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ON LOCALLY BOUNDED CATEGORIES STABLY EQUIVALENT TO THE REPETITIVE ALGEBRAS OF TUBULAR ALGEBRAS

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1. Introduction. Throughout the paper K is a fixed algebraically closed field. By an algebra we mean a finite-dimensional K-algebra, which we shall assume, without loss of generality, to be basic and connected. For an algebra A, we shall denote by $\mathrm{mod}(A)$ the category of finitely generated right A-modules, and by $\underline{\mathrm{mod}}(A)$ the stable category of $\mathrm{mod}(A)$. Recall that the objects of $\underline{\mathrm{mod}}(A)$ are the objects of $\mathrm{mod}(A)$ without projective direct summands, and for any two objects X, Y in $\underline{\mathrm{mod}}(A)$ the space of morphisms from X to Y in $\underline{\mathrm{mod}}(A)$ is $\underline{\mathrm{Hom}}_A(X,Y) = \mathrm{Hom}_A(X,Y)/\mathcal{P}(X,Y)$, where $\mathcal{P}(X,Y)$ is the subspace of $\mathrm{Hom}_A(X,Y)$ consisting of the A-homomorphisms which factorize through projective A-modules. For every $f \in \mathrm{Hom}_A(X,Y)$ we shall denote by $\underline{f}$ its coset modulo $\mathcal{P}(X,Y)$. Two algebras A and B are said to be *stably equivalent* if their stable module categories $\underline{\mathrm{mod}}(A)$ and $\underline{\mathrm{mod}}(B)$ are equivalent.

Following [5, 11] we shall say that a module T in $\mathrm{mod}(A)$ is a *tilting* (respectively, *cotilting*) module if it satisfies the following conditions:

(1) $\mathrm{Ext}^2_A(T,-) = 0$ (respectively, $\mathrm{Ext}^2_A(-,T) = 0$);
(2) $\mathrm{Ext}^1_A(T,T) = 0$;
(3) the number of nonisomorphic indecomposable summands of T equals the rank of the Grothendieck group $K_0(A)$.

Two algebras A and B are said to be *tilting-cotilting equivalent* if there exist a sequence of algebras $A = A_0, A_1, \ldots, A_m, A_{m+1} = B$ and a sequence of modules $T^i_{A_i}$, $0 \le i \le m$, such that $A_{i+1} = \mathrm{End}_{A_i}(T^i)$ and T^i is either a tilting or a cotilting module.

Following Gabriel [9], a K-category R is called *locally bounded* if the following conditions are satisfied:

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(a) different objects are not isomorphic;

(b) the algebra $R(x,x)$ of endomorphisms of x is local for every object x in R;

(c) $\sum_{y\in R}\dim_K R(x,y) < \infty$ and $\sum_{y\in R}\dim_K R(y,x) < \infty$ for every object x in R.

Interesting examples of locally bounded K-categories are the repetitive algebras introduced by Hughes and Waschbüsch in [12]. For an algebra A denote by $D = \mathrm{Hom}_K(-,K)$ the standard duality on $\mathrm{mod}(A)$. Recall that the *repetitive algebra* $\widehat{A}$ of A is the selfinjective, locally finite-dimensional matrix algebra without identity defined by

$$\widehat{A} = \begin{bmatrix} \cdot & & & & & & & 0 \\ \cdot & \cdot & & & & & & \\ & \cdot & \cdot & & & & & \\ & & \cdot & A_{i-1} & & & & \\ & & & E_{i-1} & A_i & & & \\ & & & & E_i & A_{i+1} & & \\ & & & & & \cdot & \cdot & \\ & & & & & & \cdot & \cdot \\ 0 & & & & & & \cdot & \cdot \end{bmatrix}$$

where matrices have only finitely many nonzero entries, $A_i = A$, $E_i = {}_ADA_A$ for all integers i, all the remaining coefficients are zero, and the multiplication is induced from the canonical bimodule structure of DA and the zero morphism $DA \otimes_A DA \to 0$.

One of the interesting problems concerning repetitive algebras is a classification of locally bounded K-categories which are stably equivalent to a given repetitive algebra. The problem was studied by several authors (see [1, 2, 14, 20, 21]). Wakamatsu proved in [21] that if A is tilting-cotilting equivalent to B then $\widehat{A}$ is stably equivalent to $\widehat{B}$. Peng and Xiao proved in [14] that if H is a hereditary algebra and Λ is a locally bounded K-category which is stably equivalent to $\widehat{H}$, then there is an algebra B tilting-cotilting equivalent to H such that $\widehat{B} \cong \Lambda$. We shall prove the following theorem on locally bounded K-categories stably equivalent to the repetitive algebras of tubular algebras in the sense of Ringel [18].

THEOREM. *Let A be a tubular algebra. A locally bounded K-category Λ is stably equivalent to $\widehat{A}$ if and only if Λ is isomorphic to the repetitive algebra $\widehat{B}$ of a tubular algebra B which is tilting-cotilting equivalent to A.*

Our proof of the above result rests heavily on the main results obtained in [15, 16] for trivial extension algebras. In the case when Λ is a repetitive algebra the above theorem has been proved in [2].

We shall use freely results about Auslander–Reiten sequences which can be found in [3].

2. Preliminaries

2.1. Following Ringel [18], the canonical tubular algebras of type $(2,2,2,2)$ are defined by the quiver

$$\begin{tikzcd}
 & \bullet \arrow[ddr, "\alpha_2"] & \\
 & \bullet \arrow[dr, "\beta_2"] & \\
\bullet \arrow[uur, "\alpha_2"] \arrow[ur, "\beta_2"] \arrow[dr, "\gamma_2"] \arrow[ddr, "\delta_2"'] & & \bullet \\
 & \bullet \arrow[ur, "\gamma_2"] & \\
 & \bullet \arrow[uur, "\delta_2"'] &
\end{tikzcd}$$

with the relations $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0$ and $\alpha_1\alpha_2+k\beta_1\beta_2+\delta_1\delta_2=0$, where k is some fixed element from $K\setminus\{0,1\}$. The canonical tubular algebras of type $(\mathbf{p},\mathbf{q},\mathbf{r})=(\mathbf{3},\mathbf{3},\mathbf{3})$, $(\mathbf{2},\mathbf{4},\mathbf{4})$ or $(\mathbf{2},\mathbf{3},\mathbf{6})$ are given by the quiver

$$\begin{tikzcd}
 & \bullet \arrow[r, "\alpha_2"] & \bullet \arrow[r, dash, dotted] & \bullet \arrow[dr, "\alpha_p"] & \\
\bullet \arrow[ur, "\alpha_1"] \arrow[r, "\beta_1"] \arrow[dr, "\gamma_1"'] & \bullet \arrow[r, "\beta_2"] & \bullet \arrow[r, dash, dotted] & \bullet \arrow[r, "\beta_q"] & \bullet \\
 & \bullet \arrow[r, "\gamma_2"] & \bullet \arrow[r, dash, dotted] & \bullet \arrow[ur, "\gamma_r"'] &
\end{tikzcd}$$

with $\alpha_1\alpha_2\ldots\alpha_p+\beta_1\beta_2\ldots\beta_q+\gamma_1\gamma_2\ldots\gamma_r=0$.

2.2. For the repetitive algebra $\widehat{A}$ the identity morphisms $A_i\to A_{i-1}$, $E_i\to E_{i-1}$ induce an automorphism ν_A of $\widehat{A}$ which is called the *Nakayama automorphism*. Moreover, the orbit space $\widehat{A}/(\nu_A)$ has the structure of a finite-dimensional K-algebra which is isomorphic to the trivial extension $T(A)$ of A by its minimal injective cogenerator bimodule ${}_ADA_A$. This is the algebra whose additive structure coincides with that of the group $A\oplus DA$, and whose multiplication is defined by the formula $(a,f)(b,g)=(ab,ag+fb)$ for $a,b\in A$, $f,g\in {}_ADA_A$. Thus $\widehat{A}$ is a Galois cover in the sense of [9] of the selfinjective algebra $T(A)$ with the infinite cyclic group (ν_A) generated by ν_A.

2.3. A locally bounded K-category R is said to be *locally support-finite* [6] if for every indecomposable projective R-module P, the set of isomorphism classes of indecomposable projective R-modules P' such that there exists an indecomposable finite-dimensional R-module M with $\mathrm{Hom}_R(P,M)\neq$

$0 \neq \mathrm{Hom}_R(P', M)$ is finite. Of particular interest is the fact that the repetitive algebra $\widehat{A}$ of a tubular algebra A is locally support-finite (see [13]). A locally bounded K-category is said to be *triangular* if its ordinary quiver has no oriented cycles.

2.4. Following Gabriel (see [9]), for a locally bounded K-category R and a torsion-free group G of K-automorphisms of R acting freely on the objects of R, R/G is the quotient category whose objects are the G-orbits of the objects of R. Moreover, there is a covering functor $F : R \to R/G$ which maps any object x of R to its G-orbit $G \cdot x$. F induces the push-down functor $F_\lambda : \mathrm{mod}(R) \to \mathrm{mod}(R/G)$, which preserves indecomposables and Auslander–Reiten sequences, maps projective R-modules to projective R/G-modules and preserves projective resolutions. Furthermore, if R is locally support-finite then F_λ is dense and induces a bijection between the set $(\mathrm{ind}(R)/\cong)/G$ of the G-orbits of the isomorphism classes of finite-dimensional indecomposable R-modules and the set $\mathrm{ind}(R/G)/\cong$ of the isomorphism classes of finite-dimensional indecomposable R/G-modules [6].

2.5. Let $\Omega_R : \underline{\mathrm{mod}}(R) \to \underline{\mathrm{mod}}(R)$ be Heller's loop-space functor for a selfinjective locally bounded K-category R. Then $\Omega_R \tau_R^{-1} \Omega_R(S)$ is simple for every simple R-module S, where τ_R^{-1} stands for the Auslander–Reiten translate $\mathrm{Tr}D$ on $\underline{\mathrm{mod}}(R)$. Thus we obtain a permutation of the isomorphism classes of the simple R-modules. This permutation induces a K-automorphism ν_R of R in an obvious way. We denote by (ν_R) the infinite cyclic group of K-automorphisms of R generated by ν_R.

3. Preparatory results

3.1. Throughout this section we shall assume that R_1 and R_2 are selfinjective locally bounded K-categories which are locally support-finite and have no indecomposable projective modules of length 2. Moreover, there is a fixed equivalence functor $\Phi : \underline{\mathrm{mod}}(R_1) \to \underline{\mathrm{mod}}(R_2)$.

3.2. PROPOSITION. *If M is an indecomposable nonprojective finite-dimensional R_1-module then $\Phi(\tau_{R_1}(M)) \cong \tau_{R_2}(\Phi(M))$ and $\Phi(\Omega_{R_1}(M)) \cong \Omega_{R_2}(\Phi(M))$.*

Proof. A direct adaptation of the arguments from the proofs of Proposition 2.4 and Theorem 4.4 of [4].

3.3. LEMMA. *If $\tau_{R_1}^{-1}(M) \not\cong \Omega_{R_1}^{-2}(M)$ for every indecomposable nonprojective finite-dimensional R_1-module M then (ν_{R_2}) acts freely on the objects of R_2.*

Proof. We have to show that $\Omega_{R_2} \tau_{R_2}^{-1} \Omega_{R_2}(S) \not\cong S$ for every simple R_2-module S. Suppose to the contrary that there exists a simple R_2-module S

with $\Omega_{R_2}\tau_{R_2}^{-1}\Omega_{R_2}(S) \cong S$. Then there exists a nonprojective indecomposable finite-dimensional R_1-module M such that $\Phi(M) \cong S$, and we infer by Proposition 3.2 that $\Omega_{R_1}\tau_{R_1}^{-1}\Omega_{R_1}(M) \cong M$, which contradicts our assumption, because this isomorphism implies $\tau_{R_1}^{-1}(M) \cong \Omega_{R_1}^{-2}(M)$.

3.4. LEMMA. *Let $F_1 : \mathrm{mod}(R_1) \to \mathrm{mod}(R_1)$ and $F_2 : \mathrm{mod}(R_2) \to \mathrm{mod}(R_2)$ be exact equivalences satisfying the following conditions*:

(a) *If $F_i^s : \underline{\mathrm{mod}}(R_i) \to \underline{\mathrm{mod}}(R_i)$, $i = 1, 2$, are defined by $F_i^s(X) = F_i(X)$ for $X \in \underline{\mathrm{mod}}(R_i)$, $F_i^s(\underline{f}) = \underline{F_i(f)}$ for $f : X \to Y$ in $\underline{\mathrm{mod}}(R_i)$, then F_i^s are well-defined functors which are equivalences.*

(b) *For every object $X \in \underline{\mathrm{mod}}(R_1)$, $F_1^s(X) \cong \Phi^{-1}F_2^s\Phi(X)$, where Φ^{-1} is a fixed quasi-inverse of Φ.*

Then F_1^s and $\Phi^{-1}F_2^s\Phi$ are isomorphic functors.

Proof. In the first step of the proof we show that for every short exact sequence

$$0 \to U \xrightarrow{w} X \xrightarrow{p} V \to 0$$

in $\mathrm{mod}(R_1)$ with all terms without projective direct summands there are $w' : \Phi^{-1}F_2^s\Phi(U) \to \Phi^{-1}F_2^s\Phi(X)$ and $p' : \Phi^{-1}F_2^s\Phi(X) \to \Phi^{-1}F_2^s\Phi(V)$ such that the following sequences are exact in $\mathrm{mod}(R_1)$:

$$0 \to F_1^s(U) \xrightarrow{F_1(w)} F_1^s(X) \xrightarrow{F_1(p)} F_1^s(V) \to 0,$$
$$0 \to \Phi^{-1}F_2^s\Phi(U) \xrightarrow{w'} \Phi^{-1}F_2^s\Phi(X) \xrightarrow{p'} \Phi^{-1}F_2^s\Phi(V) \to 0,$$

where $\underline{w}' = \Phi^{-1}F_2^s\Phi(\underline{w})$ and $\underline{p}' = \Phi^{-1}F_2^s\Phi(\underline{p})$. The exactness of the first sequence is obvious by the definition of F_1^s, because F_1 is exact.

In order to show the exactness of the second, we first show that w' is a monomorphism, where w' is any representative of the coset $\Phi^{-1}F_2^s\Phi(\underline{w})$. Suppose to the contrary that w' is not a monomorphism. Then $w' = w_2'w_1'$ with $w_1' : \Phi^{-1}F_2^s\Phi(U) \to \mathrm{im}(w')$ an epimorphism and $w_2' : \mathrm{im}(w') \to \Phi^{-1}F_2^s\Phi(X)$ a monomorphism. Since w is a monomorphism, we infer by [17; Lecture 3] that $\underline{w} \neq 0$. Thus $\underline{w}' = \underline{w}_2'\underline{w}_1' \neq 0$ and there are $W \in \underline{\mathrm{mod}}(R_1)$ and $w_1 : U \to W$, $w_2 : W \to X$ such that $\Phi^{-1}F_2^s\Phi(\underline{w_i}) = \underline{w}_i'$, $i = 1, 2$, because $\Phi^{-1}F_2^s\Phi$ is an equivalence. Since w_1' is a proper epimorphism, we have the following inequality for lengths: $l(\mathrm{im}(w')) < l(\Phi^{-1}F_2^s\Phi(U))$. But F_1 is an additive exact equivalence, hence F_1 preserves the lengths of R_1-modules. Therefore F_1^s preserves the lengths of R_1-modules without projective direct summands and so does $\Phi^{-1}F_2^s\Phi$, because $F_1^s(M) \cong \Phi^{-1}F_2^s\Phi(M)$ for any $M \in \underline{\mathrm{mod}}(R_1)$ by the assumption of our lemma. Consequently, $l(W) = l(\mathrm{im}(w')) < l(U)$. But $w - w_2w_1$ factorizes through a projective R_1-module, say P. Thus there are $q_1 : U \to P$ and $q_2 : P \to X$ such that $w - w_2w_1 = q_2q_1$. Since w is a monomorphism, there is $q_1' : X \to P$ such that $q_1 = q_1' w$. Then $w - w_2 w_1 = q_2 q_1 = q_2 q_1' w$ and $w - q_2 q_1' w = w_2 w_1$. Hence $(\mathrm{id}_X - q_2 q_1')w = w_2 w_1$. But $(\mathrm{id}_X - q_2 q_1')w$ is a monomorphism, because $\mathrm{id}_X - q_2 q_1'$ is an isomorphism. Therefore we obtain a contradiction, because the monomorphism $(\mathrm{id}_X - q_2 q_1')w$ factorizes through the module W of length smaller than U. Consequently, w' is a monomorphism.

Dually one proves that p' is an epimorphism, where p' is any representative of the coset $\Phi^{-1} F_2^s \Phi(\underline{p})$.

Since $\Phi^{-1} F_2^s \Phi$ preserves the lengths of R_1-modules without projective direct summands, showing that $p'w' = 0$ is sufficient to show that the considered sequence is exact. Since $pw = 0$, we have $\underline{pw} = 0$. Thus $\underline{p'}\,\underline{w'} = 0$. Hence there are a projective R_1-module P and morphisms $q_1 : \Phi^{-1} F^s \Phi(U) \to P$ and $q_2 : P \to \Phi^{-1} F_2^s \Phi(V)$ such that $p'w' = q_2 q_1$. Since w' is a monomorphism and p' is an epimorphism, there are morphisms $q_2' : P \to \Phi^{-1} F_2^s \Phi(X)$ and $q_1' : \Phi^{-1} F_2^s \Phi(X) \to P$ such that $p'w' = q_2 q_1 = p' q_2' q_1' w'$. Then putting $w'' = (\mathrm{id}_X - q_2' q_1')w'$ we obtain $p'w'' = 0$ and $\underline{w''} = \underline{w'}$.

In the second step of the proof we show that there is an isomorphism $f : F_1^s \to \Phi^{-1} F_2^s \Phi$ given by a family $(f(X))_{X \in \underline{\mathrm{mod}}(R_1)}$ of isomorphisms in $\underline{\mathrm{mod}}(R_1)$ such that for every morphism $\underline{u} : X \to Y$ in $\underline{\mathrm{mod}}(R_1)$ the diagram

$$\begin{array}{ccc}
F_1^s(X) & \xrightarrow{f(X)} & \Phi^{-1} F_2^s \Phi(X) \\
{\scriptstyle F_1^s(\underline{u})} \downarrow & & \downarrow {\scriptstyle \Phi^{-1} F_2^s \Phi(\underline{u})} \\
F_1^s(Y) & \xrightarrow{f(Y)} & \Phi^{-1} F_2^s \Phi(Y)
\end{array}$$

commutes. We construct a family $(f(X))_{X \in \underline{\mathrm{mod}}(R_1)}$ such that for every $X \in \underline{\mathrm{mod}}(R_1)$ there is an isomorphism f_X in $\mathrm{mod}(R_1)$ with $\underline{f_X} = f(X)$ and such that for every short exact sequence

$$0 \to U \xrightarrow{w} X \xrightarrow{p} V \to 0$$

in $\mathrm{mod}(R_1)$ the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & F_1^s(U) & \xrightarrow{F_1(w)} & F_1^s(X) & \xrightarrow{F_1(p)} & F_1^s(V) & \to & 0 \\
 & & \downarrow {\scriptstyle f_U} & & \downarrow {\scriptstyle f_X} & & \downarrow {\scriptstyle f_V} & & \\
0 & \to & \Phi^{-1} F_2^s \Phi(U) & \xrightarrow{w'} & \Phi^{-1} F_2^s \Phi(X) & \xrightarrow{p'} & \Phi^{-1} F_2^s \Phi(V) & \to & 0
\end{array}$$

commutes, where w', p' are as in the first step of the proof. This condition is called the *commutativity condition* for f_X.

Our construction will run inductively on the length of X in $\mathrm{mod}(R_1)$. If $l(X) = 1$ then X is a simple R_1-module. Fix an isomorphism $\underline{f_X} = f(X) : F_1^s(X) \to \Phi^{-1} F_2^s \Phi(X)$. Let $\underline{u} : X \to X$ be a nonzero morphism. Since X is simple, u is an automorphism. Thus $\Phi^{-1} F_2^s \Phi(\underline{u}) = \underline{v}$, where v is an automorphism. But u is multiplication by $k_u \in K^* = K \setminus \{0\}$. Since

$$F_1^s(\underline{\mathrm{id}_X}) = \underline{\mathrm{id}_{F_1^s(X)}} \quad \text{and} \quad \Phi^{-1}F_2^s\Phi(\underline{\mathrm{id}_X}) = \underline{\mathrm{id}_{\Phi^{-1}F_2^s\Phi(X)}},$$

it follows that for $\underline{u} = \underline{\mathrm{id}_X} \cdot k_u$ we have

$$F_1^s(\underline{u}) = \underline{\mathrm{id}_{F_1^s(X)}} \cdot k_u \quad \text{and} \quad \Phi^{-1}F_2^s\Phi(\underline{\mathrm{id}_X} \cdot k_u) = \underline{\mathrm{id}_{\Phi^{-1}F_2^s\Phi(X)}} \cdot k_u.$$

Thus for any $f(X)$ we have $f(X)F_1^s(\underline{u}) = \Phi^{-1}F_2^s\Phi(\underline{u})f(X)$.

Now consider two isomorphic simple modules X, Y such that $X \neq Y$. For every isomorphism class $[X]$ of a simple R_1-module X fix a representative, say X. For every Y isomorphic to X fix an isomorphism $u_Y : X \to Y$. Then fix an isomorphism $f_X : F_1^s(X) \to \Phi^{-1}F_2^s\Phi(X)$, and for every $Y \in [X]$ define $f_Y : F_1^s(Y) \to \Phi^{-1}F_2^s\Phi(Y)$ by the formula

$$\underline{f_Y} = f(Y) = \Phi^{-1}F_2^s\Phi(\underline{u_Y})f(X)F_1^s(\underline{u_Y^{-1}}),$$

where f_Y is an arbitrary fixed representative of the coset $f(Y)$. If $\underline{u} : Z \to Y$ is an isomorphism with $Y, Z \in [X]$ then for $Z = X$ we have $u = u_Y \cdot k_u$ for some $k_u \in K^*$. Thus $F_1^s(\underline{u}) = F_1^s(\underline{u_Y}) \cdot k_u$ and $\Phi^{-1}F_2^s\Phi(\underline{u}) = \Phi^{-1}F_2^s\Phi(\underline{u_Y}) \cdot k_u$. Therefore $f(Y) = \Phi^{-1}F_2^s\Phi(\underline{u_Y})f(X)F_1^s(\underline{u_Y^{-1}})$, which implies that

$$f(Y) = (\Phi^{-1}F_2^s\Phi(\underline{u_Y}) \cdot k_u)f(X)(F_1^s(\underline{u_Y^{-1}}) \cdot k_u^{-1}) = \Phi^{-1}F_2^s\Phi(\underline{u})f(X)F_1^s(\underline{u_Y^{-1}}).$$

Thus $f(Y)F_1^s(\underline{u}) = \Phi^{-1}F_2^s\Phi(\underline{u})f(X)$.

Now consider the case $Y = X$. Then $u = u_Z^{-1} \cdot k_u^{-1}$ for some $k_u \in K^*$. Thus $F_1^s(\underline{u}) = F_1^s(\underline{u_Z^{-1}}) \cdot k_u^{-1}$ and $\Phi^{-1}F_2^s\Phi(\underline{u}) = \Phi^{-1}F_2^s\Phi(\underline{u_Z^{-1}}) \cdot k_u^{-1}$. Therefore $f(Z) = \Phi^{-1}F_2^s\Phi(\underline{u_Z})f(X)F_1^s(\underline{u_Z^{-1}})$, which implies

$$\begin{aligned} f(Z)^{-1} &= F_1^s(\underline{u_Z})f(X)^{-1}\Phi^{-1}F_2^s\Phi(\underline{u_Z^{-1}}) \\ &= (F_1^s(\underline{u_Z}) \cdot k_u)f(X)^{-1}(\Phi^{-1}F_2^s\Phi(\underline{u_Z^{-1}}) \cdot k_u^{-1}) \\ &= F_1^s(\underline{u})^{-1}f(X)^{-1}\Phi^{-1}F_2^s\Phi(\underline{u}). \end{aligned}$$

Then

$$f(Z) = (\Phi^{-1}F_2^s\Phi(\underline{u}))^{-1}f(X)F_1^s(\underline{u})$$

and

$$\Phi^{-1}F_2^s\Phi(\underline{u})f(Z) = f(X)\Phi^{-1}F_2^s\Phi(\underline{u}).$$

Finally, consider the case $Z \neq X \neq Y$. Then $\underline{u_Y} \cdot k_u = \underline{u_Z}$ for some $k_u \in K^*$. Moreover, we infer by the above considerations that $f(Z)F_1^s(\underline{u_Z}) = \Phi^{-1}F_2^s\Phi(\underline{u_Z})f(X)$ and $f(Y)F_1^s(\underline{uu_Z}) = \Phi^{-1}F_2^s\Phi(\underline{uu_Z})f(X)$. But $F_1^s(\underline{uu_Z}) = F_1^s(\underline{u})F_1^s(\underline{u_Z})$ and $\Phi^{-1}F_2^s\Phi(\underline{uu_Z}) = \Phi^{-1}F_2^s\Phi(\underline{u})\Phi^{-1}F_2^s\Phi(\underline{u_Z})$. Then we get

$$f(Y)F_1^s(\underline{u})f(Z)^{-1}f(Z)F_1^s(\underline{u_Z}) = \Phi^{-1}F_2^s\Phi(\underline{u})\Phi^{-1}F_2^s\Phi(\underline{u_Z})f(X)$$

and $f(Y)F_1^s(\underline{u})f(Z)^{-1} = \Phi^{-1}F_2^s\Phi(\underline{u})$. Consequently,

$$f(Y)F_1^s(\underline{u}) = \Phi^{-1}F_2^s\Phi(\underline{u})f(Z),$$

and for simple R_1-modules X the family $(f(X))$ is constructed.

Assume now that a family $(f(X))$ is constructed for every $X \in \underline{\mathrm{mod}}(R_1)$ with $l(X) \le n$. Consider $Y \in \underline{\mathrm{mod}}(R_1)$ with $l(Y) = n+1$. Let S be a simple submodule of Y. For the nonsplittable short exact sequence

$$0 \to S \xrightarrow{w} Y \xrightarrow{p} Y/S \to 0,$$

where w is the inclusion monomorphism and p is the canonical epimorphism, we have the short exact sequences

$$0 \to F_1^s(S) \xrightarrow{F_1(w)} F_1^s(Y) \xrightarrow{F_1(p)} F_1^s(Y/S) \to 0,$$
$$0 \to \Phi^{-1}F_2^s\Phi(S) \xrightarrow{w'} \Phi^{-1}F_2^s\Phi(Y) \xrightarrow{p'} \Phi^{-1}F_2^s\Phi(Y/S) \to 0$$

as in the first step of our proof. Let f_S be an isomorphism such that $\underline{f_S} = f(S)$. Let $f_{Y/S}$ be an isomorphism such that $\underline{f_{Y/S}} = f(Y/S)$. Let P be the projective cover of $F_1^s(Y/S)$. Then there is an epimorphism $\pi : P \to F_1^s(Y/S)$. Furthermore, $f_{Y/S}\pi : P \to \Phi^{-1}F_2^s\Phi(Y/S)$ is an epimorphism too, because $f_{Y/S}$ is an isomorphism. Thus there are morphisms $\pi_1 : P \to F_1^s(Y)$ and $\pi_2 : P \to \Phi^{-1}F_2^s\Phi(Y)$ such that $F_1(p)\pi_1 = \pi$ and $p'\pi_2 = f_{Y/S}\pi$. The morphisms π_1, π_2 are epimorphisms, because $\mathrm{top}(F_1^s(Y)) \cong \mathrm{top}(F_1^s(Y/S))$ and $\mathrm{top}(\Phi^{-1}F_2^s\Phi(Y)) \cong \mathrm{top}(\Phi^{-1}F_2^s\Phi(Y/S))$. Moreover, there is a submodule L of P such that there is an epimorphism $\kappa : L \to F_1^s(S)$ and $F_1(w)\kappa = \pi_1|_L$. Observe that $p'\pi_2(t) = 0$ for every $t \in L$, because $p'\pi_2(t) = f_{Y/S}\pi(t) = f_{Y/S}F_1(p)\pi_1(t) = f_{Y/S}F_1(p)F_1(w)\kappa(t) = 0$. Thus $\mathrm{im}(\pi_2|_L) \subset \mathrm{im}(w')$. Then $\pi_2|_L = w' f_S \kappa \cdot k$ for some $k \in K^*$. Changing w' if necessary, we may assume that $\pi_2|_L = w' f_S\kappa$, because if $p'w' = 0$ then $p'w' \cdot k^{-1} = 0$.

We define an isomorphism $f_Y : F_1^s(Y) \to \Phi^{-1}F_2^s\Phi(Y)$ in the following way. For $y \in F_1^s(Y)$ we can find $t \in P$ such that $\pi_1(t) = y$. Then we put $f_Y(y) = \pi_2(t)$. Since $\ker(\pi_1) \subset L$ and $\ker(\pi_2) \subset L$, we have $\ker(\pi_1) = \ker(\pi_2) = \ker(\kappa)$ because $\pi_2|_L = w' f_S\kappa$ and $\pi_1|_L = F_1(w)\kappa$. Therefore f_Y is a well-defined R_1-homomorphism. Since $\ker(\pi_1) = \ker(\pi_2)$, f_Y is an isomorphism. It is easy to see that the diagram

$$\begin{array}{ccccccccc}
0 & \to & F_1^s(S) & \xrightarrow{F_1(w)} & F_1^s(Y) & \xrightarrow{F_1(p)} & F_1^s(Y/S) & \to & 0 \\
 & & \downarrow{\scriptstyle f_S} & & \downarrow{\scriptstyle f_Y} & & \downarrow{\scriptstyle f_{Y/S}} & & \\
0 & \to & \Phi^{-1}F_2^s\Phi(S) & \xrightarrow{w'} & \Phi^{-1}F_2^s\Phi(Y) & \xrightarrow{p'} & \Phi^{-1}F_2^s\Phi(Y/S) & \to & 0
\end{array}$$

commutes.

Suppose now that we have a short exact sequence $0 \to U \xrightarrow{a} Y \xrightarrow{b} V \to 0$. If $\mathrm{im}(w)$ is contained in $\mathrm{im}(a)$ then there are R_1-morphisms $i : S \to U$ and $r : Y/S \to V$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & S & \stackrel{w}{\to} & Y & \stackrel{p}{\to} & Y/S & \to & 0 \\ & & \downarrow^{i} & & || & & \downarrow^{r} & & \\ 0 & \to & U & \stackrel{a}{\to} & Y & \stackrel{b}{\to} & V & \to & 0 \end{array}$$

commutes. Moreover, we deduce from the first step of the proof that there are short exact sequences

$$0 \to F_1^s(U) \xrightarrow{F_1(a)} F_1^s(Y) \xrightarrow{F_1(b)} F_1^s(V) \to 0,$$
$$0 \to \Phi^{-1}F_2^s\Phi(U) \xrightarrow{a'} \Phi^{-1}F_2^s\Phi(Y) \xrightarrow{b'} \Phi^{-1}F_2^s\Phi(V) \to 0.$$

By the inductive assumption for some $r' : \Phi^{-1}F_2^s\Phi(Y/S) \to \Phi^{-1}F_2^s\Phi(V)$ such that $\underline{r'} = \Phi^{-1}F_2^s\Phi(\underline{r})$ we have $r'f_{Y/S} = f_V F_1(r)$. Then $r'f_{Y/S}F_1(p) = f_V F_1(r)F_1(p)$. Since $F_1(r)F_1(p) = F_1(b)$, we have $f_V F_1(b) = r'f_{Y/S}F_1(p) = r'p'f_Y$, because it was shown above that $f_{Y/S}F_1(p) = p'f_Y$. Observe that b' can be chosen in such a way that $r'p' = b'$. Indeed, since $b = rp$, we have $\underline{b'} = \Phi^{-1}F_2^s\Phi(\underline{b}) = \Phi^{-1}F_2^s\Phi(\underline{rp}) = \underline{r'}\,\underline{p'}$. Suppose that $b' - r'p' \neq 0$. Then $b' - r'p'$ factorizes through a projective R_1-module Q. Since b' is an epimorphism by the first step of our proof and $b' - r'p' = q_2q_1$ with $q_1 : \Phi^{-1}F_2^s\Phi(Y) \to Q$, $q_2 : Q \to \Phi^{-1}F_2^s\Phi(V)$, there is $q_2' : Q \to \Phi^{-1}F_2^s\Phi(Y)$ such that $q_2q_1 = b'q_2'q_1$. Therefore $r'p' = b' - b'q_2'q_1$. Thus put $b'' = b'(\mathrm{id}_{\Phi^{-1}F_2^s\Phi(Y)} - q_2'q_1)$. Then $\underline{b''} = \underline{b'}$ and b'' is an epimorphism. Moreover, if we put $a'' = (\mathrm{id}_{\Phi^{-1}F_2^s\Phi(Y)} - q_2'q_1)^{-1}$ then $\underline{a''} = \underline{a'}$ and a'' is a monomorphism with $b''a'' = 0$. Since $b'' = r'p'$, we get $f_V F_1(b) = b''f_Y$.

We deduce from the last commutative diagram by the snake lemma that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & S & \stackrel{i}{\to} & U & \stackrel{c}{\to} & U/S & \to & 0 \\ & & || & & \downarrow^{a} & & \downarrow^{v} & & \\ 0 & \to & S & \stackrel{w}{\to} & Y & \stackrel{p}{\to} & Y/S & \to & 0. \end{array}$$

By the inductive assumption $v'f_{U/S} = f_{Y/S}F_1(v)$ for some v'. Thus

$$v'f_{U/S}F_1(c) = f_{Y/S}F_1(v)F_1(c).$$

Therefore $v'f_{U/S}F_1(c) = f_{Y/S}F_1(p)F_1(a)$ and $f_{Y/S}F_1(p)F_1(a) = p'f_YF_1(a)$, since we proved that $f_{Y/S}F_1(p) = p'f_Y$. Now observe that for a suitable c' we have $f_{U/S}F_1(c) = c'f_U$ by the inductive assumption. But we may assume that $v'c' = p'a''$. Indeed, suppose to the contrary that $p'a'' - v'c' \neq 0$ but $\underline{p'a'' - v'c'} = 0$. Thus this difference factorizes through a projective R_1-module, say Q_1. Then there are $z_1 : \Phi^{-1}F_2^s\Phi(U) \to Q_1$ and $z_2 : Q_1 \to \Phi^{-1}F_2^s\Phi(Y/S)$ such that $p'a'' - v'c' = z_2z_1$. Since p' is an epimorphism by the first step of our proof, there is $z_2' : Q_1 \to \Phi^{-1}F_2^s\Phi(Y)$ such that $p'z_2' = z_2$. Then replacing a'' by $a_1' = a'' - z_2'z_1$ we obtain $p'a_1' = v'c'$.

Moreover, observe that a_1' is well-defined, because it is a monomorphism by the first step of the proof and $b''a_1' = r'p'a_1' = r'v'c' = 0$ since $r'v' = 0$.

Hence we may assume that $p'a'' - v'c' = 0$. Therefore we obtain $v'c'f_U = p'a''f_U$. Furthermore,

$$\begin{aligned} p'a''f_U &= v'c'f_U = v'f_{U/S}F_1(c) = f_{Y/S}F_1(v)F_1(c) \\ &= f_{Y/S}F_1(p)F_1(a) = p'f_YF_1(a). \end{aligned}$$

Thus $p'(a''f_U - f_YF_1(a)) = 0$. Then $d = (a''f_U - f_YF_1(a)) : U \to \Phi^{-1}F_2^s\Phi(Y)$ and $\mathrm{im}(d) \subset \ker(p') = \mathrm{im}(w')$. Thus $dF_1(i) = 0$, because $dF_1(i) = a''f_UF_1(i) - f_YF_1(a)F_1(i) = a''i'f_S - f_YF_1(w)$. But $a''i' = w'$. Indeed, if $a''i' - w' \neq 0$ then it is a monomorphism by simplicity of $\Phi^{-1}F_2^s\Phi(S)$. On the other hand, we know that $\underline{a''i' - w'} = 0$. Therefore we find that a monomorphism factorizes through a projective module, which is impossible by [17; Lecture 3]. Then $a''i'f_S - f_YF_1(w) = w'f_S - f_YF_1(w) = 0$.

Now we can consider the decompositions of K-spaces $F_1^s(Y) = \mathrm{im}(F_1(w)) \oplus Y'$ and $\Phi^{-1}F_2^s\Phi(Y) = \mathrm{im}(w') \oplus Y''$. Since f_Y is an R_1-isomorphism, f_Y is a K-linear isomorphism. Since $w'f_S = f_YF_1(w)$ and $p'f_Y = f_{Y/S}F_1(p)$, f_Y restricted to Y' is a K-linear isomorphism of Y' to Y''. But if $z \in \mathrm{im}(F_1(a)) \cap Y'$ then $f_Y(z) \in Y''$. Furthermore, we can consider the decomposition of the K-space $F_1^s(U) = \mathrm{im}(F_1(w)) \oplus U'$. Then by the inductive assumption for the decomposition $\Phi^{-1}F_2^s\Phi(U) = \mathrm{im}(i') \oplus U''$ the restriction of f_U to U' is a K-linear isomorphism between U' and U''. Since $a''i' = w'$, we get $a''f_U(z) \in Y''$, where $z \in \mathrm{im}(F_1(w)) \cap Y'$. Thus $\mathrm{im}(a''f_U - f_YF_1(a)) \subset Y''$, and so $a''f_U - f_YF_1(a) = 0$.

Now consider the case when $\mathrm{im}(a)$ does not contain $\mathrm{im}(w)$. First assume that U is simple. Then we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U & = & U & & \\
 & & & & \downarrow^{a} & & \downarrow^{a_1} & & \\
0 & \to & S & \xrightarrow{w} & Y & \xrightarrow{p} & Y/S & \to & 0 \\
 & & \| & & \downarrow^{b} & & \downarrow^{b_1} & & \\
0 & \to & S & \xrightarrow{w_1} & V & \xrightarrow{p_1} & V/S & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By the inductive assumption,

$$a_1'f_U = f_{Y/S}F_1(a_1) = f_{Y/S}F_1(p)F_1(a) = p'f_YF_1(a),$$

where $\underline{a_1'} = \Phi^{-1}F_2^s\Phi(\underline{a_1})$ satisfies the required condition by the inductive assumption. We may assume that $p_1'b' = b_1'p'$, where a', b' are so chosen that the considered column of our diagram is exact after $\Phi^{-1}F_2^s\Phi$ has been applied. Indeed, we know that $\underline{p_1'b' - b_1'p'} = 0$. Then if $p_1'b' - b_1'p' \neq 0$ then there are a projective R_1-module Q and morphisms $q_1 : \Phi^{-1}F_2^s\Phi(Y) \to Q$ and $q_2 : Q \to \Phi^{-1}F_2^s\Phi(Y)$ such that $p_1'b' - b_1'p' = p_1'b'q_2q_1$, because p_1', b' are epimorphisms by the first step of the proof. Denote by t the automorphism $\mathrm{id}_{\Phi^{-1}F_2^s\Phi(Y)} - q_2q_1$. Then putting $b'' = b't$ we get $p_1'b'' = b_1'p'$. If we put $a'' = t^{-1}a'$ then $b''a'' = 0$ and the sequence

$$0 \to \Phi^{-1}F_2^s\Phi(U) \overset{a''}{\to} \Phi^{-1}F_2^s\Phi(Y) \overset{b''}{\to} \Phi^{-1}F_2^s\Phi(V) \to 0$$

is exact again. Moreover, $p'a'' = a_1'$. Indeed, if $p'a'' - a_1' \neq 0$ then it factorizes through a projective R_1-module, since $\underline{p'a'' - a_1'} = 0$. But U is simple and hence the considered difference is a monomorphism which cannot factorize through a projective module by [17; Lecture 3]. Thus $p'a'' = a_1'$. Therefore $p'a''f_U = p'f_YF_1(a)$. Then $p'(a''f_U - f_YF_1(a)) = 0$ and for $d = a''f_U - f_YF_1(a)$ we have $\mathrm{im}(d) \subset \ker(p') = \mathrm{im}(w')$. If we consider the decompositions of the K-spaces $F_1^s(Y) = \mathrm{im}(F_1(w)) \oplus Y'$ and $\Phi^{-1}F_2^s\Phi(Y) = \mathrm{im}(w') \oplus Y''$ then f_Y, being a K-linear isomorphism, when restricted to Y' is a K-linear isomorphism between Y' and Y''. Moreover, $F_1(p)$, being a K-linear morphism, when restricted to Y' is a K-linear isomorphism between Y' and $F_1^s(Y/S)$. Furthermore, p', being a K-linear morphism, when restricted to Y'' is a K-linear isomorphism between Y'' and $\Phi^{-1}F_2^s\Phi(Y/S)$. Then $\mathrm{im}(a'') \subset Y''$ by the equality $p'a'' = a_1'$. Thus $\mathrm{im}(a''f_U) \subset Y''$. Since $\mathrm{im}(F_1(a)) \subset Y'$, we have $\mathrm{im}(f_YF_1(a)) \subset Y''$, because we already proved that $p'f_Y = f_{Y/S}F_1(p)$. Therefore $\mathrm{im}(a''f_U - f_YF_1(a)) \subset Y''$, and so it is zero. Consequently, $a''f_U = f_YF_1(a)$.

Now we infer by the inductive assumption that $p_1'f_V = f_{V/S}F_1(p_1)$. Then $p_1'f_VF_1(b) = f_{V/S}F_1(p_1)F_1(b) = f_{V/S}F_1(b_1)F_1(p) = b_1'f_{Y/S}F_1(p)$, where p_1' and b_1' are well-defined morphisms in the inductive step. Furthermore, $b_1'f_{Y/S}F_1(p) = b_1'p'f_Y$. Since $b_1'p' = p_1'b''$, we have $p_1'f_VF_1(b) = p_1'b''f_Y$. Then $p_1'(f_VF_1(b) - b''f_Y) = 0$. Then $\mathrm{im}(f_VF_1(b) - b''f_Y) \subset \ker(p_1') = \mathrm{im}(w_1')$.

Consider the decompositions of K-linear spaces $F_1^s(Y) = \mathrm{im}(F_1(w)) \oplus Y'$, $\Phi^{-1}F_2^s\Phi(Y) = \mathrm{im}(w') \oplus Y''$. Since $a''f_U = f_YF_1(a)$, we have $p'a''f_U = p'f_YF_1(a) = f_{Y/S}F_1(p)F_1(a) = f_{Y/S}F_1(a_1)$. Therefore $p'a''f_U$ is a monomorphism, and so $\mathrm{im}(a''f_U) \subset Y''$. Then we consider the decompositions of K-linear spaces $Y' = \mathrm{im}(F_1(a)) \oplus Y_1'$ and $Y'' = \mathrm{im}(a''f_U) \oplus Y_1''$. Clearly $F_1^s(V) \cong \mathrm{im}(F_1(w)) \oplus Y_1'$ and $\Phi^{-1}F_2^s\Phi(V) \cong \mathrm{im}(w') \oplus Y_1''$ as K-spaces, because $p_1'b''a''f_U = b_1'p'a''f_U = b_1'a_1'f_U = 0$. Since $w'f_S = f_YF_1(w)$ and $a''f_U = f_YF_1(a)$, the K-linear morphism f_Y restricted to $\mathrm{im}(F_1(w))$ yields an isomorphism between $\mathrm{im}(F_1(w))$ and $\mathrm{im}(w')$. Moreover, the K-linear morphism f_Y restricted to Y_1' yields an isomorphism between Y_1' and Y_1''. Moreover, $F_1(b)$ and b'' are K-linear isomorphisms between $\mathrm{im}(F_1(w)) \oplus Y_1'$

and $F_1^s(V)$, $\mathrm{im}(w') \oplus Y_1''$ and $\Phi^{-1}F_2^s\Phi(V)$, respectively. They have the property that $F_1(b)|_{Y_1'}: Y_1' \to V'$, $b''|_{Y_1''}: Y_1'' \to V''$ are isomorphisms, where $F_1^s(V) = \mathrm{im}(F_1(w_1)) \oplus V'$ and $\Phi^{-1}F_2^s\Phi(V) = \mathrm{im}(w_1') \oplus V''$ are decompositions of K-spaces. Therefore $f_V F_1(b)(z) \in V''$ for every $z \in Y_1'$, because $p_1' f_V = f_{V/S} F_1(p_1)$ by the inductive assumption and $F_1(p_1)$ is a K-linear isomorphism between V' and $F_1^s(V/S)$. Furthermore, $b'' f_Y(z) \in V''$ for every $z \in Y_1'$. Then $\mathrm{im}((f_V F_1(b) - b'' f_Y)|_{Y_1'}) = 0$, because we have already proved that $\mathrm{im}(f_V F_1(b) - b'' f_Y) \subset \mathrm{im}(w_1')$. But if $z \in \mathrm{im}(F_1(w))$ then $b'' f_Y(z) = b'' f_Y F_1(w)(z_1)$, $z_1 \in F_1^s(S)$, and

$$\begin{aligned} b'' f_Y F_1(w)(z_1) &= b'' w' f_S(z_1) = w_1' f_S(z_1) = f_V F_1(w_1)(z_1) \\ &= f_V F_1(b) F_1(w)(z_1) = f_V F_1(b)(z). \end{aligned}$$

Consequently, $f_V F_1(b) = b'' f_Y$. If U is not simple then take a simple submodule T of U. Since we proved the required condition for simple T, we may repeat the arguments from the case $\mathrm{im}(a) \supset \mathrm{im}(w)$ for U, with T instead of S. Thus we have finished the proof of the commutativity condition for f_Y.

Now we show that the required squares are commutative. First consider the case when $F_1^s(\underline{u}) : F_1^s(Y) \to F_1^s(Z)$ is an isomorphism. Then clearly so is $u : Y \to Z$. Let S be a simple direct summand in the socle of Y. We have the short exact sequence

$$0 \to S \xrightarrow{w} Y \xrightarrow{p} Y/S \to 0.$$

Denote by S_1 the simple submodule $uw(S)$ of Z. Then the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \to & S & \xrightarrow{w} & Y & \xrightarrow{p} & Y/S & \to & 0 \\ & & \downarrow^{u_1} & & \downarrow^{u} & & \downarrow^{u_2} & & \\ 0 & \to & S_1 & \xrightarrow{v} & Z & \xrightarrow{q} & Z/S_1 & \to & 0, \end{array}$$

where $u_1 = uw$, v is inclusion, q is the canonical epimorphism and u_2 is some isomorphism. By the inductive assumption, $u_1' f_S = f_{S_1} F_1(u_1)$ and $u_2' f_{Y/S} = f_{Z/S_1} F_1(u_2)$. We show that $u' f_Y = f_Z F_1(u)$ for $\underline{u}' = \Phi^{-1}F_2^s\Phi(\underline{u})$. As above, we can show that there are v' and q' such that the following diagrams are commutative:

$$\begin{array}{ccccccccc} 0 & \to & F_1^s(S) & \xrightarrow{F_1(w)} & F_1^s(Y) & \xrightarrow{F_1(p)} & F_1^s(Y/S) & \to & 0 \\ & & {\scriptstyle F_1(u_1)}\downarrow & & {\scriptstyle F_1(u)}\downarrow & & {\scriptstyle F_1(u_2)}\downarrow & & \\ 0 & \to & F_1^s(T) & \xrightarrow{F_1(v)} & F_1^s(Z) & \xrightarrow{F_1(q)} & F_1^s(Z/T) & \to & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \to & \Phi^{-1}F_2^s\Phi(S) & \xrightarrow{w'} & \Phi^{-1}F_2^s\Phi(Y) & \xrightarrow{p'} & \Phi^{-1}F_2^s\Phi(Y/S) & \to & 0 \\ & & {\scriptstyle u_1'}\downarrow & & {\scriptstyle u'}\downarrow & & {\scriptstyle u_2'}\downarrow & & \\ 0 & \to & \Phi^{-1}F_2^s\Phi(T) & \xrightarrow{v'} & \Phi^{-1}F_2^s\Phi(Z) & \xrightarrow{q'} & \Phi^{-1}F_2^s\Phi(Z/T) & \to & 0 \end{array}$$

Now consider the decompositions of K-spaces $F_1^s(Y) = \mathrm{im}(F_1(w)) \oplus Y'$, $F_1^s(Z) = \mathrm{im}(F_1(v)) \oplus Z'$, $\Phi^{-1}F_2^s\Phi(Y) = \mathrm{im}(w') \oplus Y''$, $\Phi^{-1}F_2^s\Phi(Z) = \mathrm{im}(v') \oplus Z''$. Take $y \in \mathrm{im}(F_1(w))$. Then $u'f_Y(y) = u'f_YF_1(w)(y_1)$, $y_1 \in F_1^s(S)$. Furthermore,

$$\begin{aligned} u'f_YF_1(w)(y_1) &= u'w'f_S(y_1) = v'u_1'f_S(y_1) = v'f_TF_1(u_1)(y_1) \\ &= f_ZF_1(v)F_1(u_1)(y_1) = f_ZF_1(u)F_1(w)(y_1) = f_ZF_1(u)(y). \end{aligned}$$

If $y \in Y'$ then $u'f_Y(y) = u'f_YF_1(p)^{-1}(y_1)$, where $y_1 \in F_1^s(Y/S)$ and $F_1(p)^{-1}$ is the linear inverse of $F_1(p)$ restricted to Y'. Then $u'f_YF_1(p)^{-1}(y_1) = u'(p')^{-1}f_{Y/S}(y_1)$, where $(p')^{-1}$ is the linear inverse of p' restricted to Y''. But $u'(p')^{-1} = (q')^{-1}u_2'$, where $(q')^{-1}$ is the linear inverse of q' restricted to Z''. Thus

$$\begin{aligned} u'(p')^{-1}f_{Y/S}(y_1) &= (q')^{-1}u_2'f_{Y/S}(y_1) = (q')^{-1}f_{Z/T}F_1(u_2)F_1(p)(y) \\ &= (q')^{-1}f_{Z/T}F_1(q)F_1(u)(y) = (q')^{-1}q'f_ZF_1(u)(y) \\ &= f_ZF_1(u)(y). \end{aligned}$$

Consequently, $u'f_Y = f_ZF_1(u)$, and so $\Phi^{-1}F_2^s\Phi(\underline{u})f(Y) = f(Z)F_1^s(\underline{u})$.

Now suppose that there is $0 \neq u : Y \to Z$ which is not an isomorphism and $l(Z) \le l(Y)$. Since we have a decomposition $u = a_2a_1$ with an epimorphism $a_1 : Y \to \mathrm{im}(u)$ and a monomorphism $a_2 : \mathrm{im}(u) \to Z$, it is enough to assume that u is either an epimorphism or a monomorphism. But if u is an epimorphism then there is a short exact sequence

$$0 \to V \xrightarrow{v} Y \xrightarrow{u} Z \to 0$$

with $V = \ker(u)$. Then by the commutativity condition for f_Y there is u' such that $u'f_Y = f_ZF_1(u)$. Thus $\Phi^{-1}F_2^s\Phi(\underline{u})f(Y) = f(Z)F_1^s(\underline{u})$. The same arguments can be applied for a monomorphism u. Consequently, our lemma is proved by induction.

3.5. Lemma. *Let* $F_1 : \mathrm{mod}(R_1) \to \mathrm{mod}(R_1)$ *and* $F_2 : \mathrm{mod}(R_2) \to \mathrm{mod}(R_2)$ *be exact equivalences satisfying the conditions* (a) *and* (b) *of Lemma* 3.4. *Then there is a quasi-inverse* Φ_1^{-1} *of* Φ *such that* $F_1^s(X) = \Phi_1^{-1}F_2^s\Phi(X)$ *for every object* $X \in \underline{\mathrm{mod}}(R_1)$.

Proof. First we construct a functor $\Delta : \underline{\mathrm{mod}}(R_1) \to \underline{\mathrm{mod}}(R_1)$ such that $F_1^s(X) = \Delta\Phi^{-1}F_2^s(X)$ for every $X \in \underline{\mathrm{mod}}(R_1)$. We know from Lemma 3.4 that $F_1^s \cong \Phi^{-1}F_2^s\Phi$. Fix an isomorphism $f : F_1^s \to \Phi^{-1}F_2^s\Phi$. For every $X \in \underline{\mathrm{mod}}(R_1)$ either there is $Y \in \underline{\mathrm{mod}}(R_1)$ such that $X = \Phi^{-1}F_2^s\Phi(Y)$ or X does not lie in the image of $\Phi^{-1}F_2^s\Phi$. If $X = \Phi^{-1}F_2^s\Phi(Y)$ then we put $\Delta(X) = F_1^s(Y)$. If X is not contained in the image of $\Phi^{-1}F_2^s\Phi$ then we put $\Delta(X) = X$. If $\underline{h} : X_1 \to X_2$ is a morphism in $\underline{\mathrm{mod}}(R_1)$ and $X_i = \Phi^{-1}F_2^s\Phi(Y_i)$, $i = 1, 2$, then we put $\Delta(\underline{h}) = \underline{t}$, where $\underline{t} = f(X_2)^{-1}\Phi^{-1}F_2^s\Phi(\underline{h})f(X_1)$. If $\underline{h} : X_1 \to X_2$ is a morphism in $\underline{\mathrm{mod}}(R_1)$ and X_1 does not lie in the image of $\Phi^{-1}F_2^s\Phi$ and $X_2 = \Phi^{-1}F_2^s\Phi(Y_2)$ then $\Delta(\underline{h}) = f(X_2)^{-1}\underline{h}$. If $\underline{h} : X_1 \to X_2$, $X_1 = \Phi^{-1}F_2^s\Phi(Y_1)$ and X_2 is not contained in the image of $\Phi^{-1}F_2^s\Phi$ then $\Delta(\underline{h}) = \underline{h}f(X_1)$. If $\underline{h} : X_1 \to X_2$ is a morphism in $\underline{\mathrm{mod}}(R_1)$ and X_1, X_2 do not lie in the image of $\Phi^{-1}F_2^s\Phi$ then we put $\Delta(\underline{h}) = \underline{h}$.

A simple verification shows that Δ is a well-defined functor. Moreover, Δ is dense since F_1^s is dense. Furthermore, Δ is fully faithful since F_1^s and $\Phi^{-1}F_2^s\Phi$ are. Thus Δ is an equivalence. Consequently, $\Delta\Phi^{-1} = \Phi_1^{-1}$ is a quasi-inverse of Φ. Indeed, $\Phi_1^{-1}\Phi(X) \cong \Phi^{-1}\Phi(X)$ for every $X \in \underline{\mathrm{mod}}(R_1)$ by the definition of Δ. Hence $\Phi_1^{-1}\Phi(X) \cong X$. If $\phi : 1_{\underline{\mathrm{mod}}(R_1)} \to \Phi^{-1}\Phi$ is an isomorphism of functors then fix an isomorphism $\alpha(X) : \Phi^{-1}\Phi(X) \to \Phi_1^{-1}\Phi(X)$ for every $X \in \underline{\mathrm{mod}}(R_1)$ and define $\phi_1 : 1_{\underline{\mathrm{mod}}(R_1)} \to \Phi_1^{-1}\Phi$ by $\phi_1(X) = \alpha(X)\phi(X)$ for every $X \in \underline{\mathrm{mod}}(R_1)$. Thus for every morphism $\underline{u} : X \to Z$ we have to check whether the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_1(X)} & \Phi_1^{-1}\Phi(X) \\ {\scriptstyle \underline{u}}\downarrow & & \downarrow{\scriptstyle \Phi_1^{-1}\Phi(\underline{u})} \\ Z & \xrightarrow{\phi_1(Z)} & \Phi_1^{-1}\Phi(Z) \end{array}$$

commutes. Clearly it is sufficient to prove that the diagram

$$\begin{array}{ccc} \Phi^{-1}\Phi(X) & \xrightarrow{\alpha(X)} & \Phi_1^{-1}\Phi(X) \\ {\scriptstyle \Phi^{-1}\Phi(\underline{u})}\downarrow & & \downarrow{\scriptstyle \Phi_1^{-1}\Phi(\underline{u})} \\ \Phi^{-1}\Phi(Z) & \xrightarrow{\alpha(Z)} & \Phi_1^{-1}\Phi(Z) \end{array}$$

commutes. If $\Phi^{-1}\Phi(X) = \Phi^{-1}F_2^s\Phi(Y)$ and $\Phi^{-1}\Phi(Z) = \Phi^{-1}F_2^s\Phi(W)$ then for $\alpha(X) = f(X)^{-1}$ and $\alpha(Z) = f(Z)^{-1}$ the above diagram commutes. If $\Phi^{-1}\Phi(X) = \Phi^{-1}F_2^s\Phi(Y)$ and $\Phi^{-1}\Phi(Z)$ is not contained in the image of $\Phi^{-1}F_2^s\Phi$ then for $\alpha(X) = f(X)^{-1}$ and $\alpha(Z) = 1_{\Phi^{-1}\Phi(Z)}$ the diagram commutes. If $\Phi^{-1}\Phi(X)$ is not contained in the image of $\Phi^{-1}F_2^s\Phi$ and $\Phi^{-1}\Phi(Z) = \Phi^{-1}F_2^s\Phi(W)$ then for $\alpha(X) = 1_{\Phi^{-1}\Phi(X)}$ and $\alpha(Z) = f(Z)^{-1}$ the above diagram commutes. If neither $\Phi^{-1}\Phi(X)$ nor $\Phi^{-1}\Phi(Z)$ lies in the image of $\Phi^{-1}F_2^s\Phi$ then for $\alpha(X) = 1_{\Phi^{-1}\Phi(X)}$ and $\alpha(Z) = 1_{\Phi^{-1}\Phi(Z)}$ the required commutativity holds. Thus for the isomorphism $\alpha : \Phi^{-1}\Phi \to \Phi_1^{-1}\Phi$ chosen above ϕ_1 is an isomorphism of functors. Similarly we show that there is an isomorphism $\psi_1 : 1_{\underline{\mathrm{mod}}(R_2)} \to \Phi\Phi_1^{-1}$. This finishes our proof.

3.6. PROPOSITION. *Let* $F_1 : \mathrm{mod}(R_1) \to \mathrm{mod}(R_1)$ *and* $F_2 : \mathrm{mod}(R_2) \to \mathrm{mod}(R_2)$ *be exact equivalences satisfying the following conditions*:

(a) *If* $F_i^s : \underline{\mathrm{mod}}(R_i) \to \underline{\mathrm{mod}}(R_i)$, $i = 1, 2$, *is defined by* $F_i^s(X) = F_i(X)$, $X \in \underline{\mathrm{mod}}(R_i)$, $F_i^s(\underline{f}) = \underline{F_i(f)}$, $\underline{f} : X \to Y$ *a morphism in* $\underline{\mathrm{mod}}(R_i)$, *then* F_i^s *is an equivalence.*

(b) *For every object* $X \in \underline{\mathrm{mod}}(R_1)$, $F_1^s(X) \cong \Phi^{-1}F_2^s\Phi(X)$, *where* Φ^{-1} *is a quasi-inverse of* Φ.

Then there is an equivalence $\Phi' : \underline{\mathrm{mod}}(R_1) \to \underline{\mathrm{mod}}(R_2)$ *such that* $\Phi' F_1^s = F_2^s\Phi'$.

Proof. By Lemma 3.5 there is a quasi-inverse Φ_1^{-1} of Φ such that $F_1^s(X) = \Phi_1^{-1}F_2^s\Phi(X)$ for every $X \in \underline{\mathrm{mod}}(R_1)$. We deduce from Lemma 3.4 that F_1^s and $\Phi_1^{-1}F_2^s\Phi$ are isomorphic functors. Then there is an isomorphism $f : F_1^s \to \Phi^{-1}F_2^s\Phi$. We define $\Phi' : \underline{\mathrm{mod}}(R_1) \to \underline{\mathrm{mod}}(R_2)$ by the formula $\Phi' = (F_2^s)^{-1}\Phi F_2^s$. It is easy to verify that Φ^{-1} is a quasi-inverse of Φ'. Then $f : F_1^s \to \Phi^{-1}F_2^s\Phi'$ yields the equality of functors and the proposition follows.

3.7. PROPOSITION. *If* ν_{R_1} *and* ν_{R_2} *act freely on the objects of* R_1 *and* R_2, *respectively*, *then* $R_1/(\nu_{R_1})$ *and* $R_2/(\nu_{R_2})$ *are stably equivalent.*

Proof. Observe that, under our assumptions, the action of (ν_{R_i}) on R_i induces the Nakayama functor $\mathcal{N}_{R_i} : \mathrm{mod}(R_i) \to \mathrm{mod}(R_i)$ given by the formula $\mathcal{N}_{R_i} = D\,\mathrm{Hom}_{R_i}(-, R_i)$ (see [8; 2.1]). Furthermore, $\mathcal{N}_{R_i}$ is an exact equivalence such that $\mathcal{N}_{R_i}^s : \underline{\mathrm{mod}}(R_i) \to \underline{\mathrm{mod}}(R_i)$ is an equivalence. Then $\mathcal{N}_{R_i}^s \cong \Omega_{R_i}^{-2}\tau_{R_i}$ by [8; 2.5]. Thus we deduce from Proposition 3.2 that for every object $X \in \underline{\mathrm{mod}}(R_i)$ we have $\mathcal{N}_{R_1}^s(X) \cong \Phi_1^{-1}\mathcal{N}_{R_2}^s\Phi(X)$ for some quasi-inverse Φ_1^{-1} of Φ. Therefore, by Proposition 3.6, $\Phi\mathcal{N}_{R_1}^s = \mathcal{N}_{R_2}^s\Phi$. Thus $\Phi\mathcal{N}_{R_1}^s(X) = \mathcal{N}_{R_2}^s\Phi(X)$ for every $X \in \underline{\mathrm{mod}}(R_1)$. But the push-down functor $F_{\lambda,i} : \mathrm{mod}(R_i) \to \mathrm{mod}(R_i/(\nu_{R_i}))$ is induced by $\mathcal{N}_{R_i}$. Hence $F_{\lambda,i}$ maps every $\mathcal{N}_{R_i}$-orbit of an R_i-module M onto one $R_i/(\nu_{R_i})$-module $F_{\lambda,i}(M)$. Consequently, Φ maps the $\mathcal{N}_{R_1}$-orbits of nonprojective R_1-modules onto $\mathcal{N}_{R_2}$-orbits of nonprojective R_2-modules, because $\Phi\mathcal{N}_{R_1}^s(X) = \mathcal{N}_{R_2}^s\Phi(X)$ for every $X \in \underline{\mathrm{mod}}(R_1)$. Furthermore, Φ maps the $\mathcal{N}_{R_1}^s$-orbits of morphisms in $\underline{\mathrm{mod}}(R_1)$ onto the $\mathcal{N}_{R_2}^s$-orbits of morphisms in $\underline{\mathrm{mod}}(R_2)$, because by the definition of $\mathcal{N}_{R_i}$ a morphism $f : X \to Y$ in $\mathrm{mod}(R_i)$ factorizes through a projective R_i-module iff $F_{\lambda,i}(f) : F_{\lambda,i}(X) \to F_{\lambda,i}(Y)$ factorizes through a projective $R_i/(\nu_{R_i})$-module.

Now we can define a functor $\Psi : \underline{\mathrm{mod}}(R_1/(\nu_{R_1})) \to \underline{\mathrm{mod}}(R_2/(\nu_{R_2}))$ as follows. For every indecomposable M in $\underline{\mathrm{mod}}(R_1/(\nu_{R_1}))$ there is an indecomposable R_1-module $\widetilde{M}$ which is nonprojective and satisfies $F_{\lambda,1}(\widetilde{M}) = M$. Then we put $\Psi(M) = F_{\lambda,2}\Phi(\widetilde{M})$. If $M = M_1 \oplus \ldots \oplus M_n \in \underline{\mathrm{mod}}(R_1/(\nu_{R_1}))$ with M_j indecomposable, $j = 1, \ldots, n$, then we put $\Psi(M) = \Psi(M_1) \oplus \ldots \oplus \Psi(M_n)$. If $\underline{f} : M \to N$ is a morphism in $\underline{\mathrm{mod}}(R_1/(\nu_{R_1}))$ then there is a morphism $\underline{\widetilde{f}} : \widetilde{M} \to \widetilde{N}$ in $\underline{\mathrm{mod}}(R_1)$ such that $\underline{f} = \underline{F_{\lambda,1}(\widetilde{f})}$. Then there is $\underline{h} = \Phi(\underline{\widetilde{f}})$ and we put $\Psi(\underline{f}) = \underline{F_{\lambda,2}(h)}$. Since Φ maps the $\mathcal{N}_{R_1}$-orbits of indecomposable nonprojective R_1-modules onto $\mathcal{N}_{R_2}$-orbits of indecomposable nonprojective R_2-modules and the $\mathcal{N}^s_{R_1}$-orbits of morphisms in $\underline{\mathrm{mod}}(R_1)$ onto the $\mathcal{N}^s_{R_2}$-orbits of morphisms in $\underline{\mathrm{mod}}(R_2)$, the above definition does not depend on the choice of $\widetilde{M}$ and $\widetilde{f}$.

Observe that $\Psi : \underline{\mathrm{mod}}(R_1/(\nu_{R_1})) \to \underline{\mathrm{mod}}(R_2/(\nu_{R_2}))$ is a functor. Indeed, $\Psi(\underline{\mathrm{id}_M}) = \underline{\mathrm{id}_{\Psi(M)}}$ since for $F_{\lambda,1}(\widetilde{M}) = M$ we have $F_{\lambda,1}(\mathrm{id}_{\widetilde{M}}) = \mathrm{id}_M$. Then $\Phi(\underline{\mathrm{id}_{\widetilde{M}}}) = \underline{\mathrm{id}_{\Phi(\widetilde{M})}}$ since Φ is a functor. Thus $\underline{F_{\lambda,2}(\mathrm{id}_{\Phi(\widetilde{M})})} = \underline{\mathrm{id}_{F_{\lambda,2}\Phi(\widetilde{M})}}$. If $\underline{f_1} : M \to N$ and $\underline{f_2} : N \to L$ are morphisms in $\underline{\mathrm{mod}}(R_1/(\nu_{R_1}))$ then $F_{\lambda,1}(\widetilde{f_2 f_1}) = f_2 f_1$ with $\widetilde{f_2 f_1} = \widetilde{f}_2\widetilde{f}_1$. Thus $\Phi(\underline{\widetilde{f}_2\widetilde{f}_1}) = \Phi(\underline{\widetilde{f_2 f_1}}) = \underline{h} = \underline{h_2 h_1}$ with $\Phi(\underline{\widetilde{f}_i}) = \underline{h_i}$, $i = 1, 2$. Therefore

$$\Psi(\underline{f_2 f_1}) = \underline{F_{\lambda,2}(h_2 h_1)} = \underline{F_{\lambda,2}(h_2) F_{\lambda,2}(h_1)} = \Psi(\underline{f_2})\Psi(\underline{f_1}).$$

Since R_1 and R_2 are locally support-finite, Ψ is dense.

Observe that if $0 \neq \underline{f} : M \to N$ in $\underline{\mathrm{mod}}(R_1/(\nu_{R_1}))$ then $\underline{\widetilde{f}} \neq 0$ for every $\widetilde{f}$ such that $F_{\lambda,1}(\widetilde{f}) = f$. Hence $\Phi(\underline{\widetilde{f}}) \neq 0$ since Φ is an equivalence. Thus $\Phi(\underline{\widetilde{f}}) = \underline{h} \neq 0$ and clearly $\underline{F_{\lambda,2}(h)} \neq 0$. Therefore $\Psi(\underline{f}) \neq 0$, which shows that Ψ is faithful. If $0 \neq \underline{t} : \Psi(M) \to \Psi(N)$ for some $M, N \in \underline{\mathrm{mod}}(R_1/(\nu_{R_1}))$ then there are $\widetilde{M}, \widetilde{N} \in \underline{\mathrm{mod}}(R_1)$ with $F_{\lambda,2}\Phi(\widetilde{M}) = \Psi(M)$ and $F_{\lambda,2}\Phi(\widetilde{N}) = \Psi(N)$. But there is $\widetilde{t} : \Phi(\widetilde{M}) \to \Phi(\widetilde{N})$ such that $\underline{t} = \underline{F_{\lambda,2}(\widetilde{t})}$. Since Φ is an equivalence, there is $0 \neq \underline{\widetilde{f}} : \widetilde{M} \to \widetilde{N}$ such that $\Phi(\underline{\widetilde{f}}) = \underline{\widetilde{t}}$. If we put $f = F_{\lambda,1}(\widetilde{f})$ then $\Psi(\underline{f}) = \underline{t}$. Consequently, Ψ is full and the proposition follows.

3.8. Proposition. *If R_1 and R_2 are triangular selfinjective locally support-finite K-categories with free actions of (ν_{R_1}) and (ν_{R_2}), respectively, and $R_1/(\nu_{R_1}) \cong R_2/(\nu_{R_2})$ then $R_1 \cong R_2$.*

Proof. Fix some representatives $\{P_i\}_{i\in I}$ of the isomorphism classes of indecomposable projective R_1-modules and some representatives $\{Q_j\}_{j\in J}$ of the isomorphism classes of the indecomposable projective R_2-modules. Then $R_1 \cong \mathrm{End}_{R_1}(\bigoplus_{i\in I} P_i)^{\mathrm{op}}$ and $R_2 \cong \mathrm{End}_{R_2}(\bigoplus_{j\in J} Q_j)^{\mathrm{op}}$. Let $F_{\lambda,t} : \mathrm{mod}(R_t) \to \mathrm{mod}(R_t/(\nu_{R_t}))$, $t = 1, 2$, be the push-down functors induced by the actions of (ν_{R_t}) on R_t. Fix some $i_0 \in I$. Let $LF_{\lambda,1}(P_{i_0}) = F_{\lambda,2}(Q_{j_0})$ for a fixed $j_0 \in J$, where $L : \mathrm{mod}(R_1/(\nu_{R_1})) \to \mathrm{mod}(R_2/(\nu_{R_2}))$ is the equivalence induced by a fixed isomorphism from $R_1/(\nu_{R_1})$ onto $R_2/(\nu_{R_2})$. Let $R_{1,1}$ be the subcategory of R_1 formed by P_{i_0} and the P_i, $P_{i'}$ such that the following conditions are satisfied:

(a) there is a nonzero morphism $f_i : P_i \to P_{i_0}$ of the form $f_i = f^* f_i'$, where $f_i' : P_i \to \mathrm{rad}(P_{i_0})$ satisfies $\pi_{i_0} f_i' \neq 0$ for the canonical epimorphism $\pi_{i_0} : \mathrm{rad}(P_{i_0}) \to \mathrm{top}(\mathrm{rad}(P_{i_0}))$, and $f^* : \mathrm{rad}(P_{i_0}) \to P_{i_0}$ is the identity monomorphism;

(b) there is a nonzero morphism $h_{i'} : P_{i_0} \to P_{i'}$ of the form $h''_{i'} h'_{i'}$, where $h'_{i'} : P_{i_0} \to \mathrm{rad}(P_{i'})$ satisfies $\pi_{i'} h'_{i'} \neq 0$ for the canonical epimorphism $\pi_{i'} : \mathrm{rad}(P_{i'}) \to \mathrm{top}(\mathrm{rad}(P_{i'}))$, and $h''_{i'} : \mathrm{rad}(P_{i'}) \to P_{i'}$ is the identity monomorphism.

If P, P' are objects of $R_{1,1}$ then $\mathrm{Hom}_{R_{1,1}}(P, P')$ is the subspace of $\mathrm{Hom}_{R_1}(P, P')$ generated by the isomorphisms between P and P' and the morphisms of the form $t = t_1 t_2$, where $t_1 = h_{i'}$ for some i' and t_2 is an automorphism of P_{i_0}, or $t_2 = f_i$ for some i and t_1 is an automorphism of P_{i_0}, or else $t_1 = h_{i'}$ for some i' and $t_2 = f_i$ for some i. Since R_1 is locally support-finite, $R_{1,1}$ is finite.

Let $R_{2,1}$ be the subcategory of R_2 formed by Q_{j_0} and the Q_j, $Q_{j'}$ such that the following conditions are satisfied:

(a) there is a nonzero morphism $r_j : Q_j \to Q_{j_0}$ of the form $r_j = r^* r'_j$, where $r'_j : Q_j \to \mathrm{rad}(Q_{j_0})$ satisfies $\kappa_{j_0} r'_j \neq 0$ for the canonical epimorphism $\kappa_{j_0} : \mathrm{rad}(Q_{j_0}) \to \mathrm{top}(\mathrm{rad}(Q_{j_0}))$, and $r^* : \mathrm{rad}(Q_{j_0}) \to Q_{j_0}$ is the identity monomorphism;

(b) there is a nonzero morphism $s_{j'} : Q_{j_0} \to Q_{j'}$ of the form $s''_{j'} s'_{j'}$, where $s'_{j'} : Q_{j_0} \to \mathrm{rad}(Q_{j'})$ satisfies $\kappa_{j'} s'_{j'} \neq 0$ for the canonical epimorphism $\kappa_{j'} : \mathrm{rad}(Q_{j'}) \to \mathrm{top}(\mathrm{rad}(Q_{j'}))$, and $s''_{j'} : \mathrm{rad}(Q_{j'}) \to Q_{j'}$ is the identity monomorphism.

If Q, Q' are objects of $R_{2,1}$ then $\mathrm{Hom}_{R_{2,1}}(Q, Q')$ is the subspace of $\mathrm{Hom}_{R_2}(Q, Q')$ generated by the isomorphisms between Q and Q' and the morphisms of the form $w = w_1 w_2$, where $w_1 = s_{j'}$ for some j' and w_2 is an automorphism of Q_{j_0}, or $w_2 = r_j$ for some j and w_1 is an automorphism of Q_{j_0}, or else $w_1 = s_{j'}$ for some j' and $w_2 = r_j$ for some j. Since R_2 is locally support-finite, $R_{2,1}$ is finite.

Observe that if $P_{i_1} \in R_{1,1}$ and $\mathrm{Hom}_{R_{1,1}}(P_{i_1}, P_{i_0}) \neq 0$ then there is a unique $Q_{j_1} \in R_{2,1}$ with $\mathrm{Hom}_{R_{2,1}}(Q_{j_1}, Q_{j_0}) \neq 0$ and $LF_{\lambda,1}(P_{i_1}) \cong F_{\lambda,2}(Q_{j_1})$. Indeed, if there are $Q_{j_1}, Q_{j_2} \in R_{2,1}$ with $\mathrm{Hom}_{R_{2,1}}(Q_{j_l}, Q_{j_0}) \neq 0$ and $LF_{\lambda,1}(P_{i_1}) \cong F_{\lambda,2}(Q_{j_l})$, $l = 1, 2$, then there is $z \in \mathbb{Z}$ such that $\nu_{R_2}^{z}(Q_{j_1}) = Q_{j_2}$. Furthermore, there are $0 \neq r_{j_l} : Q_{j_l} \to Q_{j_0}$, $l = 1, 2$, such that r_{j_l} factorize through $\mathrm{rad}(Q_{j_0})$ by the definition of $R_{2,1}$. Hence $\mathrm{top}(Q_{j_l})$ are direct summands in $\mathrm{top}(\mathrm{rad}(Q_{j_0}))$. Then for $z > 0$ we get a sequence $Q'_1, \ldots, Q'_z$ of indecomposable projective R_2-modules such that $\mathrm{soc}(Q'_m) \cong \mathrm{top}(Q'_{m-1})$, $m = 2, \ldots, z$, $\mathrm{top}(Q_{j_1}) \cong \mathrm{soc}(Q'_1)$, $\mathrm{top}(Q'_z) \cong \mathrm{soc}(Q_{j_2})$. But $\mathrm{top}(Q_{j_0}) \in \mathrm{supp}(Q'_1)$, R_2 is not triangular, which contradicts our assumption. Similarly we obtain a contradiction if $z < 0$. Thus $z = 0$ and $Q_{j_1} = Q_{j_2}$.

Dually one proves that if $P_{i'_1} \in R_{1,1}$ and $\mathrm{Hom}_{R_{1,1}}(P_{i_0}, P_{i'_1}) \neq 0$ then there is a unique $Q_{j'_1} \in R_{2,1}$ with $\mathrm{Hom}_{R_{2,1}}(Q_{j_0}, Q_{j'_1}) \neq 0$ and $LF_{\lambda,1}(P_{i'_1}) \cong F_{\lambda,2}(Q_{j'_1})$.

Now we define a functor $F_1 : R_{1,1} \to R_{2,1}$ putting $F_1(P_{i_0}) = Q_{j_0}$, $F_1(P_{i_1}) = Q_{j_1}$, $F_1(P_{i'_1}) = Q_{j'_1}$ for the objects of $R_{1,1}$. If $P, P' \in R_{1,1}$ then $\mathrm{Hom}_{R_{1,1}}(P, P')$ either consists of isomorphisms (if $P = P'$) or is generated by the above t. If $P = P'$ then $\mathrm{Hom}_{R_{1,1}}(P, P) \cong K \cdot \mathrm{id}_P \cong K \cdot \mathrm{id}_{F_{\lambda,1}(P)}$ as K-spaces. Then $K \cdot \mathrm{id}_{F_{\lambda,1}(P)} \cong K \cdot \mathrm{id}_{LF_{\lambda,1}(P)} \cong K \cdot \mathrm{id}_{F_1(P)}$ as K-spaces. Hence for every $f \in \mathrm{Hom}_{R_{1,1}}(P, P)$ there is exactly one $r \in \mathrm{Hom}_{R_{2,1}}(F_1(P), F_1(P))$ such that $LF_{\lambda,1}(f) = F_{\lambda,2}(r)$. Thus we put $F_1(f) = r$. If $P \neq P'$ then we construct F_1 for the morphisms of the form $t = t''t'$, where $t' : P \to \mathrm{rad}(P')$ satisfies $\pi t' \neq 0$ for the canonical epimorphism $\pi : \mathrm{rad}(P') \to \mathrm{top}(\mathrm{rad}(P'))$ and $t'' : \mathrm{rad}(P') \to P'$ is inclusion. For such a t, there is a unique $r : F_1(P) \to F_1(P')$ in $\mathrm{Hom}_{R_{2,1}}(F_1(P), F_1(P'))$ such that $LF_{\lambda,1}(t) = F_{\lambda,2}(r)$. Indeed, if r_1, r_2 satisfy $LF_{\lambda,1}(t) = F_{\lambda,2}(r_1) = F_{\lambda,2}(r_2)$ then there are $r'_1, r'_2 : F_1(P) \to \mathrm{rad}(F_1(P'))$ such that $\pi' r'_1, \pi' r'_2 \neq 0$ for the canonical projection $\pi' : \mathrm{rad}(F_1(P')) \to \mathrm{top}(\mathrm{rad}(F_1(P')))$. Furthermore, for the inclusion $r'' : \mathrm{rad}(F_1(P')) \to F_1(P')$ we have $r_1 = r''r'_1$ and $r_2 = r''r'_2$. But if $r'_1 \neq r'_2$ then $F_{\lambda,2}(r'_1) \neq F_{\lambda,2}(r'_2)$, because R_2 is triangular and $F_{\lambda,2}$ is induced by the action of (ν_{R_2}). Thus $F_{\lambda,2}(r_1) \neq F_{\lambda,2}(r_2)$ for $r_1 \neq r_2$. Consequently, $r_1 = r_2$ if $F_{\lambda,2}(r_1) = F_{\lambda,2}(r_2)$. Then we put $F_1(t) = r$. If $t = t_1 t_2$ is a composition of either an isomorphism and a morphism of the above form or two morphisms of the above form then we put $F_1(t) = F_1(t_1)F_1(t_2)$. Finally, we extend F_1 linearly to a K-functor. It is clear by the above considerations that we have obtained a functor $F_1 : R_{1,1} \to R_{2,1}$ which is dense and fully faithful. Thus F_1 yields an equivalence of categories.

Assume now that we defined a subcategory $R_{1,n}$ in R_1 such that for every pair P, P' of objects from $R_{1,n}$ either $P = P'$ and $\mathrm{Hom}_{R_{1,n}}(P, P')$ consists only of automorphisms, or $P \neq P'$ and $\mathrm{Hom}_{R_{1,n}}(P, P')$ is generated by the morphisms of the form $t = t_s \ldots t_2 t_1$ such that:

(i) $t_l : P_l \to P_{l+1}$ for some objects $P_1, \ldots, P_{s+1}$ of $R_{1,n}$, where $P_1 = P$, $P_{s+1} = P'$;

(ii) $t_l = t''_l t'_l$, $l = 1, \ldots, s$, and $t'_l : P_l \to \mathrm{rad}(P_{l+1})$ satisfies $\pi_{l+1} t'_l \neq 0$ for the canonical epimorphism $\pi_{l+1} : \mathrm{rad}(P_{l+1}) \to \mathrm{top}(\mathrm{rad}(P_{l+1}))$;

(iii) $t''_l : \mathrm{rad}(P_{l+1}) \to P_{l+1}$ is inclusion for $l = 1, \ldots, s$.

Moreover, assume that we have defined a subcategory $R_{2,n}$ of R_2 satisfying the above conditions for morphisms, and a functor $F_n : R_{1,n} \to R_{2,n}$ which is a K-linear equivalence and maps the generators of $\mathrm{Hom}_{R_{1,n}}(P, P')$ to the generators of $\mathrm{Hom}_{R_{2,n}}(F_n(P), F_n(P'))$.

Define a subcategory $R_{1,n+1}$ of R_1 in the following way. The objects of $R_{1,n+1}$ are those of $R_{1,n}$ and additionally the objects P of R_1 such that either there is a nonzero morphism $t : P \to P'$ with P' in $R_{1,n}$ and $t = t''t'$, where $t' : P \to \mathrm{rad}(P')$ satisfies $\pi' t' \neq 0$ for the canonical projection $\pi' : \mathrm{rad}(P') \to \mathrm{top}(\mathrm{rad}(P'))$ and $t'' : \mathrm{rad}(P') \to P'$ is inclusion, or there is a nonzero morphism $h : P' \to P$ with $P' \in R_{1,n}$ and $h = h''h'$, where $h' : P' \to \mathrm{rad}(P)$ satisfies $\pi h' \neq 0$ for the canonical epimorphism $\pi : \mathrm{rad}(P) \to \mathrm{top}(\mathrm{rad}(P))$ and $h'' : \mathrm{rad}(P) \to P$ is inclusion. For every P, P'' from $R_{1,n+1}$, $\mathrm{Hom}_{R_{1,n+1}}(P, P'')$ is generated by the isomorphisms between P and P'' and the compositions $h = h_s \dots h_1$ which satisfy conditions (i)–(iii) above.

In the same way we define a subcategory $R_{2,n+1}$ of R_2. Then repeating the arguments used for $R_{1,1}$ and $R_{2,1}$ we find that for every $P \in R_{1,n+1}$ such that there is a nonzero morphism $t : P \to P'$ with $P' \in R_{1,n}$ there is a unique $Q \in R_{2,n+1}$ such that there is a nonzero morphism $r : Q \to F_n(P')$ in $R_{2,n+1}$ and $LF_{\lambda,1}(P) \cong F_{\lambda,2}(Q)$. Furthermore, for every $P \in R_{1,n+1}$ such that there is a nonzero morphism $h : P' \to P$ in $R_{1,n+1}$ with $P' \in R_{1,n}$ there is a unique $Q \in R_{2,n+1}$ such that there is a nonzero morphism $r : F_n(P') \to Q$ in $R_{2,n+1}$ and $LF_{\lambda,1}(P) \cong F_{\lambda,2}(Q)$. Moreover, we also have the same uniqueness for generating morphisms $t : P \to P'$ and $h : P' \to P$ with $P' \in R_{1,n}$ and $P \in R_{1,n+1} \setminus R_{1,n}$.

Thus we define $F_{n+1} : R_{1,n+1} \to R_{2,n+1}$ in the following way. For every $P \in R_{1,n+1} \setminus R_{1,n}$ we put $F_{n+1}(P) = Q$, where Q is as above. For every $P' \in R_{1,n}$ we put $F_{n+1}(P') = F_n(P')$. For $P, P' \in R_{1,n+1}$ with $P \in R_{1,n+1} \setminus R_{1,n}$ and $P' \in R_{1,n}$, if $t : P \to P'$ is a generator of $\mathrm{Hom}_{R_{1,n+1}}(P, P')$ then we put $F_{n+1}(t) = r$, where r is the uniquely determined generator of $\mathrm{Hom}_{R_{2,n+1}}(F_{n+1}(P), F_{n+1}(P'))$. If $h : P' \to P$ is a generator of $\mathrm{Hom}_{R_{1,n+1}}(P', P)$ then we put $F_{n+1}(h) = r$, where r is the uniquely determined generator of $\mathrm{Hom}_{R_{2,n+1}}(F_{n+1}(P'), F_{n+1}(P))$. If $t : P \to P'$ is a generator of $\mathrm{Hom}_{R_{1,n+1}}(P, P')$ with $P, P' \in R_{1,n}$ then we put $F_{n+1}(t) = F_n(t)$. If $t : P \to P''$ is an isomorphism with $P, P'' \in R_{1,n+1} \setminus R_{1,n}$ then we put $F_{n+1}(t) = r$, where $LF_{\lambda,1}(t) = F_{\lambda,2}(r)$. Finally, we extend F_{n+1} to a K-linear functor $F_{n+1} : R_{1,n+1} \to R_{2,n+1}$ which is dense and fully faithful. Thus F_{n+1} yields an equivalence of categories.

Consequently, we construct inductively a functor $F : R_1 \to R_2$ which is dense and fully faithful since R_1 and R_2 are connected and locally support-finite. The proposition follows.

4. The repetitive algebras of canonical tubular algebras

4.1. For a locally bounded K-category R, we shall not distinguish between an indecomposable R-module, its isomorphism class and the vertex of Γ_R corresponding to it. Moreover, we denote by Γ_R^{s} the stable quiver of Γ_R obtained from Γ_R by removing the τ_R-orbits of all projective modules, all injective modules and the arrows attached to them. Following [7], a component $\mathbf{T}$ of Γ_R (respectively, of Γ_R^{s}) is said to be a *tube* if $\mathbf{T}$ contains a cyclic path and its geometrical realization $|\mathbf{T}|$ is homeomorphic to $S^1 \times \mathbb{R}_0^+$, where

S^1 is the unit circle and $\mathbb{R}_0^+$ is the set of nonnegative real numbers. A *stable tube* of rank $n \geq 1$ is a translation quiver of the form $\mathbb{Z}\mathbf{A}_\infty/(\tau^n)$. The stable tubes of rank one are said to be *homogeneous*. A family $\mathcal{T} = (T_i)_{i\in I}$ of tubes in Γ_R (respectively, in Γ_R^{s}) is said to be *standard* if the full subcategory of $\mathrm{mod}(R)$ (respectively, of $\underline{\mathrm{mod}}(R)$) is equivalent to the mesh-category $K(\mathcal{T})$ of $\mathcal{T}$. Finally, we say that a family of tubes $\mathcal{T} = (T_i)_{i\in I}$ in Γ_R (respectively, in Γ_R^{s}) *separates* a family of components $\mathcal{X}$ from a family of components $\mathcal{Y}$ if for any $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $i \in I$, every morphism from X to Y in $\mathrm{mod}(R)$ (respectively, in $\underline{\mathrm{mod}}(R)$) can be factorized through a module Z in the additive category $\mathrm{add}(T_i)$ and there is no nonzero morphism from Y to X in $\mathrm{mod}(R)$ (respectively, in $\underline{\mathrm{mod}}(R)$).

4.2. Let A be a canonical tubular algebra of type $\mathbb{T} = (n_1, \ldots, n_t) = (2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. To describe the structure of $\underline{\mathrm{mod}}(\widehat{A})$ we need the following types of tubular families. A family $\mathcal{T} = (T_\mu)_{\mu\in\mathbb{P}_1(K)}$, $\mathbb{P}_1(K) = K \cup \{\infty\}$, of tubes in $\Gamma_{\hat{A}}$ is said to be a *tubular* $\mathbb{P}_1(K)$-*family of type* $\mathbb{T}$ if the following conditions are satisfied:

(1) The stable part $\mathcal{T}^{\mathrm{s}}$ of $\mathcal{T}$ is a disjoint union of stable tubes $\mathcal{T}_\mu^{\mathrm{s}}$, $\mu \in \mathbb{P}_1(K)$, such that t of these tubes have ranks $n_1, \ldots, n_t$, and the remaining ones are homogeneous.

(2) One of the following conditions holds:

(a) All tubes T_μ, $\mu \in \mathbb{P}_1(K)$, are stable.

(b) The tubes T_μ, $\mu \in K$, are stable and T_∞ admits a projective-injective vertex.

(c) There are $\mu_1, \ldots, \mu_t \in \mathbb{P}_1(K)$ such that the tubes T_μ with $\mu \neq \mu_1, \ldots, \mu_t$ are stable and for each $1 \leq i \leq t$, the tube T_{μ_i} admits $n_i - 1$ projective-injective vertices.

4.3. Proposition. *Let A be a canonical tubular algebra of type $\mathbb{T}$. Then*

(a) $\Gamma_{\hat{A}} = \bigsqcup_{q\in\mathbb{Q}} \mathcal{T}_q$ *where, for each $q \in \mathbb{Q}$, $\mathcal{T}_q$ is a tubular $\mathbb{P}_1(K)$-family $\mathcal{T}_q(\mu)$, $\mu \in \mathbb{P}_1(K)$.*

(b) *For every $q \in \mathbb{Q}$, $\mathcal{T}_q$ separates $\bigsqcup_{q<i} \mathcal{T}_q$ from $\bigsqcup_{i<q} \mathcal{T}_q$.*

(c) *For each $q \in \mathbb{Q} \setminus \mathbb{Z}$, $\mathcal{T}_q$ is a standard family of stable tubes.*

(d) *For each $q \in \mathbb{Z}$, $\mathcal{T}_q$ contains finitely many projective $\widehat{A}$-modules.*

Proof. This result was obtained in [10].

4.4. In [10] the following increasing map $\sigma : \mathbb{Q} \to \mathbb{Q}$ was defined:

$$\sigma\Big(m + \frac{r}{s}\Big) = \begin{cases} m + 1 + \dfrac{s-r}{2s-3r} & \text{if } 0 \leq 2r \leq s, \\ m + 2 + \dfrac{2r-s}{3r-s} & \text{if } 1 \leq r < s \leq 2r. \end{cases}$$

We have the following lemma.

LEMMA. *Let A be a canonical tubular algebra of type $\mathbb{T}$. Then*

(a) *For every indecomposable nonprojective $\widehat{A}$-module M in $\mathcal{T}_q$ the module $\Omega_{\hat{A}}(M)$ belongs to $\mathcal{T}_{\sigma(q)}$.*

(b) *For every $q \in \mathbb{Z}$, $\mathcal{T}_{q+1/2}$ contains simple $\widehat{A}$-modules.*

(c) *If $0 \neq \underline{f} : X \to Y$ for two indecomposable nonprojective $\widehat{A}$-modules X, Y with $X \in \mathcal{T}_{q_1}$, $Y \in \mathcal{T}_{q_2}$ then $q_2 - q_1 \leq 1\frac{1}{2}$.*

P r o o f. (a) is a consequence of [10; 4.9]. (b) is a consequence of Proposition 4.3 and (a). In order to check (c) observe that if $0 \neq \underline{f} : X \to Y$ then there is a nonzero morphism $\underline{h} : \tau_{\hat{A}}^{-1}\Omega_{\hat{A}}(Y) \to X$ with $\underline{f}\underline{h} = 0$ by [4; Proposition 4.1]. Thus (c) follows from (a).

4.5. If R is a locally bounded K-category which is stably equivalent to the repetitive algebra $\widehat{A}$ of a canonical tubular algebra A then the stable Auslander–Reiten quiver Γ_R^{s} of R is isomorphic to $\Gamma_{\hat{A}}^{\mathrm{s}}$. Thus $\Gamma_R^{\mathrm{s}} = \bigsqcup_{q\in\mathbb{Q}} \mathcal{T}'_q$, and we have the following.

LEMMA. *For every $r \in \mathbb{Q}$ there are only finitely many isomorphism classes of simple R-modules in $\bigsqcup_{q\in[r,r+3]\cap\mathbb{Q}} \mathcal{T}'_q$.*

P r o o f. Suppose to the contrary that there are infinitely many nonisomorphic simple R-modules in $\bigsqcup_{q\in[r_0,r_0+3]\cap\mathbb{Q}} \mathcal{T}'_q$ for some $r_0 \in \mathbb{Q}$. Fix an equivalence $\Phi : \underline{\mathrm{mod}}(\widehat{A}) \to \underline{\mathrm{mod}}(R)$. It is easily seen that there is some $s_0 \in \mathbb{Q}$ such that for every indecomposable nonprojective $X \in \bigsqcup_{q\in[s_0,s_0+3]\cap\mathbb{Q}} \mathcal{T}_q$ we have $\Phi(X) \in \bigsqcup_{q\in[r_0,r_0+3]\cap\mathbb{Q}} \mathcal{T}'_q$. Moreover, if $S_1, \ldots, S_n$ are all pairwise nonisomorphic simple $\widehat{A}$-modules such that the top of every $X \in \bigsqcup_{q\in[s_0,s_0+3]\cap\mathbb{Q}} \mathcal{T}_q$ belongs to $\mathrm{add}(S_1, \ldots, S_n)$ then there is an epimorphism $f : X \to S$ with $S \cong S_i$, for some $i = 1, \ldots, n$. Clearly $\underline{f} \neq 0$ by [17; Lecture 3], and so $0 \neq \Phi(\underline{f}) : \Phi(X) \to \Phi(S)$. Therefore for every simple R-module T contained in $\bigsqcup_{q\in[r_0,r_0+3]\cap\mathbb{Q}} \mathcal{T}'_q$ there is an injection of T into some of the $\Phi(S_1), \ldots, \Phi(S_n)$. Moreover, for every such T there is an injection into $\Phi(S_1) \oplus \ldots \oplus \Phi(S_n)$, which contradicts the finite-dimensionality of $\Phi(S_1) \oplus \ldots \oplus \Phi(S_n)$. Consequently, the lemma follows.

4.6. COROLLARY. *For every $r \in \mathbb{Q}$ there are only finitely many isomorphism classes of R-modules of the form $P/\mathrm{soc}(P)$ in $\bigsqcup_{q\in[r,r+3]\cap\mathbb{Q}} \mathcal{T}'_q$, where P ranges over pairwise nonisomorphic indecomposable projective R-modules.*

P r o o f. Obvious by Lemma 4.5, because $P/\mathrm{soc}(P) \cong \tau_R^{-1}\Omega_R(\mathrm{top}(P))$.

4.7. PROPOSITION. *Let A be a canonical tubular algebra. If R is a locally bounded K-category which is stably equivalent to the repetitive algebra $\widehat{A}$ of A, then R is locally support-finite and selfinjective. Moreover*, (ν_R) *acts freely on R.*

P r o o f. A more general version of this proposition is proved in [19; Proposition 1]. But under our special assumptions we can give a simple proof which we present for the convenience of the reader.

We shall show that there is a natural number d such that for any indecomposable R-module M there are at most d pairwise nonisomorphic indecomposable projective R-modules $P_1, \ldots, P_d$ with $\mathrm{Hom}_R(P_i, M) \neq 0$, $i = 1, \ldots, d$. Let d denote the number of nonisomorphic indecomposable projective R-modules P such that $P/\mathrm{soc}(P) \in \bigsqcup_{q\in[r,r+3]\cap\mathbb{Q}} \mathcal{T}'_q$. If M is an indecomposable nonprojective R-module then $M \in \mathcal{T}'_{q_0}$. For every indecomposable projective P with $\mathrm{Hom}_R(P, M) \neq 0$ we have $\mathrm{Hom}_R(P/\mathrm{soc}(P), M) \neq 0$. If we consider $0 \neq f : P/\mathrm{soc}(P) \to M$ then $f = f_2 f_1$ with $f_1 : P/\mathrm{soc}(P) \to \mathrm{im}(f)$ an epimorphism and $f_2 : \mathrm{im}(f) \to M$ a monomorphism. Thus $\underline{f_1} \neq 0 \neq \underline{f_2}$ and we infer by Lemma 4.4(c) that $P/\mathrm{soc}(P) \in \bigsqcup_{q\in[q_0-3,q_0]\cap\mathbb{Q}} \mathcal{T}'_q$. Since d is finite by Corollary 4.6, it satisfies the above condition. The group (ν_R) acts freely on R by Lemma 3.2 since $\tau^{-1}_{\widehat{A}}(M) \not\cong \Omega^{-2}_{\widehat{A}}(M)$ for every indecomposable nonprojective $\widehat{A}$-module M by Lemma 4.4. Consequently, the proposition follows, because the selfinjectivity of R is clear.

5. Proof of the theorem

5.1. We start this section with the following simple fact.

LEMMA. *Let A be a canonical tubular algebra. If Λ is a locally bounded K-category which is stably equivalent to the repetitive algebra $\widehat{A}$ then Λ is triangular.*

P r o o f. It is sufficient to show that there is no oriented cycle of nonisomorphisms in Γ_Λ between projective vertices. Suppose to the contrary that there is a cycle of nonzero nonisomorphisms $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{t-1}} P_t \xrightarrow{f_t} P_1$ between indecomposable projective Λ-modules. Then by 4.5, Corollary 4.6 and Proposition 4.3, all $P_1, \ldots, P_t$ are contained in the same component $\mathcal{C}$ of Γ_Λ and f_i, $i = 1, \ldots, t$, do not factorize through a module from $\mathrm{add}(\Gamma_\Lambda \setminus \mathcal{C})$. But we deduce from Propositions 4.7 and 3.7 that $\widehat{A}/(\nu_{\widehat{A}})$ is stably equivalent to $\Lambda/(\nu_\Lambda)$. Thus there is a cycle of nonzero nonisomorphisms $Q_1 \xrightarrow{r_1} Q_2 \xrightarrow{r_2} \ldots \xrightarrow{r_t} Q_1$ in a component $\mathcal{C}_1$ of $\Gamma_{\Lambda/(\nu_\Lambda)}$ between projective $\Lambda/(\nu_\Lambda)$-modules such that r_i, $i = 1, \ldots, t$, do not factorize through a module from $\mathrm{add}(\Gamma_{\Lambda/(\nu_\Lambda)} \setminus \mathcal{C}_1)$. Furthermore, we know from [15; Theorem] that $\Lambda/(\nu_\Lambda) \cong T(B)$ for a tubular algebra B. But in $\Gamma_{T(B)}$ there is no such cycle, hence Λ is triangular.

5.2. P r o o f o f T h e o r e m. The "only if" part is due to Wakamatsu [21]. Since a tubular algebra is tilting-cotilting equivalent to a canonical tubular algebra, we may assume that A is canonical. Assume that Λ is a locally bounded K-category which is stably equivalent to the repetitive algebra $\widehat{A}$. Then Λ is selfinjective locally support-finite by Proposition 4.7. Moreover, Λ is triangular by Lemma 5.1. Thus we infer by Proposition 3.7 that $\widehat{A}/(\nu_A) \cong T(A)$ is stably equivalent to $\Lambda/(\nu_\Lambda)$. Then we deduce from [15; Theorem] that there is a tubular algebra B which is tilting-cotilting equivalent to A such that $\Lambda/(\nu_\Lambda) \cong T(B) \cong \widehat{B}/(\nu_B)$. Since $\widehat{B}$ is triangular, we conclude by Proposition 3.8 that $\Lambda \cong \widehat{B}$ and the theorem follows.

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