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Normal structure of Lorentz–Orlicz spaces

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Abstract. Let $\phi : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ be an even convex continuous function with $\phi(0) = 0$ and $\phi(u) > 0$ for all $u > 0$ and let w be a weight function. u_0 and v_0 are defined by

$$u_0 = \sup\{u : \phi \text{ is linear on } (0, u)\}, \qquad v_0 = \sup\{v : w \text{ is constant on } (0, v)\}$$

(where $\sup \emptyset = 0$). We prove the following theorem.

Theorem. *Suppose that $\Lambda_{\phi,w}(0,\infty)$ (respectively, $\Lambda_{\phi,w}(0,1)$) is an order continuous Lorentz–Orlicz space.*

(1) *$\Lambda_{\phi,w}$ has normal structure if and only if $u_0 = 0$ (respectively, $\int_0^{v_0} \phi(u_0) \cdot w < 2$ and $u_0 < \infty$).*

(2) *$\Lambda_{\phi,w}$ has weakly normal structure if and only if $\int_0^{v_0} \phi(u_0) \cdot w < 2$.*

1. Introduction. Let Ω denote either $[0, 1]$ or $[0,\infty)$ and m denote the Lebesgue measure on Ω. For a measurable function x on Ω, the *distribution function* d_x and the *decreasing rearrangement* x^* are defined by

$$d_x(t) = m(|x| > t), \qquad x^*(t) = \inf\{s > 0 : d_x(s) \le t\}.$$

An even convex continuous function $\phi : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ is said to be a *Young function* if $\phi(0) = 0$ and $\phi(u) > 0$ for all $u \ne 0$. A function $w : \Omega \to \mathbb{R}_+$ is called a *weight function* if it is a nonincreasing left continuous function and

$$\int_0^1 w(t)\, dt = 1.$$

For a Young function ϕ and a weight function w, the associated *Lorentz–Orlicz space* $\Lambda_{\phi,w}(\Omega)$ (or $\Lambda_{\phi,w}$ for short) is the set of all real measurable functions x on Ω such that

$$\varrho_\phi(\lambda x) = \int_\Omega \phi(\lambda x^*(t))w(t)\, dt \equiv \int_\Omega \phi(\lambda x^*)w < \infty$$

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for some $\lambda > 0$. The norm of $x \in \Lambda_{\phi,w}$ is defined by

$$\|x\| = \inf\{\varepsilon > 0 : \varrho_\phi(x/\varepsilon) \le 1\}.$$

Recall that a mapping $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *measure preserving transformation* if for any measurable set D, $m(D) = m(\sigma^{-1}(D))$. It is known that for any measure preserving transformation σ and any $x \in \Lambda_{\phi,w}$, $x^* = (x \circ \sigma)^*$ and

$$\int \phi(x^*)w \ge \int \phi(x) w \circ \sigma.$$

It is also known that for $z \in \Lambda_{\phi,w}$ if $m(\mathrm{supp}(z)) < \infty$ (or respectively, $m(\mathrm{supp}(z)) = \infty$), then there is (cf. [2]) a measure preserving transformation $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ (respectively, $\sigma : \mathrm{supp}(z) \to \mathbb{R}_+$) such that

(i) $\int_0^\infty \phi(z) w \circ \sigma = \int_0^\infty \phi(z^*)w$;
(ii) if $|z(t)| < |z(s)|$, then $\sigma(t) \ge \sigma(s)$.

For a Lorentz–Orlicz space $\Lambda_{\phi,w}(\Omega)$, ϕ is said to satisfy the Δ_2 *condition* if one of the following holds:

(iii) $\Omega = [0,\infty)$ and there exists $l > 0$ such that $\phi(2u) \le l\phi(u)$ for all $u > 0$.
(iv) $\Omega = [0,1]$ and there are $l > 0$ and $u_0 > 0$ such that $\phi(2u) \le l\phi(u)$ for all $u \ge u_0$.

In [7], Kamińska proved the following theorem.

THEOREM A. *For a Lorentz–Orlicz space $\Lambda_{\phi,w}$, the following are equivalent*:

(1) *$\Lambda_{\phi,w}$ is order continuous. So the Köthe dual of $\Lambda_{\phi,w}$ is the dual of $\Lambda_{\phi,w}$.*
(2) *$\Lambda_{\phi,w}$ does not contain any isometric copy of ℓ_∞.*
(3) *ϕ satisfies the Δ_2 condition and $\int_0^\infty w = \infty$ if $\Omega = (0,\infty)$.*
(4) *For any $x \in \Lambda_{\phi,w}$, $\varrho_\phi(x) = 1$ if and only if $\|x\| = 1$.*

Let X be a Banach space. For any bounded subset A of X, define

$$\begin{aligned} r(x,A) &= \sup\{\|x-y\| : y \in A\} \quad \text{for any } x \in A;\\ R(A) &= \inf\{r(x,A) : x \in A\};\\ \delta(A) &= \sup\{r(x,A) : x \in A\} = \operatorname{diam} A. \end{aligned}$$

A bounded closed convex set A is said to have *normal structure* if for any closed convex subset B of A either $R(B) = 0$ or $R(B) < \delta(B)$. X is said to have (*weakly*) *normal structure* if every bounded (weakly compact) closed convex subset of X has normal structure. Kirk [9] showed that every nonexpansive mapping on a weakly compact convex set with normal structure has the fixed point property.

Recall that a sequence $\{x_n\}$ in X is said to be a *limit-constant sequence* if for any $x \in \mathrm{co}\{x_n\}$,

$$\lim_{n\to\infty} \|x - x_n\| = \mathrm{diam}\{x_n\}.$$

Note that here we require the limit to converge to the diameter of $\mathrm{co}\{x_n : n\in\mathbb{N}\}$ (cf. [10]). A sequence $\{x_n\}$ is said to be a *unit limit-constant sequence* if $\{x_n\}$ is a limit-constant sequence with $\mathrm{diam}\{x_n\} = 1$. It is known that a Banach space X has (weakly) normal structure if and only if X contains no (weakly convergent) unit limit-constant sequence [10]. In [3], Chen showed that if ϕ is an N-function (for definition see [3]) which satisfies the Δ_2 condition, then the Orlicz space L_ϕ has weakly normal structure. Recently, Carothers, Dilworth, Hsu, Lennard and Trautman [1, 5] studied the uniform Kadec–Klee property for the Lorentz space $L_{w,1}$. They proved that $L_{w,1}$ does not have normal structure and they also gave a sufficient condition for $L_{w,1}$ to have weakly normal structure. In this article, we study (weakly) normal structure for Lorentz–Orlicz spaces and give a characterization of the Lorentz–Orlicz spaces with (weakly) normal structure. For more results about normal structure of Orlicz function (respectively, sequence) spaces and Lorentz function spaces, see [1, 3, 5, 6, 8, and 11].

It is known that L_1 does not have weakly normal structure and ℓ_∞ contains an isometric copy of L_1. Hence $\Lambda_{\phi,w}$ does not have weakly normal structure if $\Lambda_{\phi,w}$ is not order continuous. For a fixed Young function $\phi : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ and a fixed weight function w, let u_0 and v_0 be defined by

$$\begin{aligned} u_0 &= \sup\{u : \phi \text{ is linear on } (0,u)\},\\ v_0 &= \sup\{v : w \text{ is constant on } (0,v)\}, \end{aligned}$$

where $\sup\emptyset = 0$. The following are three examples of unit limit-constant sequences in Lorentz–Orlicz spaces. The first two are well-known.

Example 1 [1]. Suppose that ϕ is linear on $(0,\infty)$ and a_n is the number such that

$$\phi(n)\int_0^{a_n} w(t)\,dt = \frac{1}{2}.$$

Let $e_n = n1_{(0,a_n)}$. It is easy to see that $\{e_n\}$ is a unit limit-constant sequence. So if $\Lambda_{\phi,w}(0,1)$ has normal structure, then $u_0 < \infty$.

Suppose that $\Omega = (0,\infty)$ and ϕ is linear on $(0,u_0)$ for some $u_0 > 0$. Let b_n be the number such that

$$\phi\left(\frac{u_0}{n}\right)\int_0^{b_n} w(t)\,dt = \frac{1}{2}.$$

A similar proof shows that $\{e_n = (u_0/n)1_{(0,b_n)}\}$ is a unit limit-constant sequence. Hence if $\Lambda_{\phi,w}(0,\infty)$ has normal structure, then $u_0 = 0$.

EXAMPLE 2. Suppose that there exist two positive numbers u and v such that ϕ is linear on $(0,u)$, w is constant on $(0,v)$, and $\int_0^v \phi(u/2)w \ge 1$. Without loss of generality, we may assume that $\int_0^v \phi(u/2)w = 1$. Let

$$x_n(t) = \begin{cases} \frac{u}{2} \cdot \operatorname{sgn}\left(\sin\left(\frac{2^n \pi t}{v}\right)\right) & \text{if } t \le v, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $x \in \overline{\mathrm{co}}\{x_i : i \le k\}$ and $n > k$, $\|x - x_n\| = 1$. This implies that $\{x_n\}$ is a unit limit-constant sequence. It is known that $\Lambda_{\phi,w}(0,v)$ is not equal to $L_\infty(0,1)$ up to equivalent norm. By Proposition 2.c.10 in [13] (p. 160), $\{x_n\}$ is a weakly null sequence. Hence if $\Lambda_{\phi,w}$ has weakly normal structure, then $\int_0^{v_0} \phi(u_0) \cdot w < 2$.

EXAMPLE 3. Suppose that $u_0 > 0$ and for some $v > 0$, w is constant on (v, ∞). Then there are $0 < u < u_0$ and $v' > v$ such that

$$\int_0^{2v'} \phi(u)w = 1.$$

Let $e_n = u1_{((n-1)v', nv')}$. If $a_k \ge 0$ and $\sum_{k=1}^N a_k = 1$, then

$$\begin{aligned} \varrho_\phi\Big(e_{N+1} - \sum_{k=1}^N a_k e_k\Big) &= \int_0^{v'} \phi(u)w(t)\,dt + \sum_{k=1}^N \int_{kv'}^{(k+1)v'} \phi(a_k u)w(t)\,dt \\ &= \int_0^{2v'} \phi(u)w = 1. \end{aligned}$$

So $\{e_n\}$ is a unit limit-constant sequence.

We claim that $\{e_n\}$ is equivalent to the natural basis of ℓ_1. So it cannot be a weakly convergent sequence.

In fact, for any finite sequence $\{a_k\}_{k=1}^N$ with

$$\sum_{k=1}^N |a_k| \ge \frac{1}{\int_{v'}^{2v'} \phi(u)w},$$

we have

$$\varrho_\phi\Big(\sum_{k=1}^N a_k e_k\Big) \ge \sum_{k=1}^N |a_k| \int_{v'}^{2v'} \phi(u)w \ge 1.$$

Hence

$$\Big\|\sum_{k=1}^N a_k e_k\Big\| \ge \frac{\sum_{k=1}^N |a_k|}{\int_{v'}^{2v'} \phi(u)w}.$$

This implies that $\{e_n\}$ is equivalent to the natural basis of ℓ_1.

From the above examples, it is natural to ask the following questions:

(1) Does $\Lambda_{\phi,w}(0,\infty)$ (respectively, $\Lambda_{\phi,w}(0,1)$) have normal structure if $u_0 = 0$ (respectively, $u_0 < \infty$ and $\int_0^{v_0} \phi(u_0) \cdot w < 2$)?

(2) Does $\Lambda_{\phi,w}$ have weakly normal structure if $\int_0^{v_0} \phi(u_0) w < 2$?

The following theorem shows that the answer to the above questions is affirmative.

THEOREM 1. *Suppose that $\Lambda_{\phi,w}$ is an order continuous Lorentz–Orlicz space.*

(1) *$\Lambda_{\phi,w}$ has normal structure if $u_0 = 0$ (respectively, $\int_0^{v_0} \phi(u_0) w < 2$ and $u_0 < \infty$).*

(2) *$\Lambda_{\phi,w}$ has weakly normal structure if $\int_0^{v_0} \phi(u_0) w < 2$.*

2. Basic properties of unit limit-constant sequences in $\Lambda_{\phi,w}$. First, we need the following three lemmas. The first one easily follows from the definition and the second one was proved in [12].

LEMMA 2. *Suppose that $v > \varepsilon > 0$ and $u_2 > u_1 > 0$. If x is an element of $\Lambda_{\phi,w}$ such that*

$$m(\{t \in (0,v) : |x(t)| \le u_1\}) > \varepsilon, \qquad m(\{t \in (v,\infty) : |x(t)| \ge u_2\}) > \varepsilon,$$

then

$$\int \phi(|x|) w \le \varrho_\phi(x) - (\phi(u_2) - \phi(u_1)) \Big(\int_{v-\varepsilon}^{v} w - \int_{v}^{v+\varepsilon} w \Big).$$

Remark 1. Suppose that either w is not constant on $(v-\varepsilon, v)$ or w is not constant on $(v, v+\varepsilon)$. Then $\int_{v-\varepsilon}^{v} w - \int_v^{v+\varepsilon} w > 0$. Hence there is $\delta > 0$ such that $\varrho_\phi(x) \ge \delta + \int \phi(x) w$ whenever x satisfies the assumption of Lemma 2.

LEMMA 3. *Let $\Lambda_{\phi,w}$ be an order continuous Lorentz–Orlicz space and E be a set of positive measure and λ be a positive number. Suppose that x, y and z are three elements of $\Lambda_{\phi,w}$ such that $\varrho_\phi(x-y) \le 1$, $\varrho_\phi(x-z) \le 1$ and*

$$\phi\Big(x(t) - \frac{1}{2}(y(t) + z(t)) \Big) \le \frac{\phi\big(x(t) - y(t)\big) + \phi\big(x(t) - z(t)\big)}{2} - \lambda$$

for every $t \in E$. Then there is $\nu > 0$ such that

$$\varrho_\phi\Big(x - \frac{y+z}{2} \Big) \le 1 - \nu.$$

LEMMA 4. *Let ϕ be a Young function. For any given $\delta > 0$, there exists $\varepsilon > 0$ such that*

$$\phi\Big(d_2 - \frac{d_1}{2} \Big) < \frac{1}{2}(\phi(d_2 - d_1) + \phi(d_2))$$

whenever $d_1 > u_0 + \delta$ and $0 < d_2 < d_1 + \varepsilon$.

Proof. If $u_0 = 0$, then there is $\varepsilon < \delta/3$ such that ϕ is not linear on $(\varepsilon, 2\varepsilon)$. If $u_0 > 0$, let $\varepsilon = \frac{1}{3}\min\{u_0, \delta\}$. Then ϕ is not linear on $(\varepsilon, u_0 + \varepsilon)$. Hence if $d_3 > 2\varepsilon + u_0$ and $0 \le d_4 < \varepsilon$, then

$$\phi\left(\frac{d_3 + d_4}{2}\right) < \frac{1}{2}(\phi(d_3) + \phi(d_4)).$$

Case 1: $d_1 \le d_2$. In this case, $d_2 - d_1 < \varepsilon$, and $d_2 \ge d_1 > u_0 + \delta > u_0 + 2\varepsilon$. So

$$\phi\left(d_2 - \frac{d_1}{2}\right) = \phi\left(\frac{d_2 + d_2 - d_1}{2}\right) < \frac{1}{2}(\phi(d_2 - d_1) + \phi(d_2)).$$

Case 2: $d_1 > d_2$. If $d_2 < d_1/2$, then

$$\phi\left(d_2 - \frac{d_1}{2}\right) = \phi\left(\frac{d_1}{2} - d_2\right) \le \phi\left(\frac{d_1 - d_2}{2}\right) \le \frac{1}{2}\phi(d_1 - d_2).$$

If $d_2 \ge d_1/2$, then

$$\phi\left(d_2 - \frac{d_1}{2}\right) \le \phi\left(d_2 - \frac{d_2}{2}\right) = \phi\left(\frac{d_2}{2}\right) \le \frac{1}{2}\phi(d_2).$$

Hence

$$\phi\left(d_2 - \frac{d_1}{2}\right) < \frac{1}{2}(\phi(d_2 - d_1) + \phi(d_2)). \ \blacksquare$$

It seems that the following proposition is known. But we cannot find a reference. So we present a proof.

PROPOSITION 5. *Let $\{x_n\}$ be a sequence in the unit ball of an order continuous Köthe space E and $\{B_n\}$ be a sequence of disjoint measurable subsets. If $\{x_n 1_{B_n}\}$ is equivalent to the natural basis of ℓ_1, then $\{x_n\}$ does not converge weakly.*

Proof. Since $\{x_n 1_{B_n}\}$ is equivalent to the natural ℓ_1 basis, there is x^* in the dual of $\Lambda_{\phi,w}$ such that $\langle x^*, x_n 1_{B_n}\rangle = 1$. We claim that

$$\lim_{j\to\infty} \lim_{n\to\infty} \langle x^* 1_{B_j}, x_n\rangle = 0. \tag{1}$$

By passing to further subsequences of $\{x_n\}$, we may assume that for any $j \in \mathbb{N}$, $\lim_{n\to\infty}\langle x^* 1_{B_j}, x_n\rangle$ exists. Suppose the claim is not true. Then there exist $c > 0$, $L \ge \|x^*\|/c$, l and $F \subseteq \mathbb{N}$ such that $\mathrm{card}(F) \ge L$ and for any $j \in F$,

$$|\langle x^* 1_{B_j}, x_l\rangle| > c.$$

This implies $\langle |x^*|, |x_l|\rangle > Lc \ge \|x^*\|$, which contradicts $\|x_l\| \le 1$.

We claim that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$|\langle x^* 1_{B_{n_i}}, x_{n_l}\rangle| < \frac{1}{4} \quad \text{for any } l \ge i + 1;$$

$$\|x_{n_i}1_{\bigcup_{j=n_i}^{\infty} B_{n_j}}\| \le \frac{1}{4^{i+1}\|x^*\|}.$$

By (1), there is n_1 such that

$$\lim_{k\to\infty} |\langle x^*1_{B_{n_1}}, x_k\rangle| < \frac{1}{4}.$$

We can find an $n_2 > n_1$ such that

$$\begin{aligned} |\langle x^*1_{B_{n_1}}, x_l\rangle| &< \frac{1}{4} && \text{for any } l \ge n_2;\\ \|x_{n_1}1_{\bigcup_{j=n_2}^{\infty} B_{n_j}}\| &\le \frac{1}{4^2\|x^*\|} && \text{(since } E \text{ is order continuous);}\\ \lim_{k\to\infty} |\langle x^*1_{B_{n_2}}, x_k\rangle| &< \frac{1}{4^2} && \text{by (1).} \end{aligned}$$

Assume that $n_1, \ldots, n_i$ are selected. Then there is $n_{i+1} > n_i$ such that

$$\begin{aligned} |\langle x^*1_{B_{n_i}}, x_l\rangle| &< \frac{1}{4^i} && \text{for any } l \ge n_{i+1};\\ \|x_{n_i}1_{\bigcup_{j=n_{i+1}}^{\infty} B_{n_j}}\| &\le \frac{1}{4^{i+1}\|x^*\|} && \text{(since } E \text{ is order continuous);}\\ \lim_{k\to\infty} |\langle x^*1_{B_{n_{i+1}}}, x_k\rangle| &< \frac{1}{4^{i+1}} && \text{by (1).} \end{aligned}$$

We have constructed a subsequence $\{x_{n_k}\}$ which satisfies our claim. Let $\{a_j : 1 \le j \le N\}$ be any finite real sequence, and let

$$E_1 = \bigcup\{B_j : a_j > 0 \text{ and } j \le N\}, \qquad E_2 = \bigcup\{B_j : a_j \le 0 \text{ and } j \le N\}.$$

Then

$$\begin{aligned} \|x^*\| \cdot \Big\|\sum_{j=1}^{N} a_j x_{n_j}\Big\| &\ge \Big\langle x^*1_{E_1} - x^*1_{E_2}, \sum_{j=1}^{N} a_j x_{n_j}\Big\rangle\\ &\ge \sum_{j=1}^{N} \Big(|a_j|\langle x^*, 1_{B_j}x_{n_j}\rangle\\ &\qquad - \sum_{i=1}^{j-1} |a_j| \cdot |\langle x^*, 1_{B_i}x_{n_j}\rangle| - |a_j| \cdot \|x_{n_j}1_{\bigcup_{l=j+1}^{\infty} B_{n_l}}\|\Big)\\ &\ge \sum_{j=1}^{N} |a_j|\Big(1 - \sum_{i=1}^{j+1} \frac{1}{4^i}\Big) \ge \frac{2}{3}\sum_{j=1}^{N} |a_j|. \end{aligned}$$

This implies that $\{x_{n_k}\}$ is equivalent to the natural basis of ℓ_1. So $\{x_n\}$ cannot converge weakly. ■

Suppose that $\Lambda_{\phi,w}$ is an order continuous Lorentz–Orlicz function space without (weakly) normal structure. There exists a (weakly convergent) unit limit-constant sequence $\{x_n\}$ in $\Lambda_{\phi,w}$. (From now on, $\{x_n\}$ is a fixed (weakly convergent) unit limit-constant sequence.) Let

$$\overline{x}_n = \frac{1}{n}\sum_{k=1}^{n} x_k, \qquad z'_n = \sup\{x_1,\ldots,x_n\}, \qquad z''_n = \inf\{x_1,\ldots,x_n\}.$$

Then $\{z'_n\}$ is an increasing sequence. It converges in measure to an extended measurable function

$$z' = \sup\{x_n : n \in \mathbb{N}\} \equiv \lim_{n\to\infty} z'_n.$$

Similarly, $\{z''_n\}$ is a decreasing sequence, and it converges in measure to another extended measurable function

$$z'' = \inf\{x_n : n \in \mathbb{N}\} \equiv \lim_{n\to\infty} z''_n.$$

LEMMA 6. $m(\{t : |z'(t) - z''(t)| > u_0\}) = 0$.

Proof. If $u_0 = \infty$, then there is nothing to be proved. So we may assume that $u_0 < \infty$. Suppose that the lemma is not true. Since

$$\{t : |z'(t) - z''(t)| > u_0\} = \bigcup_{m,n\in\mathbb{N}} \{t : |x_n(t) - x_m(t)| > u_0\},$$

there are n and m such that

$$m(\{t : |x_n(t) - x_m(t)| > u_0\}) > 0.$$

By passing to a subsequence, we may assume that $x_n = x_1$ and $x_m = x_2$. Let $A = \{t : x_2(t) - x_1(t) \ge 0\}$. Replacing x_k by $(x_k - x_1)1_A - (x_k - x_1)1_{\Omega\setminus A}$, we may assume that $x_2 \ge 0$. By measure theory, there is $\delta > 0$ such that

$$m(\{t : |x_1(t) - x_2(t)| > u_0 + \delta\}) > c > 0.$$

By Lemma 4, there is $\varepsilon > 0$ such that

$$\phi\left(d_2 - \frac{d_1}{2}\right) < \frac{1}{2}(\phi(d_2 - d_1) + \phi(d_2)) \tag{2}$$

provided $d_1 > u_0 + \delta$ and $0 < d_2 < d_1 + \varepsilon$.

CLAIM. *There are a subsequence $\{y_k\}_{k=1}^{\infty}$ of $\{x_n\}$ and a decreasing sequence $\{C_k\}_{k=2}^{\infty}$ of measurable sets such that*

(a) $y_1 = x_1$, $y_2 = x_2$;
(b) $m(C_n) \ge (1/2 + 1/2^n)c$;
(c) *for any $t \in C_n$, there is $k < n$ such that*

$$\begin{aligned}|y_n(t) - y_k(t)| &\ge \varepsilon + \sup\{|y_{n-1}(t) - y_j(t)| : j < n\}\\ &= \varepsilon + \sup\{|y_i(t) - y_j(t)| : i, j < n\}.\end{aligned}$$

Suppose the claim were proved. Note that if $n > 2$, $t \in C_n$ and $y_n(t) > 0$, then

$$y_n(t) - \inf\{y_i(t) : 1 \le i \le n-1\} \ge \varepsilon + \sup\{|y_{n-1}(t) - y_i(t)| : 1 \le i < n\}.$$

Similarly, if $y_n(t) < 0$, then

$$\sup\{y_i(t) : 1 \le i \le n-1\} - y_n(t) \ge \varepsilon + \sup\{|y_{n-1}(t) - y_i(t)| : 1 \le i < n\}.$$

So for any $t \in C_n$ and $k < m \le n-1$, we have

$$(y_n(t) - y_m(t))(y_n(t) - y_k(t)) \ge 0,$$

and

$$\begin{aligned}|y_n(t) - y_m(t)| \ge\ & \sup\{|y_n(t) - y_j(t)| : j \le n-1\} \\ & - \sup\{|y_m(t) - y_j(t)| : j \le n-1\} \ge \varepsilon.\end{aligned}$$

This implies

$$\operatorname{card}(\{j \le n-1 : |y_n(t) - y_j(t)| < l\varepsilon\}) \le l-1,$$

and

$$|y_n(t) - \overline{y}_{n-1}(t)| = \frac{1}{n-1}\sum_{i=1}^{n-1} |y_n(t) - y_i(t)| \ge \frac{1}{n-1}\sum_{i=1}^{n-1} i\varepsilon = \frac{n\varepsilon}{2}.$$

Therefore,

$$\int_{\Omega} \phi((y_n - \overline{y}_{n-1})^*)w \ge \int_0^{m(C_n)} \phi\left(\frac{n\varepsilon}{2}\right)w \ge \phi\left(\frac{n\varepsilon}{2}\right)\int_0^{c/2} w,$$

which is impossible if n is large enough. Hence the lemma must be true.

P r o o f o f C l a i m. Let $C_2 = \{t : |y_1(t) - y_2(t)| > u_0 + \delta\}$. (So $m(C_2) < \infty$.) Suppose that $y_1, \ldots, y_k = x_{n_k}$ and $C_2, \ldots, C_k$ have been constructed. For $j < k$, let

$$\begin{aligned}D_j = \{t \in C_k : |y_k(t) - y_j(t)| \\ = \sup\{|y_k(t) - y_i(t)| : i < k\} > \sup\{|y_k(t) - y_i(t)| : i < j\}\}.\end{aligned}$$

Then $C_k = \bigcup_{j=1}^{k-1} D_j$.

SUBCLAIM. *There is $M_j > n_k$ such that for any $n \ge M_j$,*

$$\begin{aligned}m(\{t \in D_j :\ & \sup\{|x_n(t) - y_i(t)| : i \le k\} \\ & \ge \sup\{|y_k(t) - y_i(t)| : i < k\} + \varepsilon\}) \ge (1 - 1/2^{k+1})m(D_j).\end{aligned}$$

Suppose that the subclaim were proved. Let $n_{k+1} = \sup\{M_j : j < k\}$, $y_{k+1} = x_{n_{k+1}}$, and

$$\begin{aligned}C_{k+1} = \{t \in C_k :\ & \sup\{|y_{k+1}(t) - y_j(t)| : j \le k\} \\ & \ge \sup\{|y_k(t) - y_j(t)| : j < k\} + \varepsilon\}.\end{aligned}$$

Then C_{k+1} and y_{k+1} satisfy (b) and (c), hence the claim is proved.

Proof of Subclaim. If $m(D_j) = 0$, then let $M_j = n_k + 1$. So we may assume that $m(D_j) > 0$. By measure theory, there exists $L > \delta + u_0$ such that

$$m(\{t \in D_j : |y_k(t) - y_j(t)| \le L\}) > (1 - 1/2^{k+2})m(D_j)$$

for any $m(D_j) > 0$, $j \le k$. Note that if $t \in D_j$, then $u_0 + \delta < |y_k(t) - y_j(t)|$. Suppose the subclaim is not true. Then for any $N > n_k$, there is $m > N$ such that

$$\begin{aligned} E_{m,j} = \{t \in D_j :\ & \max\{|x_m(t) - y_k(t)|, |x_m(t) - y_j(t)|\} \\ & < |y_k(t) - y_j(t)| + \varepsilon \text{ and } u_0 + \delta < |y_k(t) - y_j(t)| \le L\} \end{aligned}$$

has measure greater than $2^{-(k+2)}m(D_j)$. For any $t \in E_{m,j}$, either $y_k(t) > y_j(t)$ or $y_k(t) < y_j(t)$. Without loss of generality, we assume that $y_k(t) > y_j(t)$ and $y_k(t) + \varepsilon \ge x_m(t) \ge y_j(t) - \varepsilon$. Let $d_1 = y_k(t) - y_j(t)$ and

$$d_2 = \begin{cases} y_k(t) - x_m(t) & \text{if } x_m(t) \le y_k(t), \\ x_m(t) - y_j(t) & \text{otherwise.} \end{cases}$$

Since $[0, L]$ is compact and ϕ is continuous, by (2), there is $\lambda > 0$ such that

$$\phi\Big(d_2 - \frac{d_1}{2}\Big) \le \frac{1}{2}(\phi(d_2 - d_1) + \phi(d_2)) - \lambda$$

whenever $L \ge d_1 > u_0 + \delta$ and $d_2 \le d_1 + \varepsilon$. So

$$\begin{aligned} \phi\Big(\frac{1}{2}(y_k(t) + y_j(t)) - x_m(t)\Big) & \\ &= \phi\Big(y_k(t) - x_m(t) - \frac{y_k(t) - y_j(t)}{2}\Big) \\ &\le \frac{1}{2}(\phi(y_k(t) - x_m(t)) + \phi(y_j(t) - x_m(t))) - \lambda. \end{aligned}$$

Note that $\varrho_\phi\big(x_m - \frac{1}{2}(y_j + y_k)\big) \le 1$ and $\int_0^\infty w = \infty$. By Lemma 3, there is $\nu > 0$ (which depends on λ, w and $m(D_j)$, but is independent of x_m) such that

$$\int_0^\infty \phi\Big(\Big(x_m - \frac{1}{2}(y_j + y_k)\Big)^*\Big)w \le 1 - \nu.$$

Since ϕ satisfies the Δ_2 condition, by Theorem A,

$$\liminf_{m\to\infty} \Big\|x_m - \frac{1}{2}(y_k + y_j)\Big\| < 1,$$

which contradicts the fact that $\{x_n\}$ is a unit limit-constant sequence. So the subclaim must be true and the proof of Lemma 6 is complete. ∎

Remark 2. Since $\{x_n\}$ is not a constant sequence, we have $u_0 > 0$.

LEMMA 7. *For any $l \in \mathbb{N}$ and $\varepsilon > 0$,*

$$\lim_{n\to\infty} m(\{t : z_l''(t) + \varepsilon < x_n(t) < z_l'(t) - \varepsilon\}) = 0.$$

Proof. Suppose the lemma is not true. By passing to a further subsequence of $\{x_n\}$, we may assume that there are $\varepsilon > 0$ and $\delta > 0$ such that for all $m > l$, the set

$$F_m = \{t : z_l''(t) + \varepsilon < x_m(t) < z_l'(t) - \varepsilon\}$$

has measure at least δ. Let σ be a measure preserving transformation such that

(i) $\int_0^\infty \phi(x_m - \overline{x}_l) w \circ \sigma = \int_0^\infty \phi((x_m - \overline{x}_l)^*) w$;
(ii) if $|(x_m - \overline{x}_l)(t)| < |(x_m - \overline{x}_l)(s)|$, then $\sigma(t) \ge \sigma(s)$.

Since for any $t \in F_m$,

$$\frac{1}{l}\sum_{k=1}^{l} |x_m(t) - x_k(t)| \ge \frac{\varepsilon}{l} + |x_m(t) - \overline{x}_l(t)|,$$

by Lemma 3, there is $\nu > 0$ (dependent only on l, δ and ε) such that

$$\int_0^\infty \phi((x_m - \overline{x}_l)^*) w = \int_0^\infty \phi\Big(x_m - \frac{1}{l}\sum_{k=1}^{l} x_k\Big) w \circ \sigma$$
$$\le \frac{1}{l}\sum_{k=1}^{l} \varrho_\phi(|x_m - x_k|) - \nu \le 1 - \nu.$$

This contradicts $\lim_{m\to\infty} \|x_m - \overline{x}_l\| = 1$. ■

LEMMA 8. *Suppose that there are two positive numbers v_1, u_1 such that*

(1) *either $w(t) < w(v_1)$ for all $t > v_1$ or $w(t) > w(v_1)$ for all $t < v_1$;*
(2) *for any $i \ne j$, $m\big(\{t : |x_i(t) - x_j(t)| \ge u_1\}\big) \le v_1$.*

Then for any $u_2 > u_1$, $m(\{t : z'(t) - z''(t) \ge u_2\}) \le v_1$.

Proof. Since the proofs are similar, we can assume that $w(t) < w(v_1)$ for all $t > v_1$. Suppose that the lemma is not true. There is $\nu > 0$ such that $u_2 - u_1 > 2\nu$ and

$$m(\{t : z'(t) - z''(t) > u_1 + 2\nu\}) > v_1 + 2\nu.$$

Let

$$F_l = \{t : z_l'(t) - z_l''(t) > u_1 + 3\nu/2\}.$$

Clearly, $m(F_k) < \infty$ for all $k \in \mathbb{N}$. Since $\{F_k\}$ is an increasing sequence and $\bigcup_{k=1}^\infty F_k \supseteq \{t : z'(t) - z''(t) > u_1 + 2t\}$, there is l such that $m(F_l) \ge v_1 + 3\nu/2$. Let

$$G_n = \{t \in F_l : x_n(t) \ge z_l'(t) - \nu/4 \text{ or } x_n(t) \le z_l''(t) + \nu/4\}.$$

By Lemma 6, $\lim_{n\to\infty} m(F_l \setminus G_n) = 0$. So there is $N_1 > l$ such that if $n > N_1$, then $m(G_n) \ge v_1 + \nu$. This implies that for any measure preserving transformation σ of Ω,

$$m(\{t \in G_n : t \in \sigma^{-1}(v_1, \infty)\}) \ge \nu.$$

Fix $n > N_1$. By the definition of G_n, for any $t \in G_n$, either $x_n(t) \ge z_l'(t) - \nu/4$ or $x_n(t) \le z_l''(t) + \nu/4$. Without loss of generality, $x_n(t) \ge z_l'(t) - \nu/4$. Let $j \le l$ such that $x_j(t) = z_l''(t)$. Then

$$|x_n(t) - x_j(t)| \ge z_l'(t) - z_l''(t) - \nu/4 > u_1 + 5\nu/4.$$

Note that $\{x_n\}$ is a unit limit-constant sequence. For any $\lambda > 0$, there are $n > N_1$ and a measure preserving transformation σ such that

(i) $\int \phi(x_n - \overline{x}_l) w \circ \sigma = \int \phi((x_n - \overline{x}_l)^*) w \ge 1 - \lambda$;
(ii) if $|(x_n - \overline{x}_l)(t)| \ge |(x_n - \overline{x}_l)(s)|$, then $\sigma(t) \ge \sigma(s)$.

For any $k \le l$, let

$$H_k = \sigma^{-1}(v_1, \infty) \cap \{t : |x_n(t) - x_k(t)| > u_1 + 5\nu/4\}.$$

Clearly, $\bigcup_{k=1}^{l} H_k \supseteq \{t \in G_n : t \in \sigma^{-1}(v_1, \infty)\}$. Hence there is $k \le l$ such that $m(H_k) \ge \nu/l$. By (2), the set $\{t \in \sigma^{-1}(0, v_1) : |x_n(t) - x_k(t)| < u_1\}$ has measure at least $m(H_k)$. By Lemma 2 and Remark 1, there is $\delta > 0$ such that δ is only dependent on u_1, ν, v_1, l, and

$$\int \phi(|x_n(t) - x_k(t)|) w \circ \sigma(t)\, dt \le \varrho_\phi(x_n - x_k) - \delta.$$

This implies, for any $\lambda > 0$,

$$\begin{aligned} 1 - \lambda \le \int \phi(x_n - \overline{x}_l) w \circ \sigma &\le \frac{1}{l} \sum_{j=1}^{l} \int \phi(x_n - x_j) w \circ \sigma \\ &\le \frac{1}{l} \sum_{j=1}^{l} \varrho_\phi(x_n - x_j) - \frac{\delta}{l} \le 1 - \frac{\delta}{l}. \end{aligned}$$

It is impossible if $\lambda < \delta/l$. ∎

We have the following two corollaries.

Corollary 9. *If $v_0 = 0$, then z' and z'' are finite almost everywhere.*

Proof. Since $v_0 = 0$ and w is left continuous, for any $\delta > \delta_1 > 0$, there are $0 < \delta_2 < \delta_1$ and $u_1 > 0$ such that $\varrho_\phi(u_1 1_{(0,\delta_2)}) > 1$ and $w(t) < w(\delta_2)$ if $t > \delta_2$. Since $\{x_n\}$ is a unit limit-constant sequence, for any m, n we have $\varrho_\phi(x_m - x_n) \le 1$. So

$$m\big(\{t : |x_n(t) - x_m(t)| > u_1\}\big) \le \delta_2 \quad \text{for all } n, m \in \mathbb{N}.$$

By Lemma 8, we have

$$m(\{t : z'(t) - z''(t) \ge u_2\}) \le \delta_2$$

for any $u_2 > u_1$. Since δ_2 is arbitrary, z' and z'' are finite almost everywhere. ■

COROLLARY 10. *Suppose that w is not constant on (v,∞) for any $v > 0$. Then for any $\varepsilon > 0$,*

$$m(\{t : z'(t) - z''(t) > 2\varepsilon\}) < \infty.$$

Proof. Since $\int_0^\infty w = \infty$ and w is not constant on (v,∞) for any $v > 0$, it follows that for any $\varepsilon > 0$, there is $L > 0$ such that $w(t) > w(L)$ for all $t > L$ and

$$m\{t : |x_n(t) - x_m(t)| \ge \varepsilon\} < L$$

for all n, m. By Lemma 8, we have

$$m(\{t : z'(t) - z''(t) > 2\varepsilon\}) < L < \infty. \ ■$$

PROPOSITION 11. *Suppose that there is $1 > \delta > 0$ such that one of the following conditions holds*:

(1) *For any $M > 0$, there is n such that $\varrho_\phi(x_n 1_{\{t:|x_n(t)|>M\}}) > \delta$.*
(2) *For any $\varepsilon > 0$ there is n such that $\varrho_\phi(x_n 1_{\{t:|x_n(t)|<\varepsilon\}}) > \delta$.*

Then $\{x_n\}$ does not converge weakly.

Proof. Since the proofs are similar, we only prove the proposition when (1) holds.

Suppose the proposition is not true. Then there is a weakly convergent unit limit-constant sequence $\{x_n\}$ satisfying (1). Lemma 6 yields $u_0 = \infty$. By assumption, there exist sequences $\{D_k\}$, $\{d_k\}$ and $\{n_k\}$ such that for all $k \in \mathbb{N}$ we have $8^k d_k < 8^k D_k < \delta d_{k+1} < \delta D_{k+1}$ and $\varrho_\phi(x_{n_k} 1_{\{t:d_k \le |x_{n_k}(t)| \le D_k\}}) > \delta$. Let

$$A_k = \{t : d_k \le |x_{n_k}(t)| \le D_k\}, \qquad B_k = A_k \setminus \bigcup_{j=k+1}^{\infty} A_j.$$

Since $\varrho_\phi(x_{n_k}) \le 1$ and $|x_{n_k}(t)| \ge d_k$ for every $t \in A_k$, $\int_0^{m(A_k)} w(t)\,dt \le 1/\phi(d_k)$. So

$$\begin{aligned}\varrho_\phi(x_{n_k} 1_{B_k}) &\ge \varrho_\phi(x_{n_k} 1_{A_k}) - \sum_{j=k+1}^{\infty} \varrho_\phi(x_{n_k} 1_{A_j}) \\ &\ge \delta - \sum_{j=k+1}^{\infty} \phi(D_k)\frac{1}{\phi(d_j)} \ge \delta - \frac{\delta}{3} = \frac{2\delta}{3}.\end{aligned}$$

We claim that $\{x_{n_k} 1_{B_k}\}$ is equivalent to the natural basis of ℓ_1. Without loss of generality, we assume that

$$B_k = \Big(\sum_{j=k+1}^{\infty} m(B_j), \sum_{j=k}^{\infty} m(B_j)\Big),$$

and $|x_{n_k}|1_{B_k}$ is decreasing on B_k. Then

$$\begin{aligned}
&\int \phi(x_{n_k}1_{B_k})w \\
&\quad = \int\limits_{B_k} \phi(x_{n_k})w \\
&\quad = \int\limits_0^{m(B_k)} \phi(x_{n_k})\Big(t + \sum_{j=k+1}^{\infty} m(B_j)\Big)w\Big(t + \sum_{j=k+1}^{\infty} m(B_j)\Big) \\
&\quad \ge \int\limits_0^{m(B_k)} \phi(x_{n_k})\Big(t + \sum_{j=k+1}^{\infty} m(B_j)\Big)w(t) - \phi(D_k)\sum_{j=k+1}^{\infty} 1/\phi(d_j) \\
&\quad \ge \varrho_\phi(x_{n_k}1_{B_k}) - \frac{\delta}{3} \ge \frac{2\delta}{3} - \frac{\delta}{3} = \frac{\delta}{3}.
\end{aligned}$$

Hence, for any sequence $\{a_n\} \in \ell_1$ with $\sum_{n=1}^{\infty} |a_n| \ge 1/(3\delta)$,

$$\begin{aligned}
\varrho_\phi\Big(\sum_{j=1}^{\infty} a_j x_{n_j} 1_{B_j}\Big) &\ge \int \phi\Big(\sum_{j=1}^{\infty} a_j x_{n_j} 1_{B_j}\Big)w = \sum_{j=1}^{\infty} \int\limits_{B_j} \phi(a_j x_{n_j})w \\
&= \sum_{j=1}^{\infty} \int\limits_{B_j} a_j \phi(x_{n_j})w \ge \sum_{j=1}^{\infty} |a_j| \frac{\delta}{3} \ge 1.
\end{aligned}$$

This implies that $\{x_{n_k}1_{B_k}\}$ is equivalent to the natural basis of ℓ_1. By Proposition 5, $\{x_n\}$ does not converge weakly. ∎

Proposition 12. *Suppose that for any $\nu > 0$, there are a sequence $\{n_i\}$ and a measurable set A such that $0 < m(A) \le v_0$ and*

$$\varrho_\phi((x_{n_k} - x_{n_j})1_A) \ge 1 - \nu \quad \text{whenever } i > j.$$

Then $\int_0^{v_0} \phi(u_0)w \ge 2$.

Proof. It is clear that $v_0 > 0$. If $u_0 = \infty$, then there is nothing to be proved. So we may assume that $u_0 < \infty$. Replacing x_n by $x_n - x_1$ if necessary, we may also assume that $x_1 \equiv 0$. By Lemma 6, both z' and z'' are bounded. Since ϕ is linear on $(0, v_0)$, without loss of generality, we further assume that $\phi(t) = t$ for all $0 < t \le u_0$ and $w(t) = 1$ for all $t \le v_0$. To prove the proposition, it is enough to show that $v_0 \ge 2/u_0$.

Let K be a fixed natural number. For any $a < b$ and any $0 \le l \le 2K$, let $\{a_k : 1 \le k \le 2K\}$ be a finite sequence such that

$$a_k = \begin{cases} a & \text{if } k \le l, \\ b & \text{otherwise.} \end{cases}$$

Then

$$\sum_{i<j\le 2K} |a_j - a_i| = (2K - l + 1)(l - 1)(b - a) \le K^2(b - a).$$

Let $0 < \delta < v_0$ be any positive number such that

$$\int_0^\delta u_0\,dt \le \frac{1}{K^4}.$$

By assumption, there are a measurable set A and a natural number N such that $0 < m(A) \le v_0$ and

$$\varrho_\phi((x_{n_k} - x_{n_j})1_A) \ge 1 - 1/K^4 \quad \text{whenever } k > j \ge N.$$

By the definition of z' and z'', there exists l such that

$$m\left\{t \in A : |z'(t) - z'_l(t)| > \frac{1}{2K^4 v_0}\right\} < \frac{\delta}{3},$$
$$m\left\{t \in A : |z''(t) - z''_l(t)| > \frac{1}{2K^4 v_0}\right\} < \frac{\delta}{3}.$$

By Lemma 7, there exists a finite subsequence $\{k_1, \ldots, k_{2K}\}$ of $\{n_k\}$ such that for any $j \le 2K$,

$$m\left(\left\{t \in A : z''_l(t) + \frac{1}{2K^4 v_0} < x_{k_j} < z'_l - \frac{1}{2K^4 v_0}\right\}\right) < \frac{\delta}{3}.$$

Let

$$B_i = \left\{t \in A : |z''(t) - x_{k_i}(t)| \ge \frac{1}{K^4 v_0} \text{ and } |x_{k_i}(t) - z'(t)| \ge \frac{1}{K^4 v_0}\right\}.$$

Then for all $i \le 2K$, B_i has measure at most δ. For each $i \le 2K$, let y_i be a measurable function such that $y_i(t) \in \{z'(t), z''(t)\}$ and for any $t \in A \setminus B_i$, $|y_i(t) - x_{k_i}(t)| < \frac{1}{K^4 v_0}$. Then

$$\begin{aligned}
K(2K-1)\left(1 - \frac{1}{K^4}\right) &\le \sum_{i<j\le 2K} \varrho_\phi((x_{k_i} - x_{k_j})1_A) \\
&= \sum_{i<j\le 2K} \int_A |x_{k_i} - x_{k_j}|\,dt \\
&\le \sum_{i<j\le 2K} \int_A |x_{k_i} - y_i| + |y_i - y_j| + |y_j - x_{k_j}|\,dt \\
&\le \sum_{i<j\le 2K} \bigg(\int_{B_i} u_0\,dt + \int_{B_j} u_0\,dt + \int_{A\setminus B_i} \frac{dt}{K^4 v_0} \\
&\qquad + \int_{A\setminus B_j} \frac{dt}{K^4 v_0} + \int_A |y_i - y_j|\,dt \bigg)
\end{aligned}$$

$$\leq K(2K-1)\frac{4}{K^4} + \int\limits_A \sum_{i<j\leq 2K} |y_i - y_j|$$
$$\leq \frac{16}{K^2} + K^2 u_0 v_0.$$

This implies

$$v_0 \geq (u_0)^{-1}\left(2K^2 - K - \frac{2}{K^2} - \frac{16}{K^2}\right)\frac{1}{K^2}.$$

Since K is arbitrary, $v_0 \geq 2/u_0$. ■

For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, define

$$p(x_{n_k}) = \sup\{u : m(\{t : \sup\{x_{n_k}\}(t) - \inf\{x_{n_k}\}(t) > u\}) = \infty\}.$$

LEMMA 13. *Suppose z' and z'' are finite almost everywhere. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for any further subsequence $\{y_k\}$ of $\{x_{n_k}\}$, $p(x_{n_k}) = p(y_k)$.*

Proof. For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, clearly, $p(x_n) \geq p(x_{n_k})$. Let

$$q(x_{n_k}) = \inf\{p(y_k) : \{y_k\} \text{ is a subsequence of } \{x_{n_k}\}\}.$$

By induction, there exists a sequence $\{x_{j,n} : n \in \mathbb{N}\}_{j=1}^{\infty}$ of sequences such that

(a) for any j, $\{x_{j,n} : n \in \mathbb{N}\}$ is a subsequence of $\{x_{j-1,n} : n \in \mathbb{N}\}$;
(b) for any j,

$$p_j = p(\{x_{j,n} : j \in \mathbb{N}\}) \leq q_{j-1} + 1/2^j$$

where $q_{j-1} = q(\{x_{j-1,n} : n \in \mathbb{N}\})$.

Note that $\{p_n\}$ is a decreasing sequence, $\{q_n\}$ is an increasing sequence and $|p_n - q_{n-1}| \leq 1/2^n$. Further,

$$u_4 = \lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n$$

exists. We claim that $p(\{x_{n,n} : n \in \mathbb{N}\}) = u_4 = q(\{x_{n,n} : n \in \mathbb{N}\})$.

Let $\{y_k\}$ be any subsequence of $\{x_{n,n} : n \in \mathbb{N}\}$. Then for any $m \in \mathbb{N}$, $\{y_k : k \geq m\}$ is a subsequence of $\{x_{m,n} : n \in \mathbb{N}\}$. So

$$p(y_k) \geq \lim_{m\to\infty} p(\{y_k : k \geq m\}) \geq \lim_{m\to\infty} q_m = u_4.$$

For any $\varepsilon > 0$, there is m such that $p_m < u_4 + \varepsilon/4$. Let

$$A = \{t : \sup\{x_{n,n} : n \geq m\}(t) - \inf\{x_{n,n} : n \geq m\}(t) \geq u_4 + \varepsilon/4\}$$
$$B = \{t : |x_{j,j}(t)| \geq \varepsilon/4 \text{ for some } j \leq m\}.$$

Since $\int_0^\infty w = \infty$ and $p(\{x_{n,n} : n \geq m\}) < u_4 + \varepsilon/4$, both A and B have finite measure.

If $j,k \le m$ and $t \notin A \cup B$, then

$$|x_{j,j}(t) - x_{k,k}(t)| \le \varepsilon/2 \le u_4 + 3\varepsilon/4;$$

$$\begin{aligned}&|\sup\{x_{n,n} : n \ge m\}(t) - x_{j,j}(t)| \\ &\qquad \le |\sup\{x_{n,n} : n \ge m\}(t) - x_{m,m}(t)| + |x_{m,m}(t) - x_{j,j}(t)| \le u_4 + 3\varepsilon/4\end{aligned}$$

and

$$\begin{aligned}&|\inf\{x_{n,n} : n \ge m\}(t) - x_{j,j}(t)| \\ &\qquad \le |\inf\{x_{n,n} : n \ge m\}(t) - x_{m,m}(t)| + |x_{m,m}(t) - x_{j,j}(t)| \le u_4 + 3\varepsilon/4.\end{aligned}$$

This implies that for any $t \notin A \cup B$, $\sup\{x_{n,n}\}(t) - \inf\{x_{n,n}\}(t) \le u_4 + 3\varepsilon/4$, and

$$p(\{x_{n,n} : n \in \mathbb{N}\}) \le u_4 + \varepsilon.$$

But ε is arbitrary, so $p(\{x_{n,n} : n \in \mathbb{N}\}) \le u_4$. ∎

LEMMA 14. *Let u_4, δ and ν be positive real numbers. Suppose that $\{x_n\}$ is a unit limit-constant sequence such that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have*

$$\begin{gathered} u_4 = p(\{x_n : n \in \mathbb{N}\}) = p(\{x_{n_k} : k \in \mathbb{N}\}), \\ m(\{t : \sup\{x_{n_k} : k \in \mathbb{N}\}(t) - \inf\{x_{n_k} : k \in \mathbb{N}\}(t) > 3\nu\}) \ge v_0 + 3\delta. \end{gathered} \tag{3}$$

Then there is a further subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for almost all t,

$$\sup\{x_{n_k}\}(t) - \inf\{x_{n_k}\}(t) \le u_4.$$

Proof. We only prove the lemma when $v_0 = 0$. Suppose the lemma is not true. Then there is $\nu/6 > \varepsilon > 0$ such that the set

$$G_1 = \{t : z'(t) - z''(t) > u_4 + \varepsilon\}$$

has measure at least ε. Replace δ by $\varepsilon/2$ if necessary. We may assume that $\delta \le \varepsilon$. Since $v_0 = 0$, there is $0 < \delta_1 < \delta/6$ such that if $t > \delta_1$, then $w(t) < w(\delta_1)$. Note that for any subsequence $\{x_{n_k}\}$, $p(x_{n_k}) = u_4$. Applying Lemma 8 and passing to subsequences, we may assume that for any $n \ne m$,

$$m(\{t : |x_n(t) - x_m(t)| \ge u_4 + 2\varepsilon/3\}) \ge \delta_1. \tag{4}$$

Let

$$G_2 = \{t : z'(t) - z''(t) > u_4 + \varepsilon/2\}.$$

Then $\delta_1 < m(G_2) < \infty$, and there is l such that

$$G_3 = \{t \in G_2 : z'(t) - z'_l(t) > \varepsilon/12 \text{ and } z''_l(t) - z''(t) > \varepsilon/12\}$$

has measure less than $\delta_1/10$. By Lemma 7, there is N_3 such that for any $n > N_3$, the set

$$G_4 = \{t \in G_2 \setminus G_3 : \text{either } |z'_l(t) - x_n(t)| < \varepsilon/6 \text{ or } |z''_l(t) - x_n(t)| < \varepsilon/6\}$$

has measure at least $m(G_2) - \delta_1/5$. Let

$$G_5 = G_4 \cap G_1 = \{t \in G_1 \setminus G_3 : \text{either } |z_l'(t) - x_n(t)| < \varepsilon/6 \\ \text{or } |z_l''(t) - x_n(t)| < \varepsilon/6\}.$$

Then $m(G_5) \geq m(G_1) - \delta_1/5$ and for any $t \in G_5$ (respectively, $t \in G_4$), there exists $k_1 \leq l$ (respectively, $k_2 \leq l$) such that

$$|x_n(t) - x_{k_1}(t)| = \max\{|x_n(t) - z_l'(t)|, |x_n(t) - z_l''(t)|\} \geq u_4 + \varepsilon - \varepsilon/3,$$

or respectively,

$$|x_n(t) - x_{k_2}(t)| = \min\{|x_n(t) - z_l'(t)|, |x_n(t) - z_l''(t)|\} \leq \varepsilon/6).$$

Since $\{x_n\}$ is a unit limit-constant sequence, for any $\lambda > 0$, there are $n > N_3$ and a measure preserving transformation σ such that

(i) $\int_0^\infty \phi(x_n - \overline{x}_l) w \circ \sigma = \int_0^\infty \phi((x_n - \overline{x}_l)^*) w \geq 1 - \lambda$;
(ii) if $|(x_n - \overline{x}_l)(t)| \geq |(x_n - \overline{x}_l)(s)|$, then $\sigma(t) \geq \sigma(s)$.

Case 1: $m(\sigma^{-1}(0, \delta_1) \cap G_4) \geq 2\delta_1/5$. For any $k \leq l$, let

$$H_k = \{t : t \in \sigma^{-1}(0, \delta_1) \text{ and } |x_n(t) - x_k(t)| \leq \varepsilon/6\}.$$

Since $\bigcup_{k=1}^l H_k \supseteq G_4 \cap \sigma^{-1}(0, \delta_1)$, there exists $k \leq l$ such that $m(H_k) \geq 2\delta_1/(5l)$. By (4),

$$m(\{t \in \sigma^{-1}(\delta_1, \infty) : |x_n(t) - x_k(t)| \geq u_4 + 2\varepsilon/3\}) \geq 2\delta_1/(5l).$$

Case 2: $m(\sigma^{-1}(0, \delta_1)) \cap G_4) < 2\delta_1/5$. Note that $G_5 \subseteq G_4 \subseteq G_2$ and $m(G_2) \leq \delta_1/5 + m(G_4)$. We have

$$m(\sigma^{-1}(0, \delta_1) \setminus G_2) \geq \delta_1 - m(\sigma^{-1}(0, \delta_1) \cap G_4) - \delta_1/5 \geq 2\delta_1/5,$$

and

$$\begin{aligned} 4\delta_1/5 &\leq m(G_1) - \delta_1/5 \\ &\leq m(G_5) = m(\sigma^{-1}(\delta_1, \infty) \cap G_5) + m(\sigma^{-1}(0, \delta_1) \cap G_5) \\ &\leq m(\sigma^{-1}(\delta_1, \infty) \cap G_5) + m(\sigma^{-1}(0, \delta_1) \cap G_4) \\ &\leq m(\sigma^{-1}(\delta_1, \infty) \cap G_5) + 2\delta_1/5. \end{aligned}$$

This yields

$$m(\sigma^{-1}(\delta_1, \infty) \cap G_5) \geq 4\delta_1/5 - 2\delta_1/5 = 2\delta_1/5. \tag{5}$$

Let

$$H_k' = \{t \in \sigma^{-1}(\delta_1, \infty) : |x_n(t) - x_k(t)| \geq u_4 + 2\varepsilon/3\}.$$

Let t be an element of $G_5 \cap \sigma^{-1}(\delta_1, \infty)$. Then

$$z'(t) - z''(t) > u_4 + \varepsilon$$

with either $|z_l'(t) - x_n(t)| < \varepsilon/6$ or $|z_l''(t) - x_n(t)| < \varepsilon/6$. So $t \in H_k'$ for some $k \leq l$. By (5), there is $k \leq l$ such that $m(H_k') \geq \delta_1/(5l)$. On the other hand, if $t \in \sigma^{-1}(0, \delta_1) \setminus G_2$, then $|x_n(t) - x_k(t)| \leq z'(t) - z''(t) \leq u_4 + \varepsilon/2$.

By Lemma 2 and Remark 1, for both cases, there is $\delta_2 > 0$ (which is dependent only on $\delta_1, l, u_4, \varepsilon$) such that

$$\varrho_\phi(x_n - x_k) \geq \int \phi(x_n - x_k) w \circ \sigma + \delta_2.$$

This implies, for any $\lambda > 0$,

$$\begin{aligned} 1 - \lambda &\leq \int \phi\biggl(x_n - \frac{1}{l}\sum_{j=1}^{l} x_j\biggr) w \circ \sigma \leq \frac{1}{l}\sum_{j=1}^{l} \int \phi(x_n - x_j) w \circ \sigma \\ &\leq \frac{1}{l}\sum_{j=1}^{l} \varrho_\phi(x_n - x_j) - \frac{\delta_2}{l} \leq 1 - \frac{\delta_2}{l}. \end{aligned}$$

This is impossible if $\lambda < \delta_2/l$. ■

3. Proof of Theorem 1. Let $\Lambda_{\phi,w}$ be an order continuous Lorentz–Orlicz space such that $\int_0^{v_0} \phi(u_0) w < 2$. We claim that if $\Lambda_{\phi,w}$ contains a unit limit-constant sequence $\{x_n\}$, then

(a) $\{x_n\}$ does not converge weakly;
(b) if $\Lambda_{\phi,w} \equiv \Lambda_{\phi,w}(0,1)$, then $u_0 = \infty$.

Condition (a) implies that if $\int_0^{v_0} \phi(u_0) w < 2$, then $\Lambda_{\phi,w}$ has weakly normal structure. By Lemma 6 (cf. Remark 2), (b) yields that $u_0 > 0$ if $\Lambda_{\phi,w}$ does not have normal structure. Moreover, if $\Lambda_{\phi,w} \equiv \Lambda_{\phi,w}(0,1)$ does not have normal structure, then either $\int_0^{v_0} \phi(u_0) w \geq 2$ or $u_0 = \infty$.

Let $\{x_n\}$ be a unit limit-constant sequence in $\Lambda_{\phi,w}$. Suppose that $\{x_n\}$ satisfies one of the following conditions:

(c) For any $M > 0$, there is n such that $\varrho_\phi(x_n 1_{\{t:|x_n(t)|>M\}}) > \delta$.
(d) For any $\varepsilon > 0$ there is n such that $\varrho_\phi(x_n 1_{\{t:|x_n(t)|<\varepsilon\}}) > \delta$.

By Proposition 11, $\{x_n\}$ does not contain any weakly covergent subsequence. By Lemma 6, (c) yields $u_0 = \infty$.

Suppose (d) holds. Since $\Lambda_{\phi,w}$ is order continuous, for any $\delta > 0$ there is $\varepsilon > 0$ such that $\varrho_\phi(\varepsilon 1_{(0,1)}) < \delta/2$. Hence, if $\varrho_\phi(x_n 1_{\{t:|x_n(t)|<\varepsilon\}}) > \delta$, then we must have $\Lambda_{\phi,w} \equiv \Lambda_{\phi,w}(0,\infty)$. Hence we may assume that neither (c) nor (d) holds.

Since $\int_0^{v_0} \phi(u_0) w < 2$, by Proposition 12, there exists $\nu > 0$ such that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$,

$$m(\{t : \sup\{x_{n_k} : k \in \mathbb{N}\}(t) - \inf\{x_{n_k} : k \in \mathbb{N}\}(t) > 3\nu\}) \geq v_0 + 3\nu. \tag{6}$$

The same assumption yields either $u_0 < \infty$ or $v_0 = 0$. By Lemma 6 and Corollary 9, both z' and z'' are finite almost everywhere. Applying Lemmas 13 and 14 and passing to further subsequences of $\{x_n\}$, we may assume that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$,

$$u_4 = p(x_n) = p(x_{n_k}),$$

and

$$\sup\{x_n\}(t) - \inf\{x_n\}(t) \le u_4. \tag{7}$$

If $u_4 = 0$, then $\{x_n\}$ contains a constant subsequence. This contradicts the fact that $\{x_n\}$ is a unit limit-constant sequence. So u_4 must be positive,

$$m(\{t : \sup\{x_n\}(t) - \inf\{x_n\}(t) > 15u_4/16\}) = \infty, \tag{8}$$

and $\Omega = (0,\infty)$. By Corollary 10, there is v such that w is constant on (v,∞). Let

$$v_1 = \inf\{v : w \text{ is constant on } (v,\infty)\}.$$

If $v_1 = 0$, then $v_0 = \infty$. This contradicts our assumption $\int_0^{v_0} \phi(u_0)w < 2$. So $v_1 \ge v_0$ and $v_1 > 0$.

By (8) and Lemma 8, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for any $j < m$,

$$m(\{t : |x_{n_j} - x_{n_m}| \ge 7u_4/8)\}) \ge v_1.$$

Replacing $\{x_k\}$ by $\{x_{n_k}\}$ if necessary, we may assume that for any $n > m$,

$$m(\{t : |x_n(t) - x_m(t)| \ge 7u_4/8)\}) \ge v_1. \tag{9}$$

Claim. *There are a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence of pairwise disjoint measurable sets $\{B_k\}$ such that $m(B_k) \ge 2v_1/3$, and for any $m \in \mathbb{N}$, $t \in B_m$,*

$$|x_{n_m}(t) - x_{n_k}(t)| \ge 3u_4/4 \quad \text{ if } k < m.$$

Suppose the claim were proved. By the proof of Example 3, $\{(x_{n_k} - x_{n_{k-1}})1_{B_k}\}$ is equivalent to the natural basis of ℓ_1. By Proposition 5, $\{x_n\}$ does not converge weakly. Hence we only need to prove our claim.

Proof of Claim. Let $n_1 = 2$. Suppose that $n_1, \ldots, n_k$ are selected. For any $l > n_k$ with $\varrho_\phi(x_l - (1/k)\sum_{j=1}^k x_{n_j}) > 1 - \lambda$, let σ be the measure preserving transformation such that

(i) $\int_0^\infty \phi(x_l - (1/k)\sum_{j=1}^k x_{n_j}) w \circ \sigma = \int_0^\infty \phi((x_l - (1/k)\sum_{j=1}^k x_{n_j})^*) w \ge 1 - \lambda$;

(ii) if $|(x_l - (1/k)\sum_{j=1}^k x_{n_j})(t)| \ge |x_l - (1/k)\sum_{j=1}^k x_{n_j})(s)|$, then $\sigma(t) \ge \sigma(s)$.

If $m(\{t \in \sigma^{-1}((0,v_1)) : |x_l(t) - x_{n_j}(t)| \le 3u_4/4\}) \ge v_1/4^j$ for some $j \le k$, then by (9), we have

$$m(\{t \in \sigma^{-1}(v_1,\infty) : |x_l(t) - x_{n_j}(t)| \ge 7u_4/8\}) \ge v_1/4^j.$$

By Lemma 2 and Remark 1, there is $\delta_3 > 0$ independent of σ such that

$$\int_0^\infty \phi\Big(x_l - \frac{1}{k}\sum_{j=1}^k x_{n_j}\Big) w \circ \sigma \le 1 - \delta_3.$$

Since $\Lambda_{\phi,w}$ is order continuous and $\{x_n\}$ is a unit limit-constant sequence, there is $n_{k+1} > n_k$ such that

$$\int_0^\infty \phi\Big(x_l - \frac{1}{k}\sum_{j=1}^k x_{n_j}\Big) w \circ \sigma \ge 1 - \frac{\delta_3}{2}.$$

The above proof shows that for any $j \le k$,

$$m(\{t \in \sigma^{-1}(0, v_1) : |x_{n_{k+1}}(t) - x_{n_j}(t)| \le 3u_4/4\}) \le v_1/4^j.$$

Let

$$B_{k+1} = \{t \in \sigma^{-1}(0, v_1) : \text{ for any } j \le k,\ |x_l(t) - x_{n_j}(t)| > 3u_4/4\}.$$

Then

$$m(B_{k+1}) \ge v_1 - \sum_{j=1}^k \frac{v_1}{4^j} \ge \frac{2v_1}{3}.$$

Let t be an element in B_k and i, j two natural numbers such that $i < j < k$. Then

$$(1 + 1/16)u_4 \ge |x_{n_k}(t) - x_{n_j}(t)| > 3u_4/4 \quad \text{by (7)}.$$

If $(x_{n_k}(t) - x_{n_j}(t))(x_{n_k}(t) - x_{n_i}(t)) < 0$, then

$$\begin{aligned}|x_{n_i}(t) - x_{n_j}(t)| &= |x_{n_k}(t) - x_{n_j}(t)| + |x_{n_k}(t) - x_{n_i}(t)| \\ &\ge 2\frac{3u_4}{4} = \frac{3u_4}{2}.\end{aligned}$$

This is impossible. So for almost all $t \in B_{k+1}$ and for $i < j < k$, $\mathrm{sgn}(x_{n_k}(t) - x_{n_j}(t)) = \mathrm{sgn}(x_{n_k}(t) - x_{n_i}(t))$, and

$$|x_{n_i}(t) - x_{n_j}(t)| \le u_4/4.$$

This implies $t \notin B_j$ and $\{B_k\}$ is pairwise disjoint. We proved our claim, and hence also Theorem 1. ■

Remark 3. (1) The results in Sections 2 and 3 are still true for Lorentz–Orlicz sequence spaces $\ell_{\phi,w}$. Hence if $\ell_{\phi,w}$ is an order continuous Lorentz–Orlicz sequence space (i.e. ϕ satisfies the Δ_2 condition for small values and $\sum_{i=1}^\infty w(i) = \infty$), then $\ell_{\phi,w}$ has normal structure if and only if $u_0 = 0$.

(2) Let $\{x_n\}$ be a limit-constant sequence in an order continuous Lorentz–Orlicz sequence space $\ell_{\phi,w}$. We claim that $\{x_n\}$ does not converge weakly. By passing to a subsequence and then translating it, we may assume that for any $n > m$,

$$\||x_n| \wedge |x_m|\|_\infty \le 1/n.$$

If for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\|x_n\|_\infty < \varepsilon$, then by Proposition 11, $\{x_n\}$ does not converge weakly. In this case, we are done. So we may assume that there is N and $\varepsilon > 0$ such that $\|x_n\|_\infty \geq \varepsilon$ for all $n > N$. By Corollary 10, there is $v \geq 0$ such that w is constant on (v, ∞). By Proposition 5 (cf. Example 3), $\{x_n\}$ does not converge weakly. Hence every order continuous Lorentz–Orlicz sequence space $\ell_{\phi,w}$ has weakly normal structure.

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