such that $e^{M(b-a)} > \eta^{p-1}$,

$$q(t,u) := \begin{cases} \alpha(t) & \text{if } u < \alpha(t),\ t \in I, \\ u & \text{if } \alpha(t) \le u \le \beta(t),\ t \in I, \\ \beta(t) & \text{if } u > \beta(t),\ t \in I, \end{cases}$$

and

$$f^*(t,u,v) := \begin{cases} f(t,u,-N) & \text{if } v < -N, \\ f(t,u,v) & \text{if } |v| \le N, \\ f(t,u,N) & \text{if } v > N. \end{cases}$$

Here we choose N so large that

$$N > \max\{|\alpha'(t)|, |\beta'(t)| : t \in I\} + A$$

and

$$\int_{g(A)}^{g(N)} \frac{|G(s)|^{(\sigma-1)/\sigma}}{H(|G(s)|)}\, ds > B^{(\sigma-1)/\sigma}\|k\|_\sigma$$

(A and B are defined by (1.4) and (1.5) respectively). (1.3) assures the existence of such an N.

LEMMA 1. *For any $u \in E := C^1(I)$, the following two statements hold*:

(1) $(d/dt)q(t,u(t))$ *exists for a.e.* $t \in I$.
(2) *If $u_0, u_j \in E$ and $u_j \to u_0$ in E, then*

$$\frac{d}{dt}q(t,u_j(t)) \to \frac{d}{dt}q(t,u_0(t)) \quad \textit{for a.e. } t \in I.$$

Proof. The proof can be found in [1, 3].

LEMMA 2. *Let u be a solution to* (2.1). *Then*

(1) $\alpha(t) \le u(t) \le \beta(t)$ *on I, and*
(2) $|u'(t)| \le N$ *for all* $t \in I$.

That is to say, the solution u is also a solution to (1.1).

Proof. We first prove that $u(t) \le \beta(t)$ on I. Let $y(t) = u(t) - \beta(t)$. Then we have

$$y(a) = \xi[q(b,u(b)) - \beta(b)] \le 0, \qquad y'(b) \le \eta y'(a). \tag{2.2}$$

Assume now that $y(t) > 0$ for some $t \in (a,b]$. Then there exists a point $t^* \in (a,b]$ such that $y(t^*)$ is the positive maximum value. We can distinguish two cases.

Case (i): $t^* < b$. In this case, $y'(t^*) = u'(t^*) - \beta(t^*) = 0$ and there exists a point $t_1 \in [a,t^*)$ such that $y(t_1) = 0$ and $y(t) > 0$ in $(t_1,t^*]$. Thus, we have

$$\begin{aligned}-[g(\beta'(t))] + Mg(\beta'(t)) &\ge f(t,\beta(t),\beta'(t)) + Mg(\beta'(t))\\ &= -[g(u'(t))]' + Mg(u'(t)) \quad \text{a.e. on } [t_1,t^*]\end{aligned}$$

(since $q(t,u(t)) = \beta(t)$ and $|\beta'(t)| \le N$ on $[t_1,t^*]$), i.e.,

$$\frac{d}{dt}\{e^{-Mt}[g(u'(t)) - g(\beta'(t))]\} \ge 0 \quad \text{a.e. on } [t_1,t^*].$$

This shows that

$$e^{-Mt}[g(u'(t)) - g(\beta'(t))] \le e^{-Mt^*}[g(u'(t^*)) - g(\beta'(t^*))] = 0 \quad \text{on } [t_1,t^*],$$

i.e., $y'(t) \le 0$ on $[t_1,t^*]$. Consequently, we get a contradiction: $0 = y(t_1) \ge y(t^*) > 0$.

Case (ii): $t^* = b$. If $y'(b) = 0$, we can get a contradiction again as in Case (i). If $y'(b) > 0$, then by (2.2),

$$y(a) = 0, \quad \eta > 0 \quad \text{and} \quad y'(a) > 0. \tag{2.3}$$

When $y(t) > 0$ in $(a,b]$, we easily obtain

$$\frac{d}{dt}\{e^{-Mt}[g(u'(t)) - g(\beta'(t))]\} \ge 0 \quad \text{a.e. on } I,$$

as in Case (i), i.e.,

$$e^{-Mb}[g(u'(b)) - g(\beta'(b))] \ge e^{-Ma}[g(u'(a)) - g(\beta'(a))].$$

Since $u'(b) = \eta u'(a)$ and $\beta'(b) \ge \eta\beta'(a)$, we have

$$(\eta^{p-1} - e^{M(b-a)})[g(u'(a)) - g(\beta'(a))] \ge 0,$$

i.e., $y'(a) = u'(a) - \beta'(a) \le 0$, which contradicts the assumption $y'(a) > 0$.

When there is a point $t_4 \in (a,b)$ such that $y(t_4) \le 0$, it follows from (2.3) that there exists an interval (t_2,t_3), $a \le t_2 < t_3 \le t_4$, such that $y(t) > 0$ in (t_2,t_3) and $y(t_2) = y(t_3) = 0$. Therefore, there is a point $t^{**} \in (t_2,t_3)$ such that $y(t^{**})$ is the positive maximum value of $y(t)$ on $[t_2,t_3]$. As in Case (i), we can get a contradiction again. This shows that $u(t) \le \beta(t)$ on I.

In very much the same way, we can prove that $\alpha(t) \le u(t)$ on I. (1) is thus proved.

The proof of (2) can be found in [1, 2]. The Nagumo condition is employed only here.

To prove the existence of solutions to problem (2.1), we define a mapping $\Phi : E \to E$ by

$$(\Phi u)(t) := \int_a^t G\Big(\tau e^{M(r-a)} - \int_a^r e^{M(r-s)}(Fu)(s)\,ds\Big)\,dr + \xi q(b, u(b)) + c$$

for $u \in E$, where the mapping $F : E \to L_\sigma(I)$ is defined by

$$(Fu)(t) := f^*\Big(t, q(t,u(t)), \frac{d}{dt}q(t,u(t))\Big) + Mg\Big(\frac{d}{dt}q(t,u(t))\Big) \quad \forall u \in E \tag{2.4}$$

and

$$\tau := [e^{M(b-a)} - \eta^{p-1}]^{-1}\int_a^b e^{M(b-s)}(Fu)(s)\,ds. \tag{2.5}$$

Obviously, F is well defined, since for any $u \in E$,

$$|(Fu)(t)| \le M^* k(t) + Mg(N) \in L_\sigma(I), \tag{2.6}$$

where $M^* := \max\{H(s) : 0 \le s \le N\}$.

Lemma 3. *Φ is a completely continuous mapping.*

Proof. Let $w(t) = (\Phi u)(t)$. From the definition of Φ, we have for $u \in E$,

$$w'(t) = G\Big(\tau e^{M(t-a)} - \int_a^t e^{M(t-s)}(Fu)(s)\,ds\Big) \in C(I)$$

and there exists an N^*, independent of u, such that

$$|\tau|, |w'(t)|, |w(t)| \le N^* \quad (t \in I).$$

This shows that $\Phi(E)$ is a bounded subset of E.

Since the set

$$\Big\{\tau e^{M(t-a)} - \int_a^t e^{M(t-s)}(Fu)(s)\,ds : u \in E\Big\}$$

is bounded and equicontinuous on I, so is the set $\{w'(t) : u \in E\}$. By the Arzelà–Ascoli theorem, $\Phi(E)$ is compact in E.

Let $u_0, u_j \in E$ and $u_j \to u_0$ in E. By Lemma 1 and the dominated convergence theorem, we conclude that as $j \to \infty$,

$$\tau_j := [e^{M(b-a)} - \eta^{p-1}]^{-1} \int_a^b e^{M(b-s)} (Fu_j)(s)\, ds$$

$$\to [e^{M(b-a)} - \eta^{p-1}]^{-1} \int_a^b e^{M(b-s)} (Fu_0)(s)\, ds =: \tau_0,$$

and hence $\Phi u_j \to \Phi u_0$ in E. This shows that Φ is continuous on E. The proof is complete.

From Lemma 3, the Schauder fixed point theorem asserts that Φ has at least one fixed point in E. Let $u \in E$ be a fixed point of Φ. Then

$$u(t) = \int_a^t G\bigg(\tau e^{M(r-a)} - \int_a^r e^{M(r-s)} \bigg[f^*\bigg(s, q(s, u(s)), \frac{d}{ds} q(s, u(s))\bigg) + Mg\bigg(\frac{d}{ds} q(s, u(s))\bigg)\bigg]\, ds\bigg)\, dr + \xi q(b, u(b)) + c \quad \text{on } I,$$

where τ is determined by (2.5). It is easy to see that the fixed point u is a solution to (2.1). Of course, the u is also a solution to (1.1).

Theorem 1 is proved.

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