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Invariant Hodge forms and equivariant splittings of algebraic manifolds

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Abstract. Let T be a complex torus acting holomorphically on a compact algebraic manifold M and let $\mathrm{ev}_* : \pi_1(T,1) \to \pi_1(M,x_0)$ be the homomorphism induced by $\mathrm{ev} : T \ni t \mapsto tx_0 \in M$. We show that for each T-invariant Hodge form Ω on M there is a holomorphic fibration $p : M \to T$ whose fibers are Ω-perpendicular to the orbits. Using this we prove that M is T-equivariantly biholomorphic to $T \times M/T$ if and only if there is a subgroup Δ of $\pi_1(M)$ and a Hodge form Ω on M such that $\pi_1(M) = \mathrm{im}\,\mathrm{ev}_* \times \Delta$ and $\int_{\beta\times\delta} \Omega = 0$ for all $\beta \in \mathrm{im}\,\mathrm{ev}_*$ and $\delta \in \Delta$.

Let $T = \mathbb{C}^n/B$ be a complex torus acting holomorphically and effectively on a closed complex algebraic manifold M. It has been shown in [2] that, if we take an appropriate action of T on T, then there is an equivariant fibration $p : M \to T$ having fibers transversal to the orbits. In particular, a finite covering space $\widehat{M}$ of M is equivariantly biholomorphic to $T \times \widehat{M}/T$. In this paper we consider a more refined variant of this result. Applying the main results of [6] we show that for each T-invariant Hodge form Ω on M there is a T-equivariant fibration $p : M \to T$ whose fibers are Ω-perpendicular to the orbits. We show that the structure group of p depends only on the appropriate periods of Ω. Using this we describe when M is T-equivariantly biholomorphic to $T \times M/T$.

Before stating the results of the paper we need some definitions. A smooth fibration $p : M \to T = \mathbb{C}^n/B$ is a *t-e fibration* if the fibers of p are transversal to the orbits and $x \mapsto p(tx)p(x)^{-1}$ depends on $t \in T$ only (cf. [6, p. 216]). For every $b \in B$ let b^π denote the corresponding element of $\pi_1(T)$. Fix a basis $b_1, \ldots, b_{2n}$ in B. Let $\mathrm{ev}_* : \pi_1(T,1) \to \pi_1(M,x_0)$ be the homomorphism induced by $T \ni t \mapsto tx_0 \in M$ and let β_j, $j = 1, \ldots, 2n$, be the image of $\mathrm{ev}_*(b_j^\pi)$ in $H_1(M,\mathbb{Z})$. Then we have the following.

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THEOREM 1. *Let $T = \mathbb{C}^n/B$ be a complex torus acting holomorphically on a closed algebraic manifold M and let Ω be a T-invariant Hodge form on M. Then there is a holomorphic t-e fibration $p : M \to T$ whose fibers are Ω-perpendicular to the orbits of the action of T. The structure group of p can be reduced to $\mathbb{Z}_a^{2n}$, where $a = |\det[\int_{\beta_i\times\beta_j}\Omega]|$.*

THEOREM 2. *Let M and T be as in Theorem 1. The following conditions are equivalent:*

(a) *M is T-equivariantly biholomorphic to $T \times M/T$,*

(b) *there is a subgroup Δ of $\pi_1(M)$ and a Hodge form Ω on M such that*

$$\pi_1(M) = \operatorname{im}\operatorname{ev}_* \times \Delta \quad \textit{and} \quad \int\limits_{\beta\times\delta}\Omega = 0 \textit{ for all } \beta \in \operatorname{im}\operatorname{ev}_* \textit{ and } \delta \in \Delta.$$

The following notation will be used in the proofs of Theorems 1 and 2. By $\varphi_u : M \to M$, $u \in \mathbb{C}^n$, we shall denote the action of $\mathbb{C}^n$ determined by the action of $T = \mathbb{C}^n/B$ on M. Let x_0 be a base point of M. For every $j = 1, \ldots, 2n$, $\varphi_{tb_j} : M \to M$, $t \in [0, 1]$, is an S^1 action on M. Let $c_j(t) = \varphi_{tb_j}(x_0)$, $t \in [0, 1]$, and let X_j be the vector field on M determined by $t \mapsto \varphi_{tb_j}$. It is easy to see that c_j belongs to β_j and $\frac{dc_j}{dt}(t) = X_j(c(t))$.

We start with the following:

LEMMA 1. *Let $T = \mathbb{C}^n/B$, M, Ω, $\beta_1, \ldots, \beta_{2n}$ be as in Theorem 1, $\gamma \in \pi_1(M)$, $i, j \in \{1, \ldots, 2n\}$, and $\eta_j : TM \ni v \mapsto \Omega(v, X_j) \in \mathbb{C}$. Then*

(a) $d\eta_j = 0$,

(b) $\int_\gamma \eta_j \in \mathbb{Z}$,

(c) $\int_{\beta_i}\eta_j = \int_{\beta_i\times\beta_j}\Omega = \Omega(X_i(x_0), X_j(x_0))$,

(d) $\det[\int_{\beta_i\times\beta_j}\Omega] \neq 0$.

Proof. (a) We have

$$i_{X_j}\Omega(v) = \Omega(X_j, v) = -\Omega(v, X_j) = -\eta_j(v).$$

By the T-invariance of Ω,

$$di_{X_j}\Omega + i_{X_j}d\Omega = L_{X_j}\Omega = 0$$

so that $d\eta_j = -di_{X_j}\Omega = 0$.

(b) Let $c : [0, 1] \to M$ be a smooth singular simplex representing the image of γ in $H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$. The formula $f : [0, 1]^2 \ni (t, s) \mapsto \varphi_{sb_j}(c(t)) \in M$ defines a singular cube on M. It is easy to see that f is a cycle and

$$\frac{\partial f}{\partial t}(t, 0) = \frac{dc}{dt}(t), \quad \frac{\partial f}{\partial s}(t, s) = X_j(f(t, s)).$$

Using this and the T-invariance of Ω we have

$$\int_0^1 \Omega\left(\frac{\partial f}{\partial t}(t,s), \frac{\partial f}{\partial s}(t,s)\right) ds = \Omega\left(\frac{dc}{dt}(t), X_j(c(t))\right),$$

so that

$$\begin{aligned}\int_c \eta_j &= \int_0^1 \Omega\left(\frac{dc}{dt}(t), X_j(c(t))\right) dt \\ &= \int_0^1\int_0^1 \Omega\left(\frac{\partial f}{\partial t}(t,s), \frac{\partial f}{\partial s}(t,s)\right) ds\, dt = \int_f \Omega \in \mathbb{Z}.\end{aligned}$$

(c) By the T-invariance of Ω, X_i and X_j we have

$$\begin{aligned}\int_{\beta_i} \eta_j = \int_{c_i} \eta_j &= \int_0^1 \eta_j\left(\frac{dc_i}{dt}(t)\right) dt = \int_0^1 \eta_j(X_i(c_i(t))\, dt \\ &= \int_0^1 \Omega(X_i(c_i(t)), X_j(c_i(t)))\, dt = \Omega(X_i(x_0), X_j(x_0)).\end{aligned}$$

Using arguments similar to those given in the proof of (b) it is easy to verify that

$$\int_{\beta_i \times \beta_j} \Omega = \Omega(X_i(x_0), X_j(x_0)).$$

Thus

$$\int_{c_i} \eta_j = \int_{\beta_i \times \beta_j} \Omega = \Omega(X_i(x_0), X_j(x_0)).$$

(d) We have $\dim_{\mathbb{C}} T(x_0) = \dim_{\mathbb{C}} T = n$, because every holomorphic, effective action of T on a closed Kähler manifold is almost free (see [2] and Remark 2 below). Let Ω_T denote the restriction of Ω to $T(x_0)$. Then Ω_T is a Hodge form on $T(x_0)$ so that $\Lambda^n \Omega_T$ is a volume form on $T(x_0)$. Since $X_1(x_0), \ldots, X_{2n}(x_0)$ is a basis of $TT(x_0)_{x_0}$ we have

$$\det\Big[\int_{\beta_i \times \beta_j} \Omega\Big] = \det[\Omega(X_i(x_0), X_j(x_0))] \neq 0.$$

This completes the proof of Lemma 1. ■

Let TO be the set of all vectors $v \in TM$ that are tangent to the orbits of the action of T. As T acts almost freely, TO is a complex vector bundle. Let $TO^{\perp} = \{v \in TM : \forall_{w \in TO}\ \Omega(v,w) = 0\}$. Since $\Omega(Jv, Jw) = \Omega(v,w)$ and $\Omega(Jv, w) = -\Omega(v, Jw)$ we have

$$TO^{\perp} = \{v \in TM : \forall_{w \in TO}\ \Omega(v, Jw) = 0\}.$$

Using this it is easy to see that $TO^\perp$ is a complex vector bundle and $TM = TO \oplus TO^\perp$.

Let $v \in TM$. Take $v_0 \in TO$ and $v_F \in TO^\perp$ such that $v = v_0 + v_F$. Let $E(v)$ be the invariant vector field on T such that v_0 is tangent to the orbit of the one-parameter subgroup generated by $E(v)$. For every $u \in \mathbb{C}^n$ let $I^{-1}(u) \in L(T)$ be the invariant vector field on T such that u is tangent to the one-parameter subgroup generated by $I^{-1}(u)$. Consider the 1-form

$$\omega : TM \ni v \mapsto (I \circ E)(v) \in \mathbb{C}^n.$$

Applying Lemma 1 we show the following:

Lemma 2. *Let* M, T, B, Ω, $X_1, \ldots, X_{2n}$, $\beta_1, \ldots, \beta_{2n}$ *be as above*, $\gamma \in \pi_1(M)$, $j \in \{1, \ldots, 2n\}$, $a_{ij} = \Omega(X_i, X_j)$, $[b_{ij}] = [a_{ij}]^{-1}$, *and let* $a = |\det[a_{ij}]|$. *Then*

(a) $\omega = \sum_{i=1}^{2n} \omega_i b_i$, *where* $\omega_i(v) = \sum_{j=1}^{2n} b_{ji}\Omega(v, X_j)$,
(b) ω *is a holomorphic* 1-*form*,
(c) $\int_\gamma a\omega \in B$,
(d) $\int_{\beta_j} \omega = b_j$.

Proof. (a) Take $x \in M$, $v \in TM_x$ and $c_1, \ldots, c_{2n} \in \mathbb{C}$ such that $v_0 = \sum_{j=1}^{2n} c_j X_j(x)$. Then

$$\Omega(v, X_i) = \Omega(v_0, X_i) = \sum_{j=1}^{2n} c_j \Omega(X_j, X_i) = \sum_{j=1}^{2n} c_j a_{ji}$$

so that

$$c_i = \sum_{j=1}^{2n} b_{ji}\Omega(v, X_j) = \omega_i(v).$$

Since $(I \circ E)(X_i(x)) = b_i$ we have

$$\omega(v) = \sum_{i=1}^{2n} c_i b_i = \sum_{i=1}^{2n} \omega_i(v) b_i.$$

(b) By (a) and by Lemma 1,

$$d\omega = \sum_{i=1}^{2n} d\omega_i b_i = \sum_{i=1}^{2n}\sum_{j=1}^{2n} b_{ji} d\eta_j b_i = 0.$$

It is easy to see that $\omega \circ J = i\omega$. As ω is closed this implies that ω is holomorphic.

(c) By Lemma 1(b), $\int_\gamma \eta_i \in \mathbb{Z}$. Since

$$ab_{ij} = |\det[a_{pq}]| b_{ij} \in \mathbb{Z}$$

we have

$$\int_\gamma a\omega_i = \sum_{j=1}^{2n} ab_{ji} \int_\gamma \eta_j \in \mathbb{Z}$$

so that

$$\int_\gamma a\omega = \sum_{i=1}^{2n} \int_\gamma a\omega_i \, b_i \in B.$$

(d) By the definition of ω, $\omega(X_j(c_j(t))) = b_j$. As $c_j : [0,1] \ni t \mapsto \varphi_{tb_j}(x_0)$ belongs to β_j,

$$\int_{\beta_j} \omega = \int_0^1 \omega\left(\frac{dc_j}{dt}(t)\right) dt = \int_0^1 \omega(X_j(c_j(t)))\, dt = \int_0^1 b_j \, dt = b_j. \ \blacksquare$$

Proof of Theorem 1. By Lemma 2 and by [6, Lemma 1] the formula $p(x) = \int_{x_0}^x a\omega \bmod B$ defines a well defined holomorphic map $p : M \to T$. Note that

$$(p_* \circ \mathrm{ev}_*)(b_j^\pi) = \Big(\int_{\beta_j} a\omega\Big)^\pi = ab_j^\pi$$

(compare [5, Lemma 1.2]). It is easy to see that ω is T-invariant. By [6, §1], p is a holomorphic t-e fibration.

Let $\mathcal{F}(p)$ be the foliation of M whose leaves are connected components of the fibers of p, let L be a leaf of $\mathcal{F}(p)$ containing the base point x_0, let $x \in L$, and let $v \in TL_x$. Take a smooth path $c : [0,1] \to M$ joining x_0 to x such that $c'(1) = v$. Then

$$\int_0^t \omega(c'(\tau))\, d\tau = \int_{x_0}^{c(t)} \omega = p(c(t)) = 0 \bmod B$$

so that $\omega(c'(t)) = 0$ for $t \in [0,1]$. In particular, $E(v) = 0$ and $v \in TO^\perp$. As p is a t-e fibration each element of T carries the leaves of p onto the leaves of p. Using this it is easy to see that the fibers of p are Ω-perpendicular to the orbits. Since $\pi_1(T)/\mathrm{im}(p \circ \mathrm{ev})_* \cong \mathbb{Z}_a^{2n}$, the structure group of p can be reduced to $\mathbb{Z}_a^{2n}$ (see [5, Proposition 2.1]).

Proof of Theorem 2. (a)$\Rightarrow$(b). Fix a biholomorphic equivariant map $M \to T \times F$. Let Ω_T and Ω_F be $(1,1)$-forms on M induced by some Hodge forms on T and F respectively. Then $\Omega = \Omega_T + \Omega_F$ is a Hodge form on M. It is clear that $\Omega(v,w) = 0$ for $v \in TT_x$, $w \in TF_x$, $x \in M$. Let $\beta_i \in H_1(T,\mathbb{Z})$, $i \in \{1,\ldots,2n\}$, be as in Theorem 1, and let $\delta \in H_1(F,\mathbb{Z})$. Arguments similar to those given in the proof of Lemma 1(b) show that $\int_{\beta_i \times \delta} \Omega = 0$.

(b)⇒(a). Averaging Ω we can assume that Ω is T-invariant. Let ω and ω_i be as in the proof of Theorem 1 and let Δ_H be the image of Δ in $H_1(M,\mathbb{Z})$. If $u \in H_1(M,\mathbb{Z})$ then $u = \sum_{j=1}^{2n} k_j\beta_j + u_F$ for some $k_1,\ldots,k_{2n} \in \mathbb{Z}$ and $u_F \in \Delta_H$. Clearly $\int_{u_F}\omega_i = 0$ so that

$$\int_u \omega_i = \sum_{j=1}^{2n} k_j \int_{\beta_j} \omega_i \in \mathbb{Z}$$

and accordingly $\int_u \omega \in B$. The arguments given in the proof of Theorem 1 show that

$$q : M \ni x \mapsto \int_{x_0}^{x} \omega \bmod B \in \mathbb{C}^n/B = T$$

is a well defined holomorphic t-e fibration. By Lemma 2, $(q\circ \mathrm{ev})_*(b_i^\pi) = (\int_{\beta_i}\omega)^\pi = b_i^\pi$ (cf. [5, Lemma 1.2]). Hence $q_*\circ \mathrm{ev}_*$ is an epimorphism and q is a trivial fibration. This completes the proof of Theorem 2. ■

Remark 1. (a) Natural examples of holomorphic toral actions on algebraic manifolds and the arising equivariant splittings were discussed in [1, 3].

(b) The theorem of complete reducibility of Poincaré (see e.g. [4, §19, Theorem 1]) is a particular case of Theorem 1. To see this let T_0 be an abelian variety and let $T = \mathbb{C}^n/B$ be a complex torus contained in T_0. Then T acts holomorphically (and freely) on T_0. By Theorem 1 there is a holomorphic t-e fibration $p : T_0 \to T$ associated with a T_0-invariant Hodge form Ω on T_0. Any connected component L of a fiber of p is a leaf of a T-invariant foliation $F(p)$. Since the leaves of $F(p)$ are covered by complex hyperplanes, L is a complex torus. It is easy to see that $\varepsilon : T\times L \ni (t,u) \mapsto tu \in T_0$ is an epimorphism with a finite kernel.

Remark 2. Using the arguments similar to those given in the proof of Lemma 1 it is possible to give a simple proof of the almost freeness of an effective holomorphic action of a complex torus on a closed Kähler manifold M. It goes as follows. Let $T = \mathbb{C}^n/B$ be a complex torus acting on M, let $b \in B$, and let $\varphi_{tb} : M \to M$, $t \in [0,1]$, be an S^1 action embedded in the action of T. Let X be the vector field on M determined by the S^1 action, let c_b be any nontrivial orbit of the action of φ_{tb}, $t \in [0,1]$, and let Ω be any invariant Kähler form on M. The formula $\eta(v) = -\frac{1}{i}\Omega(v, JX) = \frac{1}{i}\Omega(Jv, X)$ defines a closed (see Lemma 1) 1-form on M. For every $v \neq 0$ we have $\frac{1}{i}\Omega(Jv, v) > 0$ so that

$$\int_{c_b} \eta = \frac{1}{i}\int_0^1 \Omega(JX(c(t)), X(c(t)))\,dt > 0.$$

In particular, the isotropy group of every orbit of the S^1 action is discrete.

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