ANNALES
POLONICI MATHEMATICI
LXVII.2 (1997)

Only one of generalized gradients can be elliptic

by Jerzy Kalina, Antoni Pierzchalski
and Paweł Walczak (Łódź)

Abstract. Decomposing the space of k-tensors on a manifold M into the components invariant and irreducible under the action of $\mathrm{GL}(n)$ (or $\mathrm{O}(n)$ when M carries a Riemannian structure) one can define generalized gradients as differential operators obtained from a linear connection ∇ on M by restriction and projection to such components. We study the ellipticity of gradients defined in this way.

Introduction. We decompose a connection ∇ on an n-dimensional C^∞-manifold M (in particular, a Riemannian connection on a Riemannian manifold (M,g)) into the sum of first order differential operators $\nabla^{\alpha\beta}$ acting on covariant k-tensors, $k=1,2,\ldots$, and arising from the decomposition of the space T^k of k-tensors into the direct sum of irreducible $\mathrm{GL}(n)$-invariant (or, in the Riemannian case, $\mathrm{O}(n)$-invariant) subspaces. Following [SW] we shall call them $\mathrm{GL}(n)$- and $\mathrm{O}(n)$-*gradients*, respectively.

Some of the gradients $\nabla^{\alpha\beta}$ have important geometric meaning. The best known is the exterior derivative d corresponding to skew-symmetric tensors. Its role in geometry and topology of manifolds cannot be overestimated. Another one, known as the Ahlfors operator $S : T^1 \to S^2_0$, is defined for 1-forms ω by the splitting

$$\nabla\omega = \frac{1}{2}d\omega + S\omega - \frac{1}{n}\delta\omega\cdot g$$

and corresponds to the subbundle of traceless symmetric 2-tensors. It appears to play an important role in conformal and quasi-conformal geometry (see the recent papers [ØP], [P], etc.).

In Section 1, we recall (after H. Weyl [We]) the theory of Young diagrams and schemes and define our operators $\nabla^{\alpha\beta}$. In Section 2, we consider

1991 *Mathematics Subject Classification*: 53C05, 20G05.
Key words and phrases: connection, group representation, Young diagram, elliptic operator.
The authors were supported by the KBN grant 2 P301 036 04.

the ellipticity of operators corresponding to $\mathrm{GL}(n)$-invariant subspaces. We distinguish a suitable extension of a Young diagram α and show that $\nabla^{\alpha\beta}$ is elliptic if and only if β is a distinguished extension of α. In Section 3, we get some particular ellipticity results for operators corresponding to $\mathrm{O}(n)$-invariant subspaces. We end with some remarks.

Similar problems could be considered for any connection ∇,

$$\nabla : C^\infty(\xi) \to C^\infty(T^*M \otimes \xi),$$

in any vector bundle ξ over a manifold M and any Lie group G acting simultaneously in T^*M and ξ. Splitting ξ and $\widetilde{\xi} = T^*M \otimes \xi$ into the direct sums of irreducible G-invariant subbundles, $\xi = \bigoplus_\alpha \xi_\alpha$ and $\widetilde{\xi} = \bigoplus_\beta \widetilde{\xi}_\beta$, *G-gradients* could be defined as

$$\nabla^{\alpha\beta} = \widetilde{\pi}_\beta \circ \nabla \circ \iota_\alpha,$$

where $\iota_\alpha : \xi_\alpha \to \xi$ and $\widetilde{\pi}_\beta : \widetilde{\xi} \to \widetilde{\xi}_\beta$ are the canonical maps. One of interesting examples of this sort is the classical Dirac operator D which could be considered as an eliptic $\mathrm{Spin}(n)$-gradient in a spinor bundle over a manifold equipped with a spinor structure. Ellipticity of general G-gradients will be studied elsewhere.

1. Young diagrams. Let W be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) of dimension n. Fix $k \in \mathbb{N}$ and take a sequence of integers $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\alpha_1 \geq \ldots \geq \alpha_r \geq 1$, $\alpha_1 + \ldots + \alpha_r = k$. Such an α is called a *Young scheme of length* k. In some references a Young scheme is called a *decomposition*. It can be represented by the figure consisting of r rows of squares and such that the number of squares in the jth row is α_j.

A Young scheme can be filled with numbers $1, \ldots, k$ distributed in any order. A scheme filled with numbers is called a *Young diagram*. Without loss of generality we can assume that the numbers grow both in rows and columns.

Take a Young diagram α and denote by H_α and V_α the subgroups of the symmetric group S_k consisting of all permutations preserving rows and columns, respectively. α determines the linear operator (called the *Young symmetrizer*) $P_\alpha : W^k \to W^k$, $W^k = \bigotimes_k W$, given by

$$P_\alpha = \sum_{\tau \in H_\alpha,\, \sigma \in V_\alpha} \operatorname{sgn} \, \sigma \cdot \tau\sigma, \tag{1}$$

where the action of any permutation $\varrho \in S_k$ on simple tensors is given by

$$\varrho(v_1 \otimes \ldots \otimes v_k) = v_{\varrho^{-1}(1)} \otimes \ldots \otimes v_{\varrho^{-1}(k)}$$

for all $v_1, \ldots, v_k \in W$. It is well known that

$$P_\alpha^2 = m_\alpha P_\alpha \tag{2}$$

for some $m_\alpha \in \mathbb{N}$ and that $W_\alpha = \operatorname{im} P_\alpha$ is an invariant subspace of W^k for the standard representation of $\mathrm{GL}(n)$ in W^k. This representation is irreducible on W_α. Moreover,

$$W^k = \bigoplus_\alpha W_\alpha. \tag{3}$$

If W is equipped with a scalar product $g = \langle\cdot,\cdot\rangle$, then g allows defining contractions in W^k. An element w of W^k is said to be *traceless* if $C(w) = 0$ for any contraction $C : W^k \to W^{k-2}$. (In particular, all 1-tensors are traceless.) Traceless tensors form a linear subspace W_0^k of W^k. Its orthogonal complement consists of all the tensors of the form

$$\sum_{\sigma\in S_k} \sigma(g \otimes w_\sigma), \tag{4}$$

where $w_\sigma \in W^{k-2}$. For simplicity, denote the space of tensors of the form (4) by $g \otimes W^{k-2}$ so that

$$W^k = W_0^k \oplus (g \otimes W^{k-2}). \tag{5}$$

The intersection $W_\alpha^0 = W_\alpha \cap W_0^k$ is non-trivial if and only if the sum of lenghts of the first two columns of a Young diagram α is $\le n$. A diagram like this is called *admissible* and the corresponding space W_α^0 is invariant and irreducible under the $\mathrm{O}(n)$-action. Moreover,

$$W_0^k = \bigoplus_\alpha W_\alpha^0, \tag{6}$$

where α ranges over the set of all admissible Young diagrams with numbers growing both in rows and columns. Comparing (5) and (6), and proceeding with the analogous decompositions of W^{k-2}, W^{k-4}, etc., one gets the decomposition of W^k into the direct (in fact, orthogonal) sum of irreducible $\mathrm{O}(n)$-invariant subspaces.

2. GL(n)-gradients. Let $\beta = (\beta_1, \ldots, \beta_s)$ be a Young scheme of length $k+1$ obtained from α by an extension by a single square. The corresponding diagram should have $k + 1$ in the added square, while the ordering in the other part of the diagram is the same as in α. We call β a *distinguished extension* of α if

$$s = r,\ \beta_1 = \alpha_1 + 1,\ \beta_2 = \alpha_2, \ldots, \beta_s = \alpha_s. \tag{7}$$

In other words, β is distinguished when the added square is situated at the end of the first row.

Take an arbitrary $v \in W$ and consider a linear mapping $\otimes_v : W^k \to W^{k+1}$ defined by

$$\otimes_v(v_1 \otimes \ldots \otimes v_k) = v_1 \otimes \ldots \otimes v_k \otimes v. \tag{8}$$

THEOREM 1. *For $v \neq 0$ the mapping*

$$P_\beta \circ \otimes_v|_{W_\alpha} : W_\alpha \to W_\beta \tag{9}$$

is injective if and only if β is the distinguished extension of α.

Before the proof we make the following observations.

LEMMA 1. *Assume that i, j, $i \neq j$, are in the same column of a Young diagram α. Then*

$$P_\alpha(v) = 0, \tag{10}$$

whenever $v = v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_j \otimes \ldots \otimes v_{k+1}$ and $v_j = v_i$.

Proof. Denote by V_α^+ and V_α^- the subsets of V_α consisting of odd and even permutations $\sigma \in V_\alpha$, respectively, $V_\alpha^+ \cup V_\alpha^- = V_\alpha$. The mapping

$$\sigma \mapsto \widetilde{\sigma} = \sigma \circ t_{ij}, \tag{11}$$

where t_{ij} is the transposition, is a one-to-one map of V_α^+ onto V_α^-. If $v_i = v_j$, then

$$\sum_{\sigma \in V_\alpha} \sigma(v) = \sum_{\sigma \in V_\alpha^+} \sigma(v) - \sum_{\sigma \in V_\alpha^-} \sigma(v) = 0, \tag{12}$$

because the terms corresponding to σ and $\widetilde{\sigma}$ are the same. Now, the statement follows from formulae (1) and (12). ∎

LEMMA 2. *If β is the distinguished extension of α, then*

$$P_\beta = m_\alpha \Big[\mathrm{id} + \sum_{t \in T_\alpha} t \circ \mathrm{id} \Big] \tag{13}$$

on $W_\alpha \otimes W$, where T_α denotes the set of all transpositions of $k+1$ with the numbers from the first row.

Proof. Since $V_\beta = V_\alpha$ up to the canonical isomorphism and $H_\beta = H_\alpha \cup \bigcup_{t \in T_\alpha} tH_\alpha$, we have

$$P_\beta = \sum_{\tau \in H_\beta,\, \sigma \in V_\alpha} \operatorname{sgn} \sigma \cdot \tau\sigma. \tag{14}$$

Consequently,

$$\begin{aligned} P_\beta(P_\alpha v \otimes w) &= \sum_{\tau \in H_\beta} \tau \Big(\sum_{\sigma \in V_\alpha} \operatorname{sgn} \sigma \cdot \sigma(P_\alpha v) \otimes w \Big) \\ &= \sum_{\sigma \in V_\alpha,\, \tau \in H_\alpha} \operatorname{sgn} \sigma \cdot \tau\sigma(P_\alpha v) \otimes w \end{aligned}$$

$$+\sum_{t\in T_\alpha} t\Big(\sum_{\sigma\in V_\alpha,\,\tau\in H_\alpha} \operatorname{sgn}\sigma\cdot\tau\sigma(P_\alpha v)\otimes w\Big)$$
$$= P_\alpha^2 v\otimes w + \sum_{t\in T_\alpha} t(P_\alpha^2 v\otimes w),$$

for any $v\in W^k$ and $w\in W$. Now, the proof is completed by applying (2). ∎

LEMMA 3. *If $v_1,\dots,v_l\in W$ are linearly independent, ϱ is a permutation mapping the numbers $1,\dots,\alpha_1$ onto the numbers of the first row of the diagram α, $\alpha_1+1,\dots,\alpha_1+\alpha_2$ onto the numbers of the second row etc., and*

$$\omega = \varrho^{-1}(\otimes^{\alpha_1} v_1\otimes\dots\otimes^{\alpha_l} v_l), \tag{15}$$

then $P_\alpha\omega\neq 0$.

Proof. The statement follows from (1) and the following:

(i) Any two permutations σ_1 and σ_2 of V_α satisfying $\tau\sigma_1\omega=\tau\sigma_2\omega$ for some $\tau\in H_\alpha$ have the same sign.

(ii) Any two products obtained from ω by permuting factors are linearly dependent if and only if they are equal. ∎

Proof of Theorem 1. Assume first that β is the distinguished extension of α. If $\eta\in W_\alpha$ and $P_\beta(\eta\otimes w)=0$, then, by Lemma 2,

$$\eta\otimes w+\sum_t t(\eta\otimes w)=0.$$

Take $w=e_1$, $\eta=\sum\eta_{i_1\dots i_k}e_{i_1}\otimes\dots\otimes e_{i_k}$, where $\{e_1,\dots,e_k\}$ is a basis of W. Then the last equality is equivalent to

$$\sum\eta_{i_1\dots i_k}(e_{i_1}\otimes\dots\otimes e_{i_k}\otimes e_1+e_1\otimes e_{i_2}\otimes\dots\otimes e_{i_k}\otimes e_{i_1}$$
$$+\dots+e_{i_1}\otimes\dots\otimes e_{i_{k-1}}\otimes e_1\otimes e_{i_k})=0.$$

Now, if $i_1,\dots,i_k>1$, then $\eta_{i_1\dots i_k}=0$ because all the terms are linearly independent. If $i_1=1$, $i_2,\dots,i_k>1$, then

$$2\eta_{1i_2\dots i_k}e_1\otimes e_{i_2}\otimes\dots\otimes e_{i_k}\otimes e_1$$
$$+\text{(terms linearly independent of the first one)}=0,$$

so $\eta_{1i_2\dots i_k}=0$.

We can repeat the reasoning for the other coefficients. Consequently, $\eta=0$ and the mapping (9) is injective.

Assume now that β is a non-distinguished extension of α. Then, by Lemma 1,

$$P_\beta(P_\alpha\omega\otimes v_1)=0,$$

where ω is of the form (15), while, by Lemma 3, $P_\alpha\omega\neq 0$. ∎

Now, consider any connection ∇ on a manifold M and extend it to covariant k-tensor fields, $k = 1, 2, \ldots$, in the standard way:

$$\nabla\omega(X_1, \ldots, X_{k+1}) = (\nabla_{X_{k+1}}\omega)(X_1, \ldots, X_k). \tag{16}$$

For any two diagrams α and β of length k and $k+1$, respectively, denote by $\nabla^{\alpha\beta}$ the differential operator given by

$$\nabla^{\alpha\beta} = P_\beta \circ \nabla | T_\alpha, \tag{17}$$

where T_α denotes the space of all k-tensor fields ω such that $\omega(x) \in (T_x^* M)_\alpha$ for any $x \in M$. Since P_β is linear the symbol of the operator $\nabla^{\alpha\beta}$ is given by

$$\sigma(\nabla^{\alpha\beta}, w^*)(\omega) = P_\beta(\omega \otimes w^*) \tag{18}$$

for any covector $w^* \in T_x^* M$, any $\omega \in (T_x^* M)_\alpha$ and $x \in M$. Theorem 1 together with (18) yields

Corollary. *The operator $\nabla^{\alpha\beta}$ is elliptic if and only if β is the distinguished extension of α.* ■

3. $\mathbf{O}(n)$-gradients. Given two admissible Young diagrams α and β of length k and $k+1$, respectively, and a Riemannian connection ∇ on a Riemannian manifold (M, g) one can consider the differential operator $\nabla^{\alpha\beta}$ given by

$$\nabla^{\alpha\beta} = \pi \circ P_\beta \circ \nabla | W_\alpha^0, \tag{19}$$

where W_α^0 denotes the subspace of W_α consisting of all the traceless tensor fields and π is the projection of k-tensors to traceless k-tensors defined by the decomposition (5). The operator (19) differs from $\nabla^{\alpha\beta}$ of Section 2 but this should lead to no misunderstandings. Again, since π is a linear map, the symbol of $\nabla^{\alpha\beta}$ is given by the formula analogous to (18):

$$\sigma(\nabla^{\alpha\beta}, w)(\omega) = \pi(P_\beta(\omega \otimes w)) \tag{20}$$

for any traceless ω and $w \in TM$. (Hereafter, vectors and covectors are identified by the Riemannian structure.)

Note that since ∇ is Riemannian, $\nabla_X \omega$ is traceless for any vector field X and any traceless k-tensor ω while $\nabla\omega$ itself can have non-vanishing contractions of the form $C_{k+1}^i \nabla\omega$, where $i \le k$. Note also, that, in general, the distinguished extension of an admissible Young diagram is admissible again. The only exception is that of a one-column diagram of length n. These observations together with results of Section 2 motivate the following

Conjecture. *$\nabla^{\alpha\beta}$ is elliptic if and only if β is the distinguished extension of α, both α and β being admissible.*

An elementary proof of the conjecture seems unlikely, because there is no algorithm providing the traceless component of k-tensors, even of the form $\omega \otimes v$ with ω being traceless and v a single vector. However, we can prove, in an elementary way, ellipticity of $\nabla^{\alpha\beta}$ in some particular cases and the "if" part completely.

THEOREM 2. (i) *If α is trivial*, *i.e. consits of a single row or of a single column*, *β is the distinguished extension of α and both α and β are admissible*, *then the operator $\nabla^{\alpha\beta}$ is elliptic.*

(ii) *If β is a non-distinguished extension of α*, *then $\nabla^{\alpha\beta}$ is not elliptic.*

Proof. (i) Assume first that α is a single row. Then so is β and the spaces T_α and T_β consist of symmetric tensors. From (13) and (20) it follows that the ellipticity of $\nabla^{\alpha\beta}$ is equivalent to the following statement:

$(*)$ If ω is traceless and symmetric, v is a non-vanishing vector and

$$\omega \odot v \in g \otimes W^{k-1}, \tag{21}$$

then $\omega = 0$.

Since β is admissible, $n > 1$. To prove $(*)$ take an orthonormal frame $e_1, \ldots, e_n$ and assume, without loss of generality, that $v = e_1$. Since the symmetric algebra is isomorphic to the algebra of polynomials and the tensors in (21) are symmetric, we can replace (21) by the equality

$$x_1 \cdot P(x_1, \ldots, x_n) = \Big(\sum_{i=1}^{n} x_i^2\Big) \cdot Q(x_1, \ldots, x_n), \tag{22}$$

where P and Q are polynomials. From (22) it follows that Q is of the form $x_1 \cdot Q'$ for another polynomial Q' and therefore, $P = \sum x_i^2 \cdot Q'$. Since P corresponds to ω, the last equality shows that $\omega \in (g \otimes W^{k-2}) \cap W_0^k = \{0\}$.

Assume now that α is a single column. The space W_α consists of skew-symmetric tensors and β is admissible if and only if $k < n$. Assume that $\omega \in W_\alpha$ and

$$\omega \otimes v + (-1)^{k-1} v \otimes \omega \in g \otimes W^{k-1} \tag{23}$$

for some $v \neq 0$. (Note that, by Lemma 2, the tensor in (23) coincides with $P_\beta \omega$.) From (23) it follows that

$$\omega = v \wedge \eta \tag{24}$$

for some $(k-1)$-form η. In fact, otherwise $\omega \otimes v \pm v \otimes \omega$, when decomposed into a sum of simple tensors, would contain a term $w_1 \otimes \ldots \otimes w_{k+1}$ with all the factors w_i linearly independent while tensors of $g \otimes W^{k-1}$ do not admit terms of this sort. Moreover, one could choose η in (24) to be a $(k-1)$-form on the orthogonal complement $\{v\}^\perp$ of the one-dimensional space spanned by v. If so, $\omega \otimes v \pm v \otimes \omega$ would contain no non-trivial terms of the form

$$\varrho(w \otimes w \otimes w_1 \otimes \ldots \otimes w_{k-1}) \tag{25}$$

with $\varrho \in S_{k-1}$ and $w \in \{v\}^{\perp}$ while all the non-zero tensors of $g \otimes W^{k-1}$ do. Consequently, $\omega = 0$.

(ii) Assume that α is admissible and put $m = \min\{\delta_1, n/2\}$, where δ_j is the length of the jth column of α. Since $\delta_1 + \delta_2 \le n$, it follows that $\beta_2 \le m$. Split the set $\{1, 2, \ldots, k\}$ into the sum $A \cup B \cup C$ of pairwise disjoint subsets such that $\#A = \#B = m$. Set $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$ and $C = \{2m+1, \ldots, n\}$.

Fix an orthonormal frame $(e_1, \ldots, e_n)$ of W and denote by ω the sum of all the terms of the form

$$(-1)^l \cdot e_{i_1} \otimes \ldots \otimes e_{i_k}, \tag{26}$$

where $i_r \in \{a_s, b_s\}$ when r belongs to the sth row of the Young diagram α and $s \le m$, $i_r = c_s$ when r belongs to the sth row of α and $s > m$, and

$$l = \left[\tfrac{1}{2}\#\{r : i_r \in B\}\right].$$

It is easy to see that both tensors ω and $P_\alpha \omega$ are traceless while $P_\alpha(\omega) \ne 0$.

Take any non-distinguished extension β of α and denote by s the number of the column of β which contains $k+1$. Write ω in the form

$$\omega = \omega_A + \omega_B, \tag{27}$$

where ω_A (resp., ω_B) is the sum of all the terms of the form (26) for which $i_r \in A$ (resp., $i_r \in B$) for the r which appears in the first row and sth column of α. Let $v = e_{a_1} + e_{b_1}$. Then

$$\sum_{\sigma \in H_\beta} \operatorname{sgn} \sigma \cdot \sigma(\omega_A \otimes e_{a_1}) = \sum_{\sigma \in H_\beta} \operatorname{sgn} \sigma \cdot \sigma(\omega_B \otimes e_{b_1}) = 0 \tag{28}$$

by Lemma 1. Also,

$$\sum_{\sigma \in H_\beta} \operatorname{sgn} \sigma \cdot \sigma(\omega_A \otimes e_{b_1}) = -\sum_{\sigma \in H_\beta} \operatorname{sgn} \sigma \cdot \sigma(\omega_B \otimes e_{a_1}) \tag{29}$$

because for any term in the first sum there exists a unique term in the second sum with e_{a_1} and e_{b_1} interchanged. Equalities (27)–(29) together with (1) and the definition of v imply that $P_\beta(\omega \otimes v) = 0$.

Finally, following the proof of Lemma 2 one can show that

$$P_\beta = m_\alpha \sum_{t \in T^v} \sum_{t' \in T^h} \operatorname{sgn} t \cdot t' \circ (P_\alpha \otimes \mathrm{id}) \circ t, \tag{30}$$

where T^h (resp., T^v) consists of the identity and all the transpositions of $k+1$ with the elements of the row (resp., column) containing it. It follows that

$$P_\beta(P_\alpha \omega \otimes v) = m_\alpha P_\beta(\omega \otimes v) = 0. \ \blacksquare \tag{31}$$

4. Final remarks. (i) Denote by $N(k)$ the number of components in the decomposition (3). It is easy to observe that $N(1) = 1$, $N(2) = 2$, $N(3) = 4$, $N(4) = 10$, $N(5) = 26$, etc. The above observation motivates the recurrent formula

$$N(k) = N(k-1) + (k-1) \cdot N(k-2). \tag{32}$$

The authors could not find anything like this in the literature. A numerical experiment showed that (32) holds for small k, say $k \le 20$.

(ii) As we said in Section 3, there is no explicit formula for the traceless part of a tensor. In some sense, a formula of this sort could be obtained in the following way. Put

$$E = \bigoplus\nolimits_{\binom{k}{2}} T^{k-2} \tag{33}$$

and define an endomorphism $K : E \to E$ by the formula

$$K((\omega_{ij})) = \Big(C^i_j\Big(\sum_{r,s} t_r \circ t_s(g \otimes \omega rs)\Big)\Big), \tag{34}$$

where t_r (resp. t_s) is the transposition of the terms 1 and r (resp., 1 and s).

K is an isomorphism. In fact, if $K(\Omega) = 0$, $\Omega = (\omega_{ij})$, then the tensor

$$\Theta = \sum_{r,s} t_r \circ t_s(g \otimes \omega_{rs}) \tag{35}$$

is traceless and—because of its form—orthogonal to the space of traceless tensors, and therefore, it vanishes. Decomposing tensors ω_{ij} according to (6) and proceeding inductively one would get $\omega_{ij} = 0$ for all i and j, i.e. $\Omega = 0$.

The traceless part ω_0 of any k-tensor ω is given by the formula

$$\omega_0 = \omega - \Theta, \tag{36}$$

where Θ is given by (35) with $(\omega_{ij}) = K^{-1}((C^i_j\omega))$. In fact, from the definition of K it follows immediately that $C^i_j\Theta = C^i_j\omega$ for all i and j.

After submitting the paper, the authors, working jointly with B. Ørsted and G. Zhang, proved the Conjecture from Section 3 as well as formula (32). See *Elliptic gradients and highest weights*, Bull. Polish Acad. Sci. Math. 44 (1996), 527–535.

References

[ØP] B. Ørsted and A. Pierzchalski, *The Ahlfors Laplacian on a Riemannian manifold*, in: Constantin Carathéodory: An International Tribute, World Sci., Singapore, 1991, 1020–1048.

[P] A. Pierzchalski, *Ricci curvature and quasiconformal deformations of a Riemannian manifold*, Manuscripta Math. 66 (1989), 113–127.

[SW] E. M. Stein and G. Weiss, *Generalization of the Cauchy–Riemann equations and representation of the notation group*, Amer. J. Math. 90 (1968), 163–197.
[We] H. Weyl, *The Classical Groups*, Princeton Univ. Press, Princeton, 1946.

Institute of Mathematics
Technical University of Łódź
Al. Politechniki 11
93-590 Łódź, Poland
E-mail: jkalina@imul.uni.lodz.pl

Institute of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: antoni@imul.uni.lodz.pl
pawelwal@imul.uni.lodz.pl

Reçu par la Rédaction le 10.2.1995