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FUNDAMENTAL SOLUTIONS OF DIFFERENTIAL OPERATORS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE AND RELATED RIESZ TRANSFORMS

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0. Introduction. In this paper we study fundamental solutions for second order differential operators on connected, simply connected homogeneous manifolds of negative curvature. Such manifolds are solvable Lie groups S with a left-invariant Riemannian structure. By a result of Heintze [He], S is a semidirect product of its maximal nilpotent normal subgroup $N = \exp \mathcal{N}$ and $A = \mathbb{R}^+$ with the following property:

There is an H in the Lie algebra $\mathcal{A}$ of A such that the real parts of the eigenvalues of $\mathrm{ad}_H \in \mathrm{End}(\mathcal{N})$ *are all strictly positive.*

Moreover, every solvable Lie group with this property admits a left-invariant Riemannian structure with strictly negative curvature.

These groups, which will be called here *Heintze groups*, are very interesting objects from the point of view of harmonic analysis. As a particular case we recognize rank one symmetric spaces and, more generally, harmonic spaces. Harmonic analysis on harmonic spaces has been intensively studied by various authors ([ADY], [A], [ACD], [CDKR], [DR1], [DR2], [Di], [R]). The approach developed in the papers mentioned above brings new ideas also to noncompact rank one symmetric spaces incorporating them into a new picture. On harmonic manifolds there is a notion of radiality [DR1], spherical analysis is a particular case of the Jacobi function analysis [ADY], the Poisson kernel and the fundamental solution for the Laplace–Beltrami operator are given by formulas [DR1] and the heat kernel has sharp lower and upper estimates [ADY]. All this makes harmonic analysis there "very concrete" in a sense.

Nothing of that is available on general NA groups with N being a homogeneous group and A not necessarily one-dimensional. The natural questions considered in these two extreme settings are clearly different (see [DH]). Heintze groups are somewhere in between. No radiality or concrete formulas

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are available there, but some conjectures can be made on what is known for harmonic spaces, as well as analogous results (or generalizations) can be proved. A step in this direction was made in [DHZ], where sharp pointwise estimates for the Poisson kernel and its derivatives were obtained.

In the present paper we apply Ancona's potential theory of negatively curved manifolds to S. This is a theory which provides tools to describe global behavior of potentials. We use it to obtain sharp poinwise estimates for fundamental solutions for a large class of left-invariant subelliptic operators. The estimates we get imply weak type $(1,1)$ of the corresponding Riesz transforms of the first and the second order.

Let $\pi : S \to A$, $\pi(xa) = a$, be the canonical homomorphism of S onto A. We consider a left-invariant second order operator

$$L = Y_1^2 + \ldots + Y_p^2 + Y + \gamma,$$

where $Y_1, \ldots, Y_p$ generate the Lie algebra $\mathcal{S}$ of S, $\pi(L) = (a\partial_a)^2 - \alpha a\partial_a + \gamma$ and $\gamma < \alpha^2/4$. Under this condition, $L+\lambda I$ for $\lambda \le \alpha^2/4-\gamma$ admits a global Green function G^λ. Therefore Ancona's approach to the Martin boundary theory on negatively curved manifolds can be used. Following Ancona we formulate certain boundary Harnack inequalities which give sharp pointwise estimates from above and below for G^λ, $\lambda < \alpha^2/4 - \gamma$ (Theorem (2.21)). It is remarkable that using Ancona's theory we do not need any further assumptions on L like symmetry or L being the Laplace–Beltrami operator with respect to the underlying Riemannian metric. Also, the $\mathcal{N}$-part of Y is arbitrary.

In fact, all the work can be reduced to the case $\gamma = 0$ thanks to a simple conjugation of the operator, and then the case $\alpha = Q$ becomes the most interesting. It is so not only because the Laplace–Beltrami operator on S has $\alpha = Q$, but also, because this is somehow a limit case. If $\alpha > Q$ or $1 < p < \infty$ then the operator $f \to f * G$ is bounded on $L^p(m_{\mathrm{L}})$, and if λ belongs to the $L^p(m_{\mathrm{L}})$ spectrum of $-L$ then $\Re\lambda \ge (Q/p)\alpha - Q^2/p^2 > 0$. This follows from a very simple calculation, which does not require any pointwise estimates for G. If $\alpha = Q$ then $f \to f * G$ is no longer bounded on $L^1(m_{\mathrm{L}})$ but it is of weak type $(1,1)$. To prove this we use essentially our pointwise estimates for G. If $\alpha < Q$, then $f \to f * G$ is not of weak type $(1,1)$.

This is an interesting phenomenon, which gives weak type $(1,1)$ of the first and second order Riesz transforms

$$f \to \nabla^j(-L)^{j/2} f, \quad j = 1, 2,$$

for L elliptic with $\gamma = 0$, $\alpha = Q$, i.e. in the case where L has a spectral gap on $L^2(m_{\mathrm{L}})$. (For the first order Riesz transforms we assume additionally that $Y = -Qa\partial_a$ and so L is selfadjoint on $L^2(m_{\mathrm{L}})$.) Indeed, the local part is standard, and by the Harnack inequality the kernel at infinity can be dominated by G. This gives also a new proof in the case of the Laplace–Beltrami operator on harmonic spaces, namely, a proof which does not require heat kernel estimates as the one presented in [ADY].

The problem of L^p, $p > 1$, boundedness of the Riesz transforms corresponding to the Laplace–Beltrami operator on a Riemannian manifold is solved in a quite general setting. They are bounded on Riemannian manifolds with Ricci curvature bounded from below [Ba], [L1]. This is not the case with weak type $(1,1)$ and the problem is still open. To our knowledge, the best result about Riesz transforms of integrable functions on Riemannian manifolds with bounded curvature tensor together with its first and second derivatives has been obtained by N. Lohoué [L2] and it says that

$$\mu(\{x : |\nabla(-L)^{1/2} f(x)| > \beta\}) \le c\|f\|_{L^1(\mu)}(1 + |\log \beta|^{1/2})/\beta,$$

where μ is the Riemannian volume element on the manifold and L the Laplace–Beltrami operator. Heintze groups give a partial answer to the question of weak type $(1,1)$ of Riesz transforms for the Laplace–Beltrami operator.

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1. Preliminaries. Let $\mathcal{S}$ be a solvable Lie algebra which is the sum $\mathcal{S} = \mathcal{N} \oplus \mathcal{A}$ of its nilpotent ideal $\mathcal{N}$ and a one-dimensional algebra $\mathcal{A} = \mathbb{R}$. We assume that there is $H \in \mathcal{A}$ such that the real parts of all the eigenvalues of $\mathrm{ad}_H : \mathcal{N} \to \mathcal{N}$ are positive.

Let N, A, S be the connected and simply connected Lie groups whose Lie algebras are $\mathcal{N}$, $\mathcal{A}$, $\mathcal{S}$ respectively. Then $S = NA$ is a semidirect product of N and $A = \mathbb{R}^+$.

We consider a left-invariant second order operator

$$L = Y_1^2 + \ldots + Y_p^2 + Y + \gamma, \tag{1.1}$$

where $Y_1, \ldots, Y_p$ generate the Lie algebra $\mathcal{S}$. It follows from elementary linear algebra that L can be written in the form

$$L = \beta(H + Y_0')^2 + \sum_{j=1}^{m} Y_j'^2 + Y + \gamma,$$

where $Y_0', \ldots, Y_m'$ are left-invariant vector fields on S such that $Y_0'(e), \ldots$ $\ldots, Y_m'(e) \in \mathcal{N}$. We may assume $\beta = 1$.

The decomposition of S into a semidirect product of the maximal nilpotent normal subgroup N and $A = \mathbb{R}^+$ is not unique, i.e. there is no canonical choice of A. We are going to make use of this fact and select A in a convenient way.

Let $\mathcal{A}' = \operatorname{lin}(H+Y_0')$. Clearly the real parts of the eigenvalues of $\operatorname{ad}_{H+Y_0'}$ are again strictly positive.

Decomposing $s \in S$ as

$$s = xa, \quad x \in N, \quad a = \exp((\log a)(H + Y_0'))$$

we have $S = N \exp \mathcal{A}'$ and for an $\alpha \in \mathbb{R}$,

$$L = \gamma + (a\partial_a)^2 - \alpha a\partial_a + \sum_{i=1}^{m} \Phi_a(X_i)^2 + \Phi_a(X), \tag{1.2}$$

where $\Phi_a = \operatorname{Ad}_{\exp((\log a)(H+Y_0'))}$ and $X, X_1, \ldots, X_m$ are left-invariant vector fields on N.

L satisfies the following Harnack inequality [VSC]:

For every open set Ω, every compact set $K \subset \Omega$, every point $x \in \Omega$ and every multiindex I there is a constant c such that

$$\sup_{y\in K} |\partial^I f(y)| \le c f(x) \tag{1.3}$$

whenever $f \ge 0$ and $Lf = 0$ in Ω.

Since L is left-invariant, if we take $x_0\Omega$, x_0K, x_0x instead of Ω, K, x we have (1.3) with the constant c independent of $x_0 \in S$.

We are also going to use a parabolic Harnack inequality, which is satisfied by $L - \partial_t$ [VSC]:

For every open set Ω, every compact set $K \subset \Omega$, every $t_1 < t_2 < t_3 < t_4$ and every multiindex I there is a constant c such that

$$\sup_{y\in K} |\partial^I f(y, t_2)| \le c \inf_{y\in K} f(x, t_3) \tag{1.4}$$

whenever $f \ge 0$ and $(L - \partial_t)f = 0$ in $\Omega \times (t_1, t_4)$.

Again we will profit from the left-invariance of the above Harnack inequality.

Let μ_t be the semigroup of probability measures generated by L. The right convolution with μ_t,

$$T_t f(x) = \int_S f(xy^{-1})\, d\mu_t(y),$$

defines a strongly continuous semigroup of bounded operators on L^p spaces, $1 \le p \le \infty$, both with respect to the left and to the right Haar measures. Let $\mu_t = p_t\, dm_{\mathrm{R}}$. Then

$$T_t f(x) = \int_S f(xy^{-1}) p_t(y)\, dm_{\mathrm{R}}(y) = \int_S f(y) p_t(y^{-1}x)\, dm_{\mathrm{L}}(y). \tag{1.5}$$

$p_t(x)$ is a C^∞ function on $S\times\mathbb{R}^+$ and $(L-\partial_t)p_t(x)=0$. Assume for a while that $\gamma=0$. Let

$$K_1=\int_0^\infty e^{-t}\mu_t\,dt.$$

Since the right random walk with the law K_1 is transient [C], it follows that

$$G=\int_0^\infty \mu_t\,dt=\sum_{n\ge 1}K_1^{*n}$$

is a Radon measure. Moreover, G does not have an atom at e. The density of G with respect to the right Haar measure will be denoted also by G, i.e.

$$G(x)=\int_0^\infty p_t(x)\,dt. \tag{1.6}$$

Then G is a fundamental solution of L, i.e.

$$LG=-\delta_e \tag{1.7}$$

in the sense of distributions. Since S is not a unimodular group we must choose a measure to define derivatives of a distribution. For a left-invariant vector field X on S and a distribution F on S, XF is defined by

$$\langle XF,\varphi\rangle=-\langle F,X^*\varphi\rangle,\qquad \varphi\in C_{\mathrm{c}}^\infty(S), \tag{1.8}$$

where

$$\langle X\varphi,\psi\rangle=\langle\varphi,X^*\psi\rangle$$

and

$$\langle\varphi,\psi\rangle=\int_S\varphi(x)\psi(x)\,dm_{\mathrm{L}}(x),\qquad \varphi,\psi\in C_{\mathrm{c}}^\infty(S).$$

A locally integrable function F is identified with the distribution Fdm_{L}. (1.7) follows from Harnack's inequality (1.4) which allows us to dominate derivatives of $\varphi*p_t$. More precisely, for a constant c we have

$$|\partial_t(\varphi*p_t)(x)|\le c\varphi*p_{t+1}(x) \tag{1.9}$$

for every $\varphi\in C_{\mathrm{c}}(S)$, $\varphi\ge 0$, every $x\in S$ and $t\ge 1$.

We come back to L with an arbitrary γ. Let L^* be defined by

$$\langle L\varphi,\psi\rangle=\langle\varphi,L^*\psi\rangle,\qquad \varphi,\psi\in C_{\mathrm{c}}^\infty(S).$$

Since $\Phi_a(X_i)^*=-\Phi_a(X_i)$ and $(a\partial_a)^*=-a\partial_a+Q$, we have

$$\begin{aligned}L^*=\gamma+Q^2-\alpha Q+(a\partial_a)^2+(\alpha-2Q)a\partial_a\\ +\sum_{i=1}^m\Phi_a(X_i)^2-\Phi_a(X).\end{aligned} \tag{1.10}$$

We assume that L in (1.2) satisfies

$$\gamma \le \alpha^2/4. \tag{1.11}$$

Then the same condition is satisfied by L^*, i.e.

$$\gamma + Q^2 - \alpha Q \le (\alpha - 2Q)^2/4.$$

If L satisfies (1.11) then it admits a global Green function. Indeed, let

$$L'f = a^{-\beta} L(a^\beta f) = Lf + 2\beta a\partial_a f + (\beta^2 - \alpha\beta)f, \qquad f \in C^\infty(S). \tag{1.12}$$

If $\gamma \le \alpha^2/4$ we can find β such that $\beta^2 + \alpha\beta + \gamma = 0$ and so the fundamental solution G for L can be easily expressed in terms of the fundamental solution for L', which exists by (1.6), (1.7). We write

$$G(x, y) = G(y^{-1}x). \tag{1.13}$$

Then G is the Green function for L in the sense of potential theory. Moreover, we have

$$T_t^* f(x) = \int_S f(xy^{-1}) \check{p}_t(y)\, dm_{\mathrm{R}}(y) \tag{1.14}$$

and hence $G^*(x) = \check{G}(x) = G(x^{-1})$, i.e. $G^*(x, y) = G(y, x)$.

Let T_t be the semigroup with the infinitesimal generator (1.2). In what follows we will need the norms of the operators T_t acting on $L^p(m_{\mathrm{L}})$, $1 \le p \le \infty$. Let $f, g \in L^p(m_{\mathrm{L}})$. A simple calculation shows that

$$|\langle T_t f, g\rangle| \le \|f\|_{L^p} \|g\|_{L^q} \int_S a^{-Q/p}\, d\mu_t.$$

The last integral can be easily computed. Since

$$\pi_A(L) = (a\partial_a)^2 - \alpha a\partial_a + \gamma,$$

we have

$$\int_S a^{-Q/p}\, d\mu_t = e^{\gamma t} \int_{-\infty}^{\infty} e^{-rQ/p} \frac{1}{\sqrt{4\pi t}} e^{-(r-\alpha t)^2/(4t)}\, dr = e^{(\gamma - (Q/p)\alpha + Q^2/p^2)t}.$$

Therefore,

$$\|T_t\|_{L^p(m_{\mathrm{L}}) \to L^p(m_{\mathrm{L}})} \le e^{(\gamma - (Q/p)\alpha + Q^2/p^2)t}. \tag{1.15}$$

2. Estimates for the Green function. In this chapter we give sharp pointwise estimates for the fundamental solution G of L. They will follow from certain boundary Harnack inequalities due to Ancona [A1] and adapted to our case as in [D]. We start with showing that we are in the framework of Ancona's theory. In fact, this has already been elaborated in [D] for the case when the action of H on $\mathcal{N}$ is diagonal, and there is no difference between this particular case and general $S = NA$ considered here.

First of all, the sheaf of $L + \lambda I$ harmonic functions satisfies Brelot's axioms (for the details we refer to [B] and to Brelot's potential theory as presented in [A2], [B1], [B2], [H], [HH]).

Next, by (1.12) the global Green function for $L + \lambda I$ with $\gamma + \lambda \le \alpha^2/4$ exists.

Finally, there is a basis $\mathcal{R}$ of open subsets of S which are Dirichlet regular with respect to all the operators $L + \lambda I$, $\gamma + \lambda \le \alpha^2/4$ (Theorem 5.2 of [B]).

To proceed with Ancona's theory we must guarantee a certain good behaviour of elements of $\mathcal{R}$ with respect to the distance. This is immediate if L is elliptic (the case considered in [A1]) because $\mathcal{R}$ contains Riemannian balls $B(x, r) = \{y \in S : \tau(x, y) < r\}$ of sufficiently small radii r. For L as in (1.1) we need the following lemma

(2.1) Lemma.(a) *There are positive constants c_1, c_2 and r_0 such that for every $0 < r < r_0$ there is a neighbourhood V_r of e which belongs to $\mathcal{R}$ and*

$$B(e, c_1 r) \subset V_r \subset B(e, c_2 r).$$

(b) *Given any compact set K there is $\Omega \in \mathcal{R}$ such that $K \subset \Omega$.* ■

The proof of Lemma (2.1) is elementary and, provided (1.2), the same as the proof of Lemma (4.1) in [D]. Since L is left-invariant we immediately get existence of $V_r(x) = xV_r \in \mathcal{R}$ such that

$$B(x, c_1 r) \subset V_r(x) \subset B(x, c_2 r). \tag{2.2}$$

Moreover, we have uniform estimates for $G^{\lambda}_{V_r(x)}$, which is the Green function for $L + \lambda I$ on $V_r(x)$.

(2.3) Lemma. *Given $r < r_0$ there is a constant c_r such that for $0 \le \lambda \le \alpha^2/4 - \gamma$,*

$$G^{\lambda}_{V_r(x)}(y, z) \ge c_r \qquad \text{when } y, z \in \overline{B}\big(x, \tfrac{1}{2} c_1 r\big), \tag{2.4}$$

$$G^{\lambda}_{V_r(x)}(y, z) \le c_r^{-1} \qquad \text{when } \tau(y, z) \ge \tfrac{1}{4} c_1 r. \ ■ \tag{2.5}$$

Clearly it is enough to prove (2.4) for $x = e$ and $\lambda = 0$, and (2.5) for $x = e$ and $\lambda = \alpha^2/4 - \gamma$, which is standard.

Let $\Phi : [0, \infty) \to [c_0, \infty)$ with $\Phi(0) = c_0$ be a positive, increasing function such that $\lim_{t\to\infty} \Phi(t) = \infty$. By a *$\Phi$-chain* we mean a sequence of open sets $V_1 \supset \ldots \supset V_m$ together with a sequence of points $x_i \in \partial \overline{V}_i$, $i = 1, \ldots, m$, such that for every i and every $z \in \partial V_{i+1}$,

$$\tau(z, \partial V_i) \ge \Phi(\tau(z, x_{i+1})) \tag{2.6}$$

and

$$c_0 \le \tau(x_i, x_{i+1}) \le c_0^{-1}. \tag{2.7}$$

Notice that $S \setminus \overline{V_1} \subset \ldots \subset S \setminus \overline{V_m}$ together with $x_1, \ldots, x_m$ is a Φ^*-chain with some Φ^* closely related to Φ. It is called a *dual chain*. Clearly after a small modification we may have both chains with the same Φ. A sequence of points $x_1, \ldots, x_m$ is called a *Φ-chain* if there exist open subsets $V_1, \ldots, V_m$ with $x_i \in \partial\overline{V_i}$ satisfying conditions (2.6), (2.7).

Existence of a global Green function and properties (2.2), (2.4), (2.5) allow us to proceed as in [A1] to get

(2.8) THEOREM. *There is a constant c depending only on L and Φ such that for every Φ-chain $x_1, \ldots, x_m$ and every $1 < k < m$,*

$$c^{-1}G(x_m, x_k)G(x_k, x_1) \le G(x_m, x_1) \le cG(x_m, x_k)G(x_k, x_1). \quad \blacksquare \tag{2.9}$$

The crucial point in the above theorem is that c does not depend on a particular sequence $x_1, \ldots, x_m$ provided it is a Φ-chain. As a consequence of Theorem (2.8) we obtain some boundary Harnack inequalities which are going to be our main tool in proving estimates for the Green function. Let $V_1 \supset V_2$ be two open sets and $B(p, r)$ a Riemannian ball included in $V_1 \setminus \overline{V_2}$. $(V_1, V_2, B(p, r))$ will be called a *(Φ, r)-triple* if for every $x \in \partial V_1$ and every $y \in \partial V_2$ there is a Φ-chain passing through x, p, y. Proceeding as in [A1], Theorem 2, we obtain

(2.10) THEOREM. *Given Φ and r there is a constant c such that for every (Φ, r)-triple $(V_1, V_2, B(p, r))$ and any nonnegative superharmonic functions f, g with the properties*

(a) *f is harmonic on the complement of $\overline{V_2}$ and f is dominated by a potential there*,

(b) *g is harmonic in $B(p, r)$*,

we have

$$\frac{f(x)}{f(p)} \le c\frac{g(x)}{g(p)} \quad \text{for } x \notin V_1. \quad \blacksquare \tag{2.11}$$

To get convenient boundary Harnack inequalities we must recognize appriopriate Φ-chains in S.

Given an arbitrary euclidean scalar product $(\cdot, \cdot)$ in $\mathcal{N}$ let

$$\langle X, Y\rangle_H = \int_0^\infty \langle e^{-t\,\mathrm{ad}_H} X, e^{-t\,\mathrm{ad}_H} Y\rangle\, dt \quad \text{and} \quad \|X\|_H = \sqrt{\langle X, X\rangle_H}.$$

We define a norm ϱ by

$$\varrho(\exp X) = (\inf\{a > 0 : \|e^{\log a\,\mathrm{ad}_H} X\|_H \ge 1\})^{-1}. \tag{2.12}$$

Since for $X \ne 0$, $\lim_{a\to\infty} \|e^{\log a\,\mathrm{ad}_H} X\|_H = \infty$, $\lim_{a\to 0} \|e^{\log a\,\mathrm{ad}_H}\|_H = 0$ and the function $a \to \|e^{\log a\,\mathrm{ad}_H} X\|_H$ is strictly increasing, for every $X \ne 0$ there is precisely one a such that $\|e^{\log a\,\mathrm{ad}_H} X\| = 1$. Moreover,

$$\varrho(\exp e^{\log a\,\mathrm{ad}_H} X) = a\varrho(\exp X). \tag{2.13}$$

Now we proceed as in [A1] and [D], and so it is convenient to write elements of S as $s = xa$, $x \in N$, $a \in A$. Therefore we will keep this notation until the end of this chapter. Let

$$T^d = \{xa : \varrho(x) < d,\ a < d\}$$

and for $s \in S$,

$$sT^d = \{sw : w \in T^d\}.$$

In particular, for $q \in A$,

$$qT^d = \{xa : \varrho(x) < qd,\ a < qd\} = T^{qd}.$$

It turns out that $s, sq, \ldots, sq^n$ together with $sT^d, sqT^d, \ldots, sq^nT^d$ is a Φ-chain with a Φ depending only on q and d. Since left translations are isometries it is enough to prove that for $x = e$ and $n = 1$. Then the statement follows from the following estimate for τ which is due to Guivarc'h [G]:

(2.14) LEMMA. *There is a constant c such that for every $x \in N$ and $a \in A$ we have*

$$c^{-1}(\log(1 + \varrho(x)) + |\log a|) \le \tau(xa) + 1 \le C(\log(1 + \varrho(x)) + |\log a| + 1). \quad \blacksquare \tag{2.15}$$

In fact, we have more:

(2.16) LEMMA. *Let $d_1 < d_2 < d_3$. Every $y \in \partial qT^{d_1}$ and every $z \in \partial qT^{d_3}$ can be joined by a Φ-chain passing through y, qd_2 and z for some Φ which depends only on d_1, d_2, d_3 and does not depend on q.* $\blacksquare$

The proof goes along the lines suggested in [A1], Lemma 2.6, for a slightly different setting. For the group $S = NA$ with diagonal action of A on N the details of the proof are given in [D] (Lemma 5.3).

The above lemma and Theorem (2.10) imply

(2.17) COROLLARY. *There is a constant c such that for every q and for every $y \in \frac{3}{4}qT^1$ and $z_1, z_2 \notin 2qT^1$,*

$$c^{-1}\frac{G(y, z_2)}{G(q, z_2)} \le \frac{G(y, z_1)}{G(q, z_1)} \le c\frac{G(y, z_2)}{G(q, z_2)}. \quad \blacksquare \tag{2.18}$$

Before proving estimates for the Green function we need one more lemma.

(2.19) LEMMA. *Let $a_n \to \infty$ be a sequence such that*

$$\lim_{n\to\infty} \frac{G(s, a_n)}{G(e, a_n)} = h(s)$$

exists. Then for every $x \in N$,

$$h(xs) = h(s). \tag{2.20}$$

Proof. Let $x \in N$. We have

$$\frac{G(xs,a_n)}{G(e,a_n)} = \frac{G(s,a_n a_n^{-1}x^{-1}a_n)}{G(s,a_n)} \cdot \frac{G(s,a_n)}{G(e,a_n)}$$

and $\lim_{n\to\infty} a_n^{-1}x^{-1}a_n = e$. By the Harnack inequality (1.3) for L^* there is c independent of n such that

$$|G(s,a_n a_n^{-1}x^{-1}a_n) - G(s,a_n)| \le c\tau(a_n^{-1}xa_n)G(s,a_n)$$

provided $\tau(a_n^{-1}s) > 1$ and $\tau(a_n^{-1}xa_n) < 1/2$. Therefore

$$\lim_{n\to\infty} \frac{G(s,a_n a_n^{-1}xa_n)}{G(s,a_n)} = 1$$

and (2.20) follows. ∎

Now we are ready to formulate our main result.

(2.21) THEOREM. *Let L be as in (1.2) with $\gamma = 0$, $\alpha > 0$. Given $B(e,r) = B$ there is a constant c such that*

$$c^{-1}h(s) \le G(s) \le ch(s), \qquad s \notin B(e,r),$$

where $s = xa$ and h is the function

$$h(xa) = \begin{cases} a^{\alpha} & \text{if } \varrho(x) \le 1,\ a \le 1, \\ \varrho(x)^{-Q-\alpha}a^{\alpha} & \text{if } \varrho(x) \ge 1,\ \varrho(x) \ge a, \\ a^{-Q} & \text{if } \varrho(x) \le a,\ a \ge 1. \end{cases} \tag{2.22}$$

Remark 1. Theorem (2.21) gives estimates for the fundamental solution of the operator (1.2) satisfying $\gamma < \alpha^2/4$.

Indeed, taking $\beta = (\alpha - \sqrt{\alpha^2 - 4\gamma})/2$ in (1.12) we obtain

$$L'f = a^{-\beta}(L(a^{\beta}f)) = (a\partial_a)^2 - (\alpha - 2\beta)a\partial_a + \sum_{i=1}^{m} \Phi_a(X_i)^2 + \Phi_a(X)$$

with $\alpha - 2\beta > 0$. Moreover, if G and G' are fundamental solutions for L and L' respectively then $G(xa) = G'(xa)a^{\beta}$. Since L' satisfies the assumptions of Theorem (2.21), for G' we have estimates (2.22) with $\alpha - 2\beta$ instead of α and so appropriate estimates for G.

Remark 2. If S is a harmonic space and L the Laplace–Beltrami operator then $h(xa) \approx e^{-Q\tau(xa)}$, where τ is the Riemannian metric ([DR1]). This reflects radiality properties of L. In the general case the comparison of the Green function for the Laplace–Beltrami operator with the function $e^{-\delta\tau}$, although possible, is clearly not good enough. More precisely, on pinched manifolds we have a trivial estimate [AS]

$$c^{-1}e^{-(1/\delta)\tau} \le G \le ce^{-\delta\tau}$$

outside a ball around the origin but it is far from being optimal. As an example we can take the operators considered in this paper such that $\alpha \ne Q$.

Therefore, for Heintze groups it seems much better to formulate estimates in terms of x and a coordinates, as well as to compare with a^α rather than with an exponential of the distance. This is important also for applications (see Theorem (2.38)).

Proof of Theorem (2.21). In the proof various constants are denoted by c. First we prove that

$$G(xa) \le ca^\alpha \quad \text{for } \varrho(x) \le 3,\ a \le 3. \tag{2.23}$$

We take $V_1 = 3T^1$, $V_2 = 2T^1$, $f = G$, $g = a^\alpha$ in Theorem (2.10). Then there is a constant c such that

$$G(xa) \le ca^\alpha \tag{2.24}$$

for $xa \in \partial V_1$. To extend (2.24) to $V_1 \setminus B$ we use the Harnack inequality for L^* and $G^* = \check{G}$. Let $ya \in V_1 \setminus B$ and $xa \in \partial V_1$. Then $\tau(a^{-1}y^{-1}, a^{-1}x^{-1}) = \tau(xy^{-1})$, which is bounded by a constant d when $\varrho(x), \varrho(y) \le 3$. Therefore by left-invariance of L^*,

$$G(ya) = G^*(a^{-1}y^{-1}) \le cG^*(a^{-1}x^{-1}) = G(xa)$$

with a constant c depending only on B and d, which proves (2.24).

Applying (2.24) to L^*, G^* and a^Q instead of a^α we obtain

$$G^*(a) \le ca^Q \quad \text{for } a \le 1. \tag{2.25}$$

Hence

$$G(a) \le ca^{-Q} \quad \text{for } a \ge 1. \tag{2.26}$$

Let $q \ge 1$. We take $V_1 = 3qT^1$, $V_2 = 2qT^1$, $f = G$, $g = a^\alpha$ in Theorem (2.10). Then

$$G(qxb) \le cG(2.5q)b^\alpha$$

for $xb \in \partial T^3$ with c independent of q. Therefore by (2.26),

$$G(qxb) \le cq^{-Q}b^\alpha, \quad xb \in \partial T^3. \tag{2.27}$$

If $b = 3$, (2.27) implies

$$G(xa) \le ca^{-Q} \quad \text{if } \varrho(x) \le a,\ a \ge 3.$$

Since for $1 \le a \le 3$ the above inequality is obvious, we obtain

$$G(xa) \le ca^{-Q} \quad \text{if } \varrho(x) \le a,\ a \ge 1. \tag{2.28}$$

If $b < 3$ in (2.27) then $\varrho(x) = 3$ and $\varrho(qxq^{-1}) = 3q$. Therefore (2.27) implies

$$G(qxq^{-1}qb) \le c(qb)^\alpha \varrho(qxq^{-1})^{-Q-\alpha}$$

and so, combining the above inequality with (2.23), we have

$$G(xa) \le c\varrho(x)^{-Q-\alpha}a^\alpha \tag{2.29}$$

for $\varrho(x) \ge 1$, $\varrho(x) \ge a$.

Now we pass to lower estimates. The first one is

$$G(xa) \geq c^{-1}a^{\alpha} \quad \text{if } \varrho(x) \leq 2,\ a \leq 2. \tag{2.30}$$

To prove (2.30) we use the dual chain to the one considered above. Namely, let $V_2 = \left(\frac{3}{4}\overline{T^1}\right)^c$, $V_1 = \left(\frac{1}{4}\overline{T^1}\right)^c$, $f = a^{\alpha}$, $g = G$ in Theorem (2.10). Before we proceed further we must check that a^{α} is dominated by a potential in $\frac{3}{4}T^1$. Let $a_n \to \infty$ be a sequence such that

$$\lim_{a_n\to\infty} \frac{G(xa, a_n)}{G(e, a_n)} = h(xa)$$

exists. By Lemma (2.19), $h(xa) = h(a)$ and in view of (2.18), h is dominated by a potential in $\frac{3}{4}T^1$. It remains to prove that

$$h(a) = a^{\alpha}. \tag{2.31}$$

By Lemma (2.16) and Theorem (2.8)

$$\frac{G(a, a_n)}{G(e, a_n)} \leq c\frac{G(a, e)G(e, a_n)}{G(e, a_n)} = cG(a).$$

Now by (2.23),

$$G(a) \leq ca^{\alpha}, \quad a \in \tfrac{3}{4}T^1.$$

Moreover, h is L-harmonic. Therefore $h(xa) = a^{\alpha}$.

Now Theorem (2.10) implies (2.30) for $xa \in \frac{1}{4}T^1$. To extend (2.30) for $xa \in 2T^1$ we must take care only of the points xa with $a \leq \frac{1}{4}$. We use again the Harnack inequality for L^* and G^*. Let $ya \in 2T^1$ and $xa \in \frac{1}{4}T^1$. Then

$$\tau(a^{-1}y^{-1}, a^{-1}x^{-1}) = \tau(xy^{-1}),$$

which is bounded whenever $\varrho(x), \varrho(y) \leq 2$. Therefore

$$G(xa) = G^*(a^{-1}x^{-1}) \leq cG^*(a^{-1}y^{-1}) = cG(ya)$$

for a constant c. The next estimate is

$$G(a) \geq c^{-1}a^{-Q} \quad \text{for } a \geq 1. \tag{2.32}$$

For (2.32) we prove, as above, that a^Q is dominated by an L^*-potential in $\frac{3}{4}T^1$. There are two L^*-harmonic functions depending only on a: a^Q and $a^{Q-\alpha}$. Proceeding as before and using (2.25) we see that if the limit $\lim_{a_n\to\infty} G(a, a_n)/G(e, a_n)$ exists then it must be equal to a^Q. Therefore we may apply (2.30) to G^* and a^Q, which gives

$$G^*(a) \geq c^{-1}a^Q \quad \text{for } a \leq 1 \tag{2.33}$$

and so (2.32) follows.

Let now $q \geq 1$ and take $V_2 = (q\overline{T^3})^c$, $V_1 = (q\overline{T^2})^c$. If $xb \in \partial qT^2$ then by Theorem (2.10) applied to $f = a^{\alpha}$, $g = G$,

$$b^{\alpha} \leq c\frac{G(qxb)}{G\left(\frac{5}{2}q\right)}. \tag{2.34}$$

If $b = 2$ then $\varrho(qxq^{-1}) \le qb$ so (2.34) together with (2.32) implies

$$G(xa) \ge c^{-1}a^{-Q} \quad \text{for } a \ge \varrho(x),\ a \ge 1. \tag{2.35}$$

If $b < 2$ in (2.34) then $\varrho(qxq^{-1}) = 2q$ and (2.34) together with (2.32) gives

$$G(qxq^{-1}qb) \ge c^{-1}\varrho(qxq^{-1})^{-Q-\alpha}(qb)^{\alpha}$$

and so

$$G(xa) \ge c^{-1}\varrho(x)^{-Q-\alpha}a^{\alpha} \quad \text{for } \varrho(x) \ge a,\ \varrho(x) \ge 2. \tag{2.36}$$

For $1 \le \varrho(x) \le 2$, (2.36) follows from (2.30). ∎

(2.37) Corollary. *Under the assumptions of Theorem* (2.21) *there is a constant c such that*

$$G(xa) \le ca^{\alpha}(1+\varrho(x))^{-Q-\alpha}. \ ∎$$

(2.38) Theorem. *Let K be a function on S which satisfies the estimate*

$$|K(xa)| \le ca^{Q}(1+\varrho(x))^{-Q-\varepsilon}$$

for an $\varepsilon > 0$. Then the operator $Tf(s) = f * K(s)$ is of weak type $(1,1)$.

Proof. The proof comes back to the ideas of Strömberg [St]. (See also [ADY].) We consider

$$Tf(s) = \int_S f(s(ya)^{-1})(1+\varrho(y))^{-Q-\varepsilon}a^{Q}\,dy\,da.$$

Then T is a composition of two operators

$$T_1f(s) = \int_N f(sy^{-1})(1+\varrho(y))^{-Q-\varepsilon}\,dy$$

and

$$T_2f(s) = \int_A f(sa^{-1})a^{Q}\,da.$$

T_1 is bounded on L^1. To prove that, it is convenient to write elements of S as $s = bx$. Then $dx\,db$ is the left Haar measure. Let $f \in L^1(m_{\mathrm{L}})$. Then

$$\begin{aligned}\|T_1f\|_{L^1(m_{\mathrm{L}})} &\le \int_N\int_S |f(bxy^{-1})|(1+\varrho(y))^{-Q-\varepsilon}\,dx\,db\,dy\\ &\le \|f\|_{L^1(m_{\mathrm{L}})}\int_N (1+\varrho(y))^{-Q-\varepsilon}\,dy\end{aligned}$$

and the boundedness of T_1 follows.

For T_2 we have

$$T_2f(s) = \int_A f(xca^{-1})a^{-Q}\,da = c^{Q}\int_A f(xb)b^{-Q}\,db = c^{Q}\psi(x)$$

where $s = xc$. Hence

$$m_{\mathrm{L}}(\{s : |T_2 f(s)| > \lambda\}) = \int\limits_{|T_2 f(s)|>\lambda} c^{-Q}\, dc\, dx = \int\limits_{c\ge \lambda^{1/Q}\psi(x)^{-1/Q}} c^{-Q}\, dc\, dx. \tag{2.39}$$

Given x, let $c_0(x) = \lambda^{1/Q}\psi(x)^{-1/Q}$. Then

$$\int\limits_{c_0}^{\infty} c^{-Q}\, dc = \frac{1}{Q} c_0^{-Q} = \frac{1}{\lambda Q}\psi(x).$$

Integrating first over c then over x in (2.39) we obtain

$$m_{\mathrm{L}}(\{s : |T_2 f(s)| > \lambda\}) = \frac{1}{\lambda Q}\int\limits_N \psi(x)\, dx = \frac{1}{\lambda Q}\|f\|_{L^1(m_{\mathrm{L}})},$$

which proves weak type $(1,1)$ of T_2 and hence of T. ∎

(2.40) Corollary. *If $\gamma = 0$ and $\alpha = Q$ in* (1.2) *then the operator $f \to f * G$ is of weak type* $(1,1)$.

Theorem (2.21) implies sharp pointwise lower and upper estimates for the Poisson kernel P corresponding to L with $\alpha > 0$. P is a smooth, bounded, integrable function on N, $\int_N P(x)\, dx = 1$, such that all bounded L-harmonic functions F on S are given by the Poisson integrals ([DH], [Ra])

$$F(xa) = \int\limits_N f(xaua^{-1})P(u)\, du = \int\limits_N f(u)a^{-Q}P(a^{-1}(x^{-1}u)a)\, du, \quad f \in L^\infty(N).$$

(If $\alpha \le 0$ then there are no bounded L-harmonic functions on S [BR].) In particular,

$$P_u(xa) = a^{-Q}\frac{P(a^{-1}(x^{-1}u)a)}{P(u)}$$

is L-harmonic as a function of xa. If the action of ad_H on $\mathcal{N}$ is diagonal, sharp pointwise estimates for P were described in [D]. It turns out that analogous estimates hold for general Heintze groups and the argument is, in fact, the same as presented in [D]. However, for the reader's convenience, we outline the proof.

(2.41) Theorem. *Let $x_n a_n$ be a sequence of points in S such that $x_n \to u \in N$ and $a_n \to 0$. Then*

$$P_u(xa) = \lim_{n\to\infty}\frac{G(xa, x_n a_n)}{G(e, x_n a_n)}.$$

Moreover, *there is c such that*

$$c^{-1}(1+\varrho(x))^{-Q-\alpha} \le P(x) \le c(1+\varrho(x))^{-Q-\alpha}, \quad x \in N. \tag{2.42}$$

Proof. First we notice as in [D], Proposition 2.12, that for every $u \in N$, $P_u(xa)$ is a minimal L-harmonic function. Secondly, in view of Corollary (2.17) and Lemma (2.19), if $\varrho(x_n) + a_n \to \infty$ then a minimal function obtained as $\lim_{n\to\infty} G(xa, x_n a_n)/G(e, x_n a_n)$ does not depend on x. Indeed, by Corollary (2.17), all the potentials $G(\cdot, y)/G(q, y)$ are comparable on $\frac{3}{4}qT^1$ as long as $y \notin 2qT^1$ with a constant independent of q. It follows that there is a constant c such that for every q and every $y_1, y_2 \notin 2qT^1$,

$$c^{-1}\frac{G(xa, y_2)}{G(e, y_2)} \le \frac{G(xa, y_1)}{G(e, y_1)} \le c\frac{G(xa, y_2)}{G(e, y_2)}$$

and so by Lemma (2.19) the only minimal function we can obtain this way is $h(xa) = a^\alpha$. On the other hand, if $x_n \to u$ and $a_n \to 0$ then

$$\lim_{n\to\infty} \frac{G(xa, x_n a_n)}{G(e, x_n a_n)} = K_u(xa) \tag{2.43}$$

exists. This follows from Theorem 7 of [A1] applied to the Φ-chain uq^nT^1 for a $q < 1$. Moreover, $K_u(xa) = K(u^{-1}xa)$, P_e must be one of these functions and clearly the others correspond to the translates P_u of P_e normalized at e. By (2.22) and (2.43),

$$K_e(xa) \le \varrho(x)^{-Q-\alpha}a^\alpha \quad \text{if } \varrho(x) > a.$$

This proves that given a neighbourhood U of e in N we have

$$\lim_{a\to 0} \int_{U^{\mathrm{c}}} K_e(xa)\, dx = 0,$$

which is possible only if $K_e = P_e$. Now (2.42) follows from (2.22).

3. Riesz transforms for elliptic operators. In this section we assume that L is elliptic. Then

$$|\Im\langle Lf, f\rangle| \le c(\Re\langle -Lf, f\rangle + \|f\|^2_{L^2}), \qquad f \in C_{\mathrm{c}}^\infty(S),$$

for a constant c and so T_t is an analytic semigroup ([P], §2.5) on $L^p(m_{\mathrm{L}})$ and its infinitesimal generator will be denoted L. Let $\mathcal{L} = -L$ and for $\delta > 0$ let

$$\mathcal{L}^{-\delta} = \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}T_t\, dt. \tag{3.1}$$

In view of (1.15), if $\gamma-(Q/p)\alpha+Q^2/p^2 < 0$ then $\mathcal{L}^{-\delta}$ is a bounded one-to-one operator on $L^p(m_{\mathrm{L}})$ and $\mathcal{L}^{-(\delta+\eta)} = \mathcal{L}^{-\delta} \circ \mathcal{L}^{-\eta}$ ([P], §2.5). We define

$$\mathcal{L}^\delta = (\mathcal{L}^{-\delta})^{-1}.$$

Then $\mathcal{L}^\delta$ is a closed operator with domain $D(\mathcal{L}^\delta) = \mathcal{R}(\mathcal{L}^{-\delta})$ and moreover,

$$\mathcal{L}^\delta \circ \mathcal{L}^\beta f = \mathcal{L}^{\delta+\beta} f$$

for every $f \in D(\mathcal{L}^\gamma)$, $\gamma = \max(\delta, \beta, \delta + \beta)$. The kernel K^δ of $\mathcal{L}^{-\delta}$ is given by

$$K^\delta(x) = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} p_t(x)\, dt, \tag{3.2}$$

i.e.

$$\mathcal{L}^{-\delta} f = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} f * p_t(x)\, dt, \qquad f \in L^p(m_{\mathrm{L}}).$$

Clearly if $\gamma = 0$ and $\alpha > Q$ then convolution with K^δ, $\delta > 0$, is a bounded operator on all $L^p(m_{\mathrm{L}})$, $p \geq 1$. If $\alpha = Q$ it is so for $p > 1$ and in view of Corollary (2.40) convolution with K^1 is of weak type $(1,1)$. This phenomenon has a very nice application to the Riesz transforms. We assume that L is elliptic and $\gamma = 0$. Given left-invariant vector fields Y_1, Y_2 we consider the operator

$$Rf = Y_1 Y_2 (f * G), \qquad f \in C_{\mathrm{c}}^\infty(S),$$

where $G = K^1$ is the kernel of $\mathcal{L}^{-1}$. By (3.1) and the following lemma, R is bounded on $L^2(m_{\mathrm{L}})$ if $\alpha > Q/2$.

(3.3) Lemma. *If $Y^I f \in L^2(m_{\mathrm{L}})$ for every multiindex I such that $|I| \leq 2$, then*

$$\|Y^I f\|_{L^2(m_{\mathrm{L}})} \leq c(\|Lf\|_{L^2(m_{\mathrm{L}})} + \|f\|_{L^2(m_{\mathrm{L}})}) \tag{3.4}$$

whenever $|I| \leq 2$.

For L^p boundedness or weak type $(1,1)$ of R it is convenient to write $G = G_1 + G_2$, where $G_1 = \varphi G$ with $\varphi \in C_{\mathrm{c}}^\infty(S)$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in a neighbourhood of e. Since the behaviour of the "local" part $R_1 f = Y_1 Y_2 (f * G_1)$ is well understood, only the "global" part $R_2 f = Y_1 Y_2 (f * G_2)$ matters. In view of (3.1) and the Harnack inequality (1.3), R_2 is bounded on all L^p if $\alpha > Q$. Therefore if $\alpha > Q$ then the second order Riesz transforms R are trivially bounded on $L^p(m_{\mathrm{L}})$, $p \geq 1$, and only the case $\alpha = Q$ is interesting. It contains, in particular, the Laplace–Beltrami operator for a left-invariant Riemannian metric on S. If $\alpha < Q$ our methods do not give any decisive results. Therefore, we formulate our next theorem under the assumption $\gamma = 0$, $\alpha \geq Q$. $B(x, r)$ is, as before, the Riemannian ball with centre x and radius r.

(3.5) Theorem. *Assume $\gamma = 0$ and $\alpha \geq Q$. Then the operator R is bounded on L^p, $p > 1$, and of weak type $(1,1)$. Let $G_\varepsilon = \varphi_\varepsilon G$, where $\varphi_\varepsilon \in C^\infty(G)$, $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon(x) = 0$ if $x \in B(e, \varepsilon)$, $\varphi_\varepsilon(x) = 1$ if $x \notin B(e, 2\varepsilon)$. Then given $f \in L^p$, $1 \leq p < \infty$, the limit $\lim_{\varepsilon \to 0} R_\varepsilon f(x) = \lim_{\varepsilon \to 0} f *$*

$Y_1Y_2G_\varepsilon(x)$ exists for almost every x in S and

$$Rf = \lim_{\varepsilon\to 0} R_\varepsilon f.$$

P r o o f. Let $\varphi \in C_c^\infty(S)$, $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighbourhood of e. Let

$$G_0(x) = \varphi(x)\int_0^1 p_t(x)\,dt,$$
$$G_\infty(x) = \varphi(x)\int_1^\infty p_t(x)\,dt + (1-\varphi)G(x).$$

We split R as

$$R = R_0 + R_\infty, \tag{3.6}$$

where

$$R_0 f = Y_1Y_2(f * G_0), \qquad R_\infty f = Y_1Y_2(f * G_\infty).$$

By the Harnack inequality for both L and $L - \partial_t$ (see (1.9)),

$$|Y_1Y_2G_\infty(x)| \le cG(x) \tag{3.7}$$

and

$$R_\infty f = f * Y_1Y_2G_\infty.$$

Therefore boundedness of R_∞ on L^p and weak type $(1,1)$ follow from (3.1) and Corollary (2.40). In particular, R_0 is bounded on $L^2(m_{\mathrm{L}})$.

To prove weak type $(1,1)$ of R_0 and boundedness on L^p, $1 < p \le 2$, we show that it is a Calderón–Zygmund type singular operator with kernel $K = Y_1Y_2G_0$. For $f \in C_c^\infty$ and $x \notin \operatorname{supp} f$, we have $R_0 f(x) = f * K(x)$. In view of [Heb],

$$|Y^I p_t(x)| \le c_1 t^{-(n+1)/2-|I|/2} e^{-c_2\tau(x)^2/t}, \qquad t \le 1, \tag{3.8}$$

where $n+1 = \dim S$ and so, for a constant c,

$$|\nabla K(x)| \le c\|x\|^{-n-2}.$$

This shows that if $\operatorname{supp} f$ is contained in a fixed compact set U the assumptions of Theorem 3, §4, Chapter 1 in [S] are satisfied and so R_0 is of weak type $(1,1)$ and bounded on $L^p(U)$, $1 < p \le 2$, with a constant depending on U. Moreover, the maximal function associated with the truncated singular integral $R_\varepsilon f$,

$$Mf = \sup_{\varepsilon>0} |R_\varepsilon f|,$$

is of weak type $(1,1)$ and bounded on $L^p(U)$, $1 < p \le 2$ (see [S], Chapter 1, §7). Since $\lim_{\varepsilon\to 0} R_\varepsilon f = Rf$ for $f \in C_c^\infty(S)$, $\lim_{\varepsilon\to 0} R_\varepsilon f(x)$ exists for every

$f \in L^p(U)$, $1 \le p \le 2$, and so

$$Rf = \lim_{\varepsilon \to 0} R_\varepsilon f, \qquad f \in L^p(U). \tag{3.9}$$

To get rid of the support assumption and to extend (3.9) to $L^p(S)$, $1 \le p \le 2$, we use the following lemma:

(3.10) LEMMA. *Given $\varepsilon, \delta > 0$ there exist a sequence $x_1, x_2, \ldots$ of points of S and positive integers m_1, m_2 such that*

(i) *$S = \bigcup_k x_k B(e, \varepsilon)$.*
(ii) *Each point $x \in S$ belongs to at most m_1 of the sets $x_k B(e, \varepsilon)$.*
(iii) *Each point $x \in S$ belongs to at most m_2 of the sets $x_k B(e, \varepsilon + \delta)$.*

For the proof of Lemma (3.10) see [An], [GQS].

To prove boundedness on L^p, $p > 2$, we use the adjoint kernel $\breve{K} = (Y_1 Y_2 G_0)\breve{} = \widetilde{Y}_1 \widetilde{Y}_2 \breve{G}_0$, where $\widetilde{Y}_j$ is a right-invariant vector field. Since the support of G_0 is compact and $G^* = G(x^{-1})$ (see (1.14)), L^p boundedness and weak type $(1,1)$ of the convolution with $\breve{K}$ follow from the above argument applied to L^*. Although $L^* 1 \ne 0$ if $\alpha > Q$, the condition $\gamma - (Q/p)\alpha + Q^2/p^2 < 0$ is satisfied provided it is satisfied by L, and so the argument for L^* is the same as for L. ∎

For the first order Riesz transforms we restrict ourselves to L_0 of the form

$$L_0 = (a\partial_a)^2 - Qa\partial_a + \sum_{i=1}^n \Phi_a(X_i)^2,$$

where $X_1, \ldots, X_n$ is a basis of $\mathcal{N}$. Given a left-invariant vector field Y let

$$\widetilde{R} f = Y(f * K^{1/2}), \quad \text{where} \quad K^{1/2} = (-L_0)^{-1/2}.$$

(3.11) THEOREM. *The operator $\widetilde{R}$ is bounded on L^p, $p > 1$, and of weak type $(1,1)$. Let $K_\varepsilon^{1/2} = \varphi_\varepsilon K^{1/2}$, where φ_ε is as in Theorem (3.5). Then given $f \in L^p$, $1 \le p < \infty$, the limit*

$$\lim_{\varepsilon \to 0} \widetilde{R}_\varepsilon f(x) = \lim_{\varepsilon \to 0} f * Y K_\varepsilon^{1/2}(x)$$

exists for almost every x in S and $\widetilde{R} f = \lim_{\varepsilon \to 0} \widetilde{R}_\varepsilon f$.

Proof. Let $Y_0 = a\partial_a$ and $Y_i = \Phi_a(X_i)$. First we notice that

$$\sum_{i=0}^n \|Y_i f\|^2_{L^2(m_{\mathrm{L}})} = \langle -L_0 f, f \rangle, \qquad f \in C_{\mathrm{c}}^\infty, \tag{3.12}$$

which implies that $\widetilde{R}$ is bounded on L^2. Indeed, substituting $f * K^{1/2} \in$

$D(L_0)$ in (3.12) we obtain

$$\sum_{i=0}^{n} \|Y_i(f * K^{1/2})\|_{L^2(m_{\mathrm{L}})} = \langle -L_0(f * K^{1/2}), f * K^{1/2}\rangle$$
$$= \|(-L_0)^{1/2}(f * K^{1/2})\|_{L^2}^2 = \|f\|_{L^2}^2.$$

As before we split $\widetilde{R}$ into two parts. Let φ be as in the proof of Theorem (3.5) and let

$$K_0(x) = \frac{\varphi(x)}{\Gamma(1/2)} \int_0^1 t^{-1/2} p_t(x)\, dt,$$
$$K_\infty(x) = \frac{1-\varphi(x)}{\Gamma(1/2)} \int_0^1 t^{-1/2} p_t(x)\, dt + \frac{1}{\Gamma(1/2)} \int_1^\infty t^{-1/2} p_t(x)\, dt.$$

Then $\widetilde{R} = \widetilde{R}_0 + \widetilde{R}_\infty$, where $\widetilde{R}_0 f = Y(f * K_0)$ and $\widetilde{R}_\infty f = Y(f * K_\infty)$. For $\widetilde{R}_0$ we proceed as before. For $\widetilde{R}_\infty$ we notice that by (3.8) and (1.4) given a compact set U there are constants c_1, c_2 such that

$$|YK_\infty(xu)| \le c_1(G(x) + e^{-c_2\tau(x)^2}) \quad \text{for } u \in U,\ x \in S.$$

Therefore,

$$\widetilde{R}_\infty f = f * YK_\infty, \qquad f \in C_c^\infty(S),$$

and by (1.15) and Corollary (2.40), $\widetilde{R}_\infty$ is bounded on L^p, $p > 1$, and of weak type $(1,1)$.

REFERENCES

[A1] A. Ancona, *Negatively curved manifolds, elliptic operators, and the Martin boundary*, Ann. of Math. 125 (1987), 495–536.

[A2] —, *Théorie du potentiel sur les graphes et les variétés*, in: A. Ancona, D. Geman and N. Ikeda, École d'Été de Probabilités de Saint-Flour XVIII-1988, Lecture Notes in Math. 1427, Springer, Berlin, 1990, 1–112.

[AS] M. T. Anderson and R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. of Math. 121 (1985), 429–461.

[An] J. P. Anker, *A short proof of a classical covering lemma*, Monatsh. Math. 107 (1989), 5–7.

[ADY] J. P. Anker, E. Damek and C. Yacoub, *Spherical analysis on harmonic AN groups*, Ann. Scuola Norm. Sup. Pisa, to appear.

[A] F. Astengo, *Multipliers for a distinguished Laplacean on solvable extensions of H-type groups*, Monatsh. Math. 120 (1995), 179–188.

[ACD] F. Astengo, R. Camporesi and B. Di Blasio, *The Helgason Fourier transform on a class of nonsymmetric harmonic spaces*, preprint.

[Ba] D. Bakry, *Etude des transformées de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*, in: Séminaire de Probabilités XXI, Lecture Notes in Math. 1247, Springer, Berlin, 1987, 137–172.

[BR] L. Birgé et A. Raugi, *Fonctions harmoniques sur les groupes moyennables*, C. R. Acad. Sci. Paris 278 (1974), 1287–1289.

[B] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble) 19 (1) (1969), 277–304.

[B1] M. Brelot, *Éléments de la théorie classique du potentiel*, Centre de Documentation Universitaire, Paris, 1961.

[B2] —, *Axiomatique des fonctions harmoniques*, Les Presses de l'Université de Montréal, Montréal, 1966.

[CDKR] M. Cowling, A. H. Dooley, A. H. Korányi and F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. in Math. 87 (1991), 1–41.

[C] P. Crepel, *Récurrence des marches aléatoires sur les groupes de Lie*, in: Théorie Ergodique Rennes 1973/4, Lecture Notes in Math. 532, Springer, 1976, 50–69.

[D] E. Damek, *Pointwise estimates for the Poisson kernel on NA groups by the Ancona method*, Ann. Fac. Sci. Toulouse Math. , to appear.

[DH] E. Damek and A. Hulanicki, *Boundaries for left-invariant subelliptic operators on semidirect products of nilpotent and abelian groups*, J. Reine Angew. Math. 411 (1990), 1–38.

[DHZ] E. Damek, A. Hulanicki and J. Zienkiewicz, *Estimates for the Poisson kernels and their derivatives on rank one NA groups*, preprint.

[DR1] E. Damek and F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. 2 (1992), 213–248.

[DR2] —, —, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. 27 (1992), 139–142.

[Di] B. Di Blasio, *Paley–Wiener type theorems on harmonic extensions of H-type groups*, preprint.

[GQS] G. I. Gaudry, T. Qian and P. Sjögren, *Singular integrals associated to the Laplacian on the affine group* $ax + b$, Ark. Mat. 30 (1992), 259–281.

[G] Y. Guivarc'h, *Sur la loi des grands nombres et la rayon spectral d'une marche aléatoire*, in: Journées sur les marches aléatoires, Astérisque 74 (1980), 47–98.

[Heb] W. Hebisch, *Estimates on the semigroups generated by left-invariant operators on Lie groups*, J. Reine Angew. Math. 423 (1992), 1–45.

[He] E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. 211 (1974), 23–34.

[H] R.-M. Hervé, *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier (Grenoble) 12 (1962), 415–571.

[HH] R.-M. Hervé et M. Hervé, *Les fonctions surharmoniques dans l'axiomatique de M. Brelot associeés à un opérateur elliptique dégénéré*, ibid. 22 (2) (1972), 131–145.

[L1] N. Lohoué, *Comparaison des champs de vecteurs et des puissances du Laplacien sur une variété riemannienne à courbure non positive*, J. Funct. Anal. 61 (1985), 164–201.

[L2] —, *Transformées de Riesz et fonctions sommables*, Amer. J. Math. 114 (1992), 875–922.

[P] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[Ra] A. Raugi, *Fonctions harmoniques sur les groupes localement compacts à base dénombrable*, Bull. Soc. Math. France Mém. 54 (1977), 5–118.

[R] F. Ricci, *The spherical transform on harmonic extensions of H-type groups*, Rend. Sem. Mat. Univ. Politec. Torino 50 (1992), 381–392.

[S] E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N.J., 1993.

[Str] J.-O. Strömberg, *Weak type L^1 estimates for maximal functions on non-compact symmetric spaces*, Ann. of Math. 114 (1981), 115–126.

[VSC] N. T. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Tracts in Math. 100, Cambridge Univ. Press, 1992.

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