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ON PASCAL'S TRIANGLE MODULO p^2

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1. Introduction. Let n be a nonnegative integer. The nth row of Pascal's triangle consists of the $n+1$ binomial coefficients

$$\binom{n}{0} \binom{n}{1} \binom{n}{2} \cdots \binom{n}{n}.$$

We denote by $N_n(t,m)$ the number of these binomial coefficients which are congruent to t modulo m, where t and m (≥ 1) are integers.

If p is a prime we write the p-ary representation of the positive integer n as

$$n = a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k,$$

where $k \geq 0$, each $a_i = 0, 1, \ldots, p-1$ and $a_k \neq 0$. We denote the number of r's occurring among $a_0, a_1, \ldots, a_k$ by n_r ($r = 0, 1, \ldots, p-1$). We set $\omega = e^{2\pi i/(p-1)}$ and let g denote a primitive root (mod p). We denote the index of the integer $t \not\equiv 0 \pmod p$ with respect to g by $\operatorname{ind}_g t$; that is, $\operatorname{ind}_g t$ is the unique integer j such that $t \equiv g^j \pmod p$. Hexel and Sachs [2, Theorem 3] have shown in a different form that for $t = 1, 2, \ldots, p-1$,

$$N_n(t,p) = \frac{1}{p-1} \sum_{s=0}^{p-2} \omega^{-s \operatorname{ind}_g t} \prod_{r=1}^{p-1} B(r,s)^{n_r}, \tag{1.1}$$

where for any integer r not exceeding $p-1$ and any integer s,

$$B(r,s) = \sum_{c=0}^{r} \omega^{s \operatorname{ind}_g \binom{r}{c}}. \tag{1.2}$$

In this paper we make use of the Hexel–Sachs formula (1.1) to determine the analogous formula for $N_n(tp, p^2)$ for $t = 1, 2, \ldots, p-1$. We prove

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THEOREM 1.1. *For* $t = 1, 2, \ldots, p-1$,

$$N_n(tp, p^2) = \frac{1}{p-1} \sum_{i=0}^{p-2} \sum_{j=1}^{p-1} n_{ij} \sum_{s=0}^{p-2} \omega^{-s(\mathrm{ind}_g t + \mathrm{ind}_g (i+1) - \mathrm{ind}_g j)} \times B(p-2-i, -s) B(j-1, s) \prod_{r=1}^{p-1} B(r,s)^{n_r - \delta(r-i) - \delta(r-j)}, \tag{1.3}$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$

and n_{ij} *denotes the number of occurrences of the pair* ij *in the string* $a_0 a_1 \ldots a_k$.

The proof of this theorem is given in §3 after a preliminary result is proved in §2. We consider the special cases $p = 2$ and $p = 3$ of the theorem in §4 and §5 respectively.

The proof of (1.1) given by Hexel and Sachs [2] is quite long so we conclude this introduction by giving a short proof of their result.

Proof of (1.1). For $t = 1, 2, \ldots, p-1$ we have

$$\begin{aligned} N_n(t,p) &= \sum_{\substack{r=0 \\ \binom{n}{r} \equiv t \,(\mathrm{mod}\, p)}}^{n} 1 = \sum_{\substack{r=0 \\ \binom{n}{r} \equiv t \,(\mathrm{mod}\, p) \\ p \nmid \binom{n}{r}}}^{n} 1 = \sum_{\substack{r=0 \\ p \nmid \binom{n}{r} \\ p-1 \mid \mathrm{ind}_g \binom{n}{r} - \mathrm{ind}_g t}}^{n} 1 \\ &= \frac{1}{p-1} \sum_{\substack{r=0 \\ p \nmid \binom{n}{r}}}^{n} \sum_{s=0}^{p-2} \omega^{(\mathrm{ind}_g \binom{n}{r} - \mathrm{ind}_g t)s} \\ &= \frac{1}{p-1} \sum_{s=0}^{p-2} \omega^{-s \, \mathrm{ind}_g t} \sum_{\substack{r=0 \\ p \nmid \binom{n}{r}}}^{n} \omega^{s \, \mathrm{ind}_g \binom{n}{r}}. \end{aligned}$$

It remains to show that

$$\sum_{\substack{r=0 \\ p \nmid \binom{n}{r}}}^{n} \omega^{s \, \mathrm{ind}_g \binom{n}{r}} = \prod_{r=1}^{p-1} B(r,s)^{n_r}.$$

We express r $(0 \le r \le n)$ in base p as

$$r = b_0 + b_1 p + \ldots + b_k p^k,$$

where each $b_i = 0, 1, \ldots, p-1$. By Lucas' theorem [5, p. 52], we have

$$\binom{n}{r} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \ldots \binom{a_k}{b_k} \pmod{p}.$$

If $p\nmid\binom{n}{r}$, we have $p\nmid\binom{a_i}{b_i}$ $(i=0,1,\ldots,k)$ so that $b_i\le a_i$ $(i=0,1,\ldots,k)$. Conversely, if $b_i\le a_i$ $(i=0,1,\ldots,k)$ then $p\nmid\binom{a_i}{b_i}$ $(i=0,1,\ldots,k)$ so that $p\nmid\binom{n}{r}$. Hence

$$\begin{aligned}\sum_{\substack{r=0\\ p\nmid\binom{n}{r}}}^{n}\omega^{s\operatorname{ind}_g\binom{n}{r}} &= \sum_{b_0,\ldots,b_k=0}^{a_0,\ldots,a_k}\omega^{s\sum_{i=0}^{k}\operatorname{ind}_g\binom{a_i}{b_i}}\\ &= \prod_{i=0}^{k}\Big\{\sum_{b_i=0}^{a_i}\omega^{s\operatorname{ind}_g\binom{a_i}{b_i}}\Big\} = \prod_{r=0}^{p-1}\prod_{\substack{i=0\\ a_i=r}}^{k}\Big\{\sum_{b_i=0}^{r}\omega^{s\operatorname{ind}_g\binom{r}{b_i}}\Big\}\\ &= \prod_{r=0}^{p-1}\Big\{\sum_{b_i=0}^{r}\omega^{s\operatorname{ind}_g\binom{r}{b}}\Big\}^{n_r} = \prod_{r=0}^{p-1}B(r,s)^{n_r}.\end{aligned}$$

As $B(0,s)=1$ the term $r=0$ contributes 1 to the product.

2. A preliminary result. We begin by recalling Wilson's theorem in the form

$$h!(p-h-1)!\equiv(-1)^{h+1}\pmod p\qquad(h=0,1,\ldots,p-1).\tag{2.1}$$

We make use of (2.1) in the proof of the following result.

LEMMA 2.1. *Let p be a prime and let g be a primitive root of p. Set $\omega=e^{2\pi i/(p-1)}$. Let s be an integer. Then*

(i) $$\sum_{b=0}^{a-1}\omega^{s\operatorname{ind}_g(b!(a-1-b)!/a!)}=\omega^{-s\operatorname{ind}_g a}B(a-1,-s)$$

for $a=1,2,\ldots,p-1$, and

(ii) $$\sum_{b=a+1}^{p-1}\omega^{s\operatorname{ind}_g(b!(a+p-b)!/a!)}=\omega^{s\operatorname{ind}_g(-1)}\omega^{s\operatorname{ind}_g(a+1)}B(p-a-2,s)$$

for $a=0,1,2,\ldots,p-2$.

Proof. (i) We have

$$\begin{aligned}\sum_{b=0}^{a-1}\omega^{s\operatorname{ind}_g(b!(a-1-b)!/a!)} &= \omega^{-s\operatorname{ind}_g a}\sum_{b=0}^{a-1}\omega^{s\operatorname{ind}_g(b!(a-1-b)!/(a-1)!)}\\ &= \omega^{-s\operatorname{ind}_g a}\sum_{b=0}^{a-1}\omega^{-s\operatorname{ind}_g((a-1)!/(b!(a-1-b)!))}\\ &= \omega^{-s\operatorname{ind}_g a}\sum_{b=0}^{a-1}\omega^{-s\operatorname{ind}_g\binom{a-1}{b}}\\ &= \omega^{-s\operatorname{ind}_g a}B(a-1,-s).\end{aligned}$$

(ii) By Wilson's theorem (2.1), we have for $b = a+1, \dots, p-1$,

$$\begin{aligned}\frac{b!(a+p-b)!}{a!} &\equiv \frac{(-1)^{b+1}}{(p-b-1)!} \cdot \frac{(-1)^{a+p-b+1}}{(b-a-1)!} \cdot \frac{(p-a-1)!}{(-1)^{a+1}} \\ &\equiv (p-a-1)\binom{p-a-2}{b-a-1} \pmod p,\end{aligned}$$

as $1 \equiv (-1)^{p+1} \pmod p$. Thus we have

$$\begin{aligned}\sum_{b=a+1}^{p-1} \omega^{s\,\mathrm{ind}_g(b!(a+p-b)!/a!)} &= \sum_{b=a+1}^{p-1} \omega^{s\,\mathrm{ind}_g((p-a-1)\binom{p-a-2}{b-a-1})} \\ &= \sum_{l=0}^{p-a-2} \omega^{s\,\mathrm{ind}_g((p-a-1)\binom{p-a-2}{l})} \\ &= \omega^{s\,\mathrm{ind}_g(p-a-1)} \sum_{l=0}^{p-a-2} \omega^{s\,\mathrm{ind}_g\binom{p-a-2}{l}} \\ &= \omega^{s\,\mathrm{ind}_g(-a-1)} B(p-a-2, s).\end{aligned}$$

The asserted result now follows as

$$\omega^{s\,\mathrm{ind}_g(-a-1)} = \omega^{s\,\mathrm{ind}_g(-1)+s\,\mathrm{ind}_g(a+1)}.$$

Remark. We adopt the convention that (i) holds when $a = 0$ and (ii) holds when $a = p-1$ as $B(-1, \pm s) = 0$.

3. Proof of the theorem. Let n be a fixed positive integer. Let

$$n = \sum_{j=0}^{k} a_j p^j \tag{3.1}$$

be the p-ary representation of n so that $k, a_0, \dots, a_k$ are fixed integers satisfiying

$$k \ge 0, \quad 0 \le a_j \le p-1 \quad (j = 0, 1, \dots, k), \quad a_k \ne 0. \tag{3.2}$$

Let r denote an arbitrary integer between 0 and n. We express r and $n-r$ in base p as follows:

$$r = \sum_{j=0}^{k} b_j p^j, \qquad n-r = \sum_{j=0}^{k} c_j p^j, \tag{3.3}$$

where each b_j and c_j is one of the integers $0, 1, \dots, p-1$. Let $c(n, r)$ denote the number of carries when r is added to $n-r$ in base p. Kazandzidis

[4, pp. 3–4] (see also Singmaster [6]) has shown that

$$\binom{n}{r} \equiv (-p)^{c(n,r)} \prod_{j=0}^{k} \frac{a_j!}{b_j!c_j!} \pmod{p^{c(n,r)+1}}. \tag{3.4}$$

If $c(n,r)=0$ then $b_j+c_j=a_j$ for $j=0,1,\dots,k$. Conversely, if $b_j+c_j=a_j$ for $j=0,1,\dots,k$, then $c(n,r)=0$. Hence, for $t=1,2,\dots,p-1$, we have

$$\binom{n}{r} \equiv t \pmod{p} \tag{3.5}$$

$$\Leftrightarrow b_j+c_j=a_j \ (j=0,1,\dots,k) \text{ and } \prod_{j=0}^{k} \frac{a_j!}{b_j!c_j!} \equiv t \pmod{p}.$$

Thus

$$N_n(t,p) = \sum_{\substack{r=0 \\ \binom{n}{r}\equiv t \,(\text{mod } p)}}^{n} 1 = \sum_{\substack{b_0,c_0,\dots,b_k,c_k=0 \\ b_j+c_j=a_j \ (j=0,1,\dots,k) \\ \prod_{j=0}^{k} a_j!/(b_j!c_j!)\equiv t \,(\text{mod } p)}}^{p-1} 1. \tag{3.6}$$

Suppose now that $c(n,r)=1$. If the unique carry occurs in the jth place $(0 \le j \le k-1)$, then, for $i=0,1,\dots,k$, the pair (b_i,c_i) satisfies

$$b_i+c_i = \begin{cases} a_i & \text{if } i \ne j, j+1, \\ a_j+p & \text{if } i=j, \\ a_{j+1}-1 & \text{if } i=j+1. \end{cases} \tag{3.7}$$

Conversely, if each pair (b_i,c_i) satisfies (3.7) then $c(n,r)=1$, and the carry occurs in the jth place. By Kazandzidis' theorem (3.4) we have

$$\binom{n}{r} \equiv tp \pmod{p^2} \Leftrightarrow c(n,r)=1 \text{ and } \prod_{l=0}^{k} \frac{a_l!}{b_l!c_l!} \equiv -t \pmod{p}. \tag{3.8}$$

As

$$N_n(tp,p^2) = \sum_{\substack{r=0 \\ \binom{n}{r}\equiv tp \,(\text{mod } p^2)}}^{n} 1,$$

appealing to (3.8), we obtain

$$N_n(tp,p^2) = \sum_{\substack{r=0 \\ c(n,r)=1 \\ \prod_{l=0}^{k} a_l!/(b_l!c_l!)\equiv -t \,(\text{mod } p)}}^{n} 1 = \sum_{j=0}^{k-1} \sum_{\substack{r=0 \\ \text{carry in } j\text{th place} \\ \prod_{l=0}^{k} a_l!/(b_l!c_l!)\equiv -t \,(\text{mod } p)}}^{n} 1.$$

Appealing to (3.1), (3.3) and (3.7), we deduce that

$$N_n(tp,p^2) = \sum_{j=0}^{k-1} \sum_{\substack{b_j,c_j,b_{j+1},c_{j+1}=0\\ b_j+c_j=a_j+p\\ b_{j+1}+c_{j+1}=a_{j+1}-1}}^{p-1} \sum_{\substack{b_0,c_0,\dots,b_{j-1},c_{j-1},b_{j+2},c_{j+2},\dots,b_k,c_k=0\\ b_l+c_l=a_l\ (l\neq j,j+1)\\ \prod a_l!/(b_l!c_l!)\equiv -t(b_j!c_j!b_{j+1}!c_{j+1}!)/(a_j!a_{j+1}!)\ (\mathrm{mod}\, p)}}^{p-1} 1,$$

where the product is over $l=0,\dots,j-1,j+2,\dots,k$. Next, appealing to (3.6), we see that the inner sum is

$$N_{n-a_jp^j-a_{j+1}p^{j+1}}\left(\frac{-tb_j!c_j!b_{j+1}!c_{j+1}!}{a_j!a_{j+1}!},p\right),$$

where the quotient is taken as an integer modulo p. Then

$$\begin{aligned} N_n(tp,p^2) &= \sum_{j=0}^{k-1} \sum_{\substack{b_j,c_j,b_{j+1},c_{j+1}=0\\ b_j+c_j=a_j+p\\ b_{j+1}+c_{j+1}=a_{j+1}-1}}^{p-1} N_{n-a_jp^j-a_{j+1}p^{j+1}}\left(\frac{-tb_j!c_j!b_{j+1}!c_{j+1}!}{a_j!a_{j+1}!},p\right)\\ &= \sum_{j=0}^{k-1} \sum_{b_j=a_j+1}^{p-1} \sum_{b_{j+1}=0}^{a_{j+1}-1} K_j, \end{aligned}$$

where

$$K_j = N_{n-a_jp^j-a_{j+1}p^{j+1}}\left(\frac{-tb_j!(a_j+p-b_j)!b_{j+1}!(a_{j+1}-1-b_{j+1})!}{a_j!a_{j+1}!},p\right).$$

The next step is to apply Hexel and Sachs' theorem (see (1.1)) to $n-a_jp^j-a_{j+1}p^{j+1}$. The number of r's in the p-ary representation of $n-a_jp^j-a_{j+1}p^{j+1}$ is $n_r-\delta(r-a_j)-\delta(r-a_{j+1})$. Hence

$$\begin{aligned} K_j &= \frac{1}{p-1}\sum_{s=0}^{p-2} \omega^{-s\,\mathrm{ind}_g(-tb_j!(a_j+p-b_j)!b_{j+1}!(a_{j+1}-1-b_{j+1})!)/(a_j!a_{j+1}!))}\\ &\quad\times \prod_{r=1}^{p-1} B(r,s)^{n_r-\delta(r-a_j)-\delta(r-a_{j+1})}. \end{aligned}$$

Thus

$$\begin{aligned} N_n(tp,p^2) &= \frac{1}{p-1}\sum_{s=0}^{p-2}\omega^{-s\,\mathrm{ind}_g(-t)} \sum_{j=0}^{k-1}\Big\{\sum_{b_j=a_j+1}^{p-1} \omega^{-s\,\mathrm{ind}_g(b_j!(a_j+p-b_j)!/a_j!)}\Big\}\\ &\quad\times\Big\{\sum_{b_{j+1}=0}^{a_{j+1}-1}\omega^{-s\,\mathrm{ind}_g(b_{j+1}!(a_{j+1}-1-b_{j+1})!/a_{j+1}!)}\Big\}\\ &\quad\times\prod_{r=1}^{p-1} B(r,s)^{n_r-\delta(r-a_j)-\delta(r-a_{j+1})}. \end{aligned}$$

Appealing to Lemma 2.1, we obtain

$$
\begin{aligned}
&N_n(tp,p^2)\\
&\quad = \frac{1}{p-1}\sum_{s=0}^{p-2}\omega^{-s\,\mathrm{ind}_g(-1)}\omega^{-s\,\mathrm{ind}_g t}\\
&\qquad \times \sum_{\substack{j=0\\ a_j\le p-2\\ a_{j+1}\ge 1}}^{k-1}\{\omega^{-s\,\mathrm{ind}_g(-1)}\omega^{-s\,\mathrm{ind}_g(a_j+1)}B(p-a_j-2,-s)\}\\
&\qquad \times \{\omega^{s\,\mathrm{ind}_g(a_{j+1})}B(a_{j+1}-1,s)\}\prod_{r=1}^{p-1}B(r,s)^{n_r-\delta(r-a_j)-\delta(r-a_{j+1})}\\
&\quad = \frac{1}{p-1}\sum_{s=0}^{p-2}\omega^{-s\,\mathrm{ind}_g t}\sum_{\substack{j=0\\ a_j\le p-2\\ a_{j+1}\ge 1}}^{k-1}\omega^{s(\mathrm{ind}_g a_{j+1}-\mathrm{ind}_g(a_j+1))}\\
&\qquad \times B(p-a_j-2,-s)B(a_{j+1}-1,s)\prod_{r=1}^{p-1}B(r,s)^{n_r-\delta(r-a_j)-\delta(r-a_{j+1})}\\
&\quad = \frac{1}{p-1}\sum_{s=0}^{p-2}\omega^{-s\,\mathrm{ind}_g t}\sum_{u=0}^{p-2}\sum_{v=1}^{p-1}\sum_{\substack{j=0\\ a_j=u\\ a_{j+1}=v}}^{k-1}\omega^{s(\mathrm{ind}_g v-\mathrm{ind}_g(u+1))}\\
&\qquad \times B(p-u-2,-s)B(v-1,s)\prod_{r=1}^{p-1}B(r,s)^{n_r-\delta(r-u)-\delta(r-v)}\\
&\quad = \frac{1}{p-1}\sum_{u=0}^{p-2}\sum_{v=1}^{p-1}n_{uv}\sum_{s=0}^{p-2}\omega^{-s(\mathrm{ind}_g t+\mathrm{ind}_g(u+1)-\mathrm{ind}_g v)}\\
&\qquad \times B(p-2-u,-s)B(v-1,s)\prod_{r=1}^{p-1}B(r,s)^{n_r-\delta(r-u)-\delta(r-v)}.
\end{aligned}
$$

4. Case $p=2$. Here $\omega=1$ and $g=1$. From (1.2) we obtain

$$B(0,s)=1,\quad B(1,s)=2.$$

Taking $p=2$ and $t=1$ in the theorem, we deduce that

$$N_n(2,4)=n_{01}B(0,0)^2B(1,0)^{n_1-1}=n_{01}2^{n_1-1}.$$

This result is due to Davis and Webb [1, Theorem 7].

5. Case $p = 3$**.** Here $\omega = -1$ and $g = 2$. From (1.2) we have

$$B(0,s) = 1, \quad B(1,s) = 2, \quad B(2,s) = 2 + (-1)^s.$$

Taking $p = 3$ and $t = 1,2$ in the theorem, we obtain

$$\begin{aligned} N_n(3t,9) = n_{01}(2^{n_1-1}3^{n_2} - (-1)^t 2^{n_1-1}) &+ n_{02}(2^{n_1+1}3^{n_2-1} + (-1)^t 2^{n_1+1}) \\ &+ n_{11}(2^{n_1-3}3^{n_2} + (-1)^t 2^{n_1-3}) \\ &+ n_{12}(2^{n_1-1}3^{n_2-1} - (-1)^t 2^{n_1-1}). \end{aligned}$$

This result is due to Huard, Spearman and Williams [3].

6. Concluding remarks. As

$$\sum_{t=1}^{p-1} \omega^{-s \operatorname{ind}_g t} = \begin{cases} p-1 & \text{if } s = 0, \\ 0 & \text{if } s \neq 0, \end{cases}$$

and

$$B(r,0) = r+1,$$

summing (1.1) and (1.3) over $t = 1, 2, \ldots, p-1$, we obtain

$$n + 1 - N_n(0,p) = \sum_{t=1}^{p-1} N_n(t,p) = \prod_{r=1}^{p-1} (r+1)^{n_r}$$

and

$$\begin{aligned} N_n(0,p) - N_n(0,p^2) &= \sum_{t=1}^{p-1} N_n(tp,p^2) \\ &= \sum_{i=0}^{p-2} \sum_{j=1}^{p-1} n_{ij}(p-1-i)j \prod_{r=1}^{p-1} (r+1)^{n_r - \delta(r-i) - \delta(r-j)}, \end{aligned}$$

so that

$$\begin{aligned} (6.1) \quad & N_n(0,p^2) \\ &= n + 1 - \prod_{r=1}^{p-1} (r+1)^{n_r} - \sum_{i=0}^{p-2} \sum_{j=1}^{p-1} n_{ij}(p-1-i)j \prod_{r=1}^{p-1} (r+1)^{n_r - \delta(r-i) - \delta(r-j)}. \end{aligned}$$

We conclude this paper by observing that our theorem shows that $N_n(tp,p^2)$ $(p \nmid t)$ depends only on t, n_i $(i = 1, 2, \ldots, p-1)$ and n_{ij} $(i = 0, 1, \ldots, p-2;\ j = 1, 2, \ldots, p-1)$. This result should be compared to that of Webb [7, Theorem 3] for $N_n(t,p^2)$ $(p \nmid t)$.

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