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A new large cardinal and Laver sequences for extendibles

by

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Abstract. We define a new large cardinal axiom that fits between A_3 and A_4 in the hierarchy of axioms described in [SRK]. We use this new axiom to obtain a Laver sequence for extendible cardinals, improving the known large cardinal upper bound for the existence of such sequences.

1. Introduction. In [SRK], the authors define a hierarchy of large cardinal axioms, A_1–A_7, having consistency strength strictly between huge and extendible. These axioms were shown to have the property that, for $1 \le i \le 6$, A_i implies A_{i+1} in a very strong sense; in particular, it was shown that for $2 \le i \le 5$, $A_i(\kappa)$ *strongly implies* $A_{i+1}(\kappa)$, that is, if $A_i(\kappa)$ is true, then not only is $A_{i+1}(\kappa)$ true, but also, for some normal ultrafilter D over κ, $\{\alpha < \kappa : A_{i+1}(\alpha) \text{ holds}\} \in D$.

In this note, we introduce an axiom, which we call $A_{3.5}$, that lies strictly between A_3 and A_4. The axiom A_3 is commonly known as *almost hugeness*; the axiom $A_4(\kappa)$ asserts that κ is λ-supercompact for a particular $\lambda > \kappa$ such that for some normal ultrafilter U over $P_\kappa\lambda$, $i_U(g)(\kappa) < \lambda$ for all $g \in {}^\kappa\kappa$, where i_U is the canonical embedding defined from U. The axiom $A_{3.5}(\kappa)$ is obtained from the notion of α-extendibility in the same way as $A_4(\kappa)$ is obtained from λ-supercompactness; it asserts that there is an inaccessible $\alpha > \kappa$ and an elementary embedding $i : V_\alpha \to V_\eta$ such that $\alpha < i(\kappa) < \eta$ and for all $g \in {}^\kappa\kappa$, $i(g)(\kappa) < \alpha$. For a technical reason to be explained later, we also require that $V_\kappa \prec V_\alpha$. In Section 2 of this note, we show that $A_3(\kappa)$ strongly implies $A_{3.5}(\kappa)$, and that $A_{3.5}(\kappa)$ strongly implies $A_4(\kappa)$.

In Section 3, we apply a global version of $A_{3.5}(\kappa)$, which we term *hyperextendibility*, to build a Laver sequence for extendible cardinals. Laver introduced the notion of a Laver sequence in [L] and used it to prove that it is consistent, relative to a supercompact κ, that supercompactness cannot

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be destroyed by κ-directed closed forcing. Gitik and Shelah [GS] obtained an analogous result for strong cardinals, using a Laver sequence for strong cardinals. (Details for constructing such a sequence appear in [C].) In [C], Corazza shows that Laver sequences can be constructed for most of the commonly used large cardinals not weaker than strong cardinals. However, although Laver sequences for strong and supercompact cardinals can be obtained under the assumption of a strong or supercompact cardinal, respectively, relatively stronger hypotheses were required for the construction of Laver sequences for extendible, almost huge and huge cardinals. In particular, Corazza's argument requires a superhuge cardinal to obtain a Laver sequence for extendibles. In Section 3, we show how to obtain such a Laver sequence assuming only a global version of $A_{3.5}(\kappa)$ (which will be easily seen to be much weaker than superhuge).

We assume modest familiarity with the notions of supercompact, extendible, almost huge, and superhuge cardinals, and of Laver sequences. For excellent background information on large cardinals, see [SRK] or [K]; for a detailed study of Laver sequences, see [C]. We wish to thank the referee for simplifying the proof of Proposition 2.1 and the definition of Laver sequence in the present context.

2. The axiom $A_{3.5}$. We begin with the definitions of $A_3(\kappa), A_4(\kappa)$, and $A_{3.5}(\kappa)$:

$A_3(\kappa)$: There is an elementary embedding $j : V \to M$ with critical point κ so that ${}^{\lambda}M \subseteq M$ for every $\lambda < j(\kappa)$.

$A_4(\kappa)$: There is a $\lambda > \kappa$ and a normal ultrafilter U over $P_\kappa\lambda$ so that if $M \cong V^{P_\kappa\lambda}/U$ and $g \in {}^{\kappa}\kappa$, then $M \vDash j(g)(\kappa) < \lambda$.

$A_{3.5}(\kappa)$: There exist $\alpha > \kappa$, $\eta > \alpha$, and an elementary embedding $i : V_\alpha \to V_\eta$ with critical point κ such that

1. α is inaccessible;
2. $\alpha < i(\kappa) < \eta$;
3. for all $g \in {}^{\kappa}\kappa, i(g)(\kappa) < \alpha$; and
4. $V_\kappa \prec V_\alpha$.

We will call an elementary embedding $i : V_\alpha \to V_\eta$—as in the definition of $A_{3.5}(\kappa)$—having critical point κ and satisfying (1)–(4), a *κ-good embedding*.

2.1. Proposition. *$A_3(\kappa)$ strongly implies $A_{3.5}(\kappa)$.*

Proof. Let $j : V \to M$ be an almost huge embedding with critical point κ. We recall the properties of the embedding $j \cdot j : M \to N$: $j \cdot j = \bigcup_{\beta\in\mathrm{ON}} j(j|V_\beta)$; $j \circ j = (j \cdot j) \circ j$; $\mathrm{cp}(j \cdot j) = j(\kappa)$; $(j \cdot j)(j(\kappa)) = j^2(\kappa)$.

Since $j(\kappa)$ is Mahlo in M and $\{\alpha < j(\kappa) : V_\kappa \prec V_\alpha \prec V_{j(\kappa)}\}$ is club in M, it follows that $\{\alpha < j(\kappa) : \alpha \text{ is inaccessible and } V_\kappa \prec V_\alpha\}$ is (in V)

unbounded in $j(\kappa)$. Pick any α in this set such that $\alpha > \sup(\{j(g)(\kappa) : g \in {}^{\kappa}\kappa\})$. Let D be the normal ultrafilter over κ derived from j. By almost hugeness, $j|V_\alpha : V_\alpha \to V_\eta^M \in M$. It follows that $M \models A_{3.5}(\kappa)$, whence $\{\beta < \kappa : A_{3.5}(\beta)\} \in D$.

To see that $A_{3.5}(\kappa)$ holds (in V), first notice that, using the α, η defined in the last paragraph, $\eta = j(\alpha) < j^2(\kappa)$. Thus

$$N \models \exists \alpha, \eta < j^2(\kappa)\, \exists i : V_\alpha \to V_\eta\, [i \text{ is } \kappa\text{-good}],$$

whence, pulling back with $j \cdot j$,

$$M \models \exists \alpha, \eta < j(\kappa)\, \exists i : V_\alpha \to V_\eta\, [i \text{ is } \kappa\text{-good}].$$

Now, if $\alpha, \eta, i : V_\alpha \to V_\eta$ are witnesses in M, by almost hugeness, i is κ-good (in V), and $A_{3.5}(\kappa)$ holds. ∎

The proof given above that $A_3(\kappa) \Rightarrow A_{3.5}(\kappa)$ actually yields somewhat more, and this will be useful to know in Section 3; we actually showed that there are arbitrarily large $\alpha < j(\kappa)$ for which there are $\eta, i : V_\alpha \to V_\eta$ satisfying the conditions of $A_{3.5}(\kappa)$. Thus, if we define a cardinal κ to be *hyper-extendible* if for each γ there are $\alpha > \gamma, \eta, i : V_\alpha \to V_\eta$ satisfying the conditions of $A_{3.5}(\kappa)$, then our argument above shows that whenever $j : V \to M$ is an almost huge embedding with critical point κ, then $V_{j(\kappa)} \models$ "κ is hyper-extendible".

We can take the argument one step further. Notice that the definition of hyper-extendibility takes a "local" definition (namely, $A_{3.5}(\kappa)$—see [SRK] for a definition of *local*) and "globalizes" it. We show in [C] (see remarks following 2.15) that the resulting property must be Π_3^{ZFC}. Now, if we "globalize" the definition of almost hugeness, we obtain the notion of a *super-almost-huge* cardinal: We will say that κ is super-almost-huge if for each $\gamma > \kappa$ there is an almost huge embedding $i : V \to N$ with critical point κ and with $i(\kappa) > \gamma$. Now suppose κ is super-almost-huge; in [C, 2.18] it is shown that κ must be extendible. Thus, if $j : V \to M$ is an almost huge embedding with critical point κ, then $M \models$ "$j(\kappa)$ is extendible". Recall that Σ_3 formulas relativize down below extendible cardinals (see [K, 23.10]). It follows that because (as we showed in the last paragraph) $V_{j(\kappa)} \models$ "κ is hyper-extendible", in fact $M \models$ "κ is hyper-extendible". (See [K, Chapters 22, 23] for similar arguments.) Summing up,

2.2. Corollary. *If κ is almost huge and $j : V \to M$ is an almost huge embedding with critical point κ, then $V_{j(\kappa)} \models$ "κ is hyper-extendible". Moreover, if κ is super-almost-huge and $j : V \to M$ is an almost huge embedding with critical point κ, then $M \models$ "κ is hyper-extendible".* ∎

We now show that $A_{3.5}$ strongly implies A_4.

2.3. Proposition. *$A_{3.5}(\kappa)$ strongly implies $A_4(\kappa)$.*

Proof. Let $j : V_\alpha \to V_\eta$ be as in $A_{3.5}(\kappa)$. Let λ be such that

$$\sup(\{j(g)(\kappa) : g \in {}^\kappa\kappa\}) < \lambda < \alpha.$$

Let U be the normal ultrafilter over $P_\kappa\lambda$ derived from j. Let

$$M_\lambda \cong V_\alpha^{P_\kappa\lambda}/U, \quad M \cong V^{P_\kappa\lambda}/U,$$

and let $i_\lambda : V_\alpha \to M_\lambda$ and $i : V \to M$ be the corresponding canonical embeddings. As usual, there is $k : M_\lambda \to V_\beta$ such that $k|\lambda = \mathrm{id}_\lambda$ and $k \circ i_\lambda = j$; it follows that for all $g \in {}^\kappa\kappa$, $j(g)(\kappa) = i_\lambda(g)(\kappa)$ (see [C, Section 2] for details). Note that for any $h : P_\kappa\lambda \to V_\alpha$, $[h]_U^{V_\alpha} = [h]_U$ (see [C, Lemma 2.22] for details). It follows that for any $g \in {}^\kappa\kappa$, $i_\lambda(g)(\kappa) = i(g)(\kappa)$. Thus, for all such g, $i(g)(\kappa) < \lambda$; this proves $A_4(\kappa)$.

Let $X = \{\beta < \kappa : A_4(\beta)\}$ and let D be the normal ultrafilter over κ derived from j. To complete the proof, we show that $X \in D$. First, notice that, since $i_\lambda \in V_\eta$, we have

$$\begin{aligned} V_\eta \models \exists\alpha\,\exists\lambda\,\exists U\,[&\alpha \text{ is inaccessible } \wedge \kappa < \lambda < \alpha \\ &\wedge U \text{ is a normal ultrafilter over } P_\kappa\lambda \wedge \forall g \in {}^\kappa\kappa\ [i_U^{V_\alpha}(g)(\kappa) < \lambda]]. \end{aligned}$$

It follows that $S \in D$, where

$$\begin{aligned} S = \{\beta < \kappa : \exists\alpha\,\exists\lambda\,\exists U\,[&\alpha \text{ is inaccessible } \wedge \beta < \lambda < \alpha \\ &\wedge U \text{ is a normal ultrafilter over } P_\beta\lambda \wedge \forall g \in {}^\beta\beta\ [i_U^{V_\alpha}(g)(\beta) < \lambda]]\}. \end{aligned}$$

We will be done if we can show $S \subseteq X$. Let $\beta \in S$ and let α, λ, U witness that $\beta \in S$. As in the last paragraph, let $i_U : V \to M_U \cong V^{P_\beta\lambda}/U$ be the canonical embedding. As before, for each $g \in {}^\beta\beta$,

$$i(g)(\beta) = i^{V_\alpha}(g)(\beta) < \lambda.$$

Thus $\beta \in X$, as required. ■

The reader will notice that the requirement "$V_\kappa \prec V_\alpha$" in the definition of $A_{3.5}(\kappa)$ is never used in Proposition 2.3; thus, a somewhat more natural version of $A_{3.5}(\kappa)$ would omit this requirement, and the proofs of 2.1–2.3 would go through virtually without change. The reason we included this condition in $A_{3.5}(\kappa)$ is for the sake of our application of the axiom; our proof of Theorem 3.1 below makes (it seems) essential use of this condition.

3. Laver sequences for extendibles. A *Laver sequence at* κ is a function $f : \kappa \to V_\kappa$ such that for each set x and each $\lambda \geq \kappa \cdot |\mathrm{TC}(x)|$ there is a normal ultrafilter U over $P_\kappa\lambda$ such that $i_U(f)(\kappa) = x$ (where i_U is the canonical embedding defined from U). In [C] we generalize this definition so that Laver sequences are defined for classes of (set) embeddings rather than just for particular large cardinals; the notion of a Laver sequence relative to a specific large cardinal is obtained as a special case. As in [C], let

$$\mathcal{E}_\kappa^{\rm ext} = \{i : V_\alpha \to V_\eta : {\rm cp}(i) = \kappa \wedge \alpha < i(\kappa) < \eta\}.$$

We shall say that $f : \kappa \to V_\kappa$ is an $\mathcal{E}_\kappa^{\rm ext}$*-Laver sequence at* κ if for each set x there are arbitrarily large α such that there exists $i : V_\alpha \to V_\eta$, for some η, such that $i(f)(\kappa) = x$. (This is a slight simplification of the general definition given in [C], but is easily seen to be equivalent in the present context of extendible cardinals.)

We also define the formula $\phi(g, x, \lambda)$ to be the following assertion about g, x, λ:

"there exists a cardinal δ with $g : \delta \to V_\delta$,
and for all $\beta > \lambda$ and all $i \in \mathcal{E}_\delta^{\rm ext}$ with ${\rm dom}\, i = V_\beta$, $i(g)(\delta) \neq x$".

Note that a function $f : \kappa \to V_\kappa$ is $\mathcal{E}_\kappa^{\rm ext}$-Laver at κ iff $\forall x\, \forall \lambda\, [\neg\phi(f, x, \lambda)]$. In [C], we give a construction of an $\mathcal{E}_\kappa^{\rm ext}$-Laver sequence under the assumption that κ is superhuge. Theorem 3.1 below improves the result by requiring only that κ be hyper-extendible (and Corollary 2.2 above shows that this is a significant weakening of the hypothesis).

3.1. Theorem. *Assume κ is hyper-extendible. Then there is a $\mathcal{E}_\kappa^{\rm ext}$-Laver sequence at κ.*

Proof. We begin with a construction given in [C, Section 5]. Let $R \subseteq V_\kappa$ be a well-ordering of V_κ. Inside the structure $\langle V_\kappa, \in, R\rangle$, define $f : \kappa \to V_\kappa$ by

$$f(\alpha) = \begin{cases} \emptyset & \text{if } f|\alpha \text{ is a } \mathcal{E}_\alpha^{\rm ext}\text{-Laver sequence at } \alpha, \text{ or } \alpha \text{ is not a cardinal;} \\ x & \text{if } \alpha \text{ is a cardinal and } f|\alpha \text{ is not } \mathcal{E}_\alpha^{\rm ext}\text{-Laver at } \alpha, \\ & \text{where } x \text{ is } R\text{-least such that } \exists\lambda\ [\phi(f|\alpha, x, \lambda)]. \end{cases}$$

Assume that the f defined above is not $\mathcal{E}_\kappa^{\rm ext}$-Laver at κ. Let x, λ be such that $\phi(f, x, \lambda)$. Let $\alpha > \lambda \cdot {\rm rank}(x)$ and $j : V_\alpha \to V_\eta$ be κ-good (since κ is hyper-extendible). Let D be the normal ultrafilter over κ derived from j.

If $\{\beta : \langle V_\kappa, \in, R\rangle \models$ "$f|\beta$ is $\mathcal{E}_\beta^{\rm ext}$-Laver"$\} \in D$, then $\langle V_{j(\kappa)}, \in, j(R)\rangle \models$ "f is $\mathcal{E}_\kappa^{\rm ext}$-Laver at κ". Thus, there are $\gamma < j(\kappa)$, $i \in \mathcal{E}_\kappa^{\rm ext} \cap V_{j(\kappa)}$ such that ${\rm dom}\, i = V_\gamma$, $\gamma > \lambda$, and $i(f)(\kappa) = x$, contradicting the choice of x.

Thus $\{\beta : \langle V_\kappa, \in, R\rangle \models \exists\lambda\, [\phi(f|\beta, f(\beta), \lambda)]\} \in D$. Hence, there is $\lambda < j(\kappa)$ such that

$$(*) \qquad \langle V_{j(\kappa)}, \in, j(R)\rangle \models \phi(f, j(f)(\kappa), \lambda).$$

Let $y = j(f)(\kappa)$. Define $g : \kappa \to \kappa$ by $g(\beta) = {\rm rank}(f(\beta))$. Then

$${\rm rank}(y) = {\rm rank}(j(f)(\kappa)) = j(g)(\kappa) < \alpha.$$

Let γ be such that $\lambda\cdot\mathrm{rank}(y) < \gamma < \alpha$, and let $i = j|V_\gamma \in V_\eta$. Since $V_\kappa \prec V_\alpha$, we have $V_{j(\kappa)} \prec V_\eta$. But now

$$V_\eta \models \exists e \in \mathcal{E}_\kappa^{\mathrm{ext}}\, \exists\gamma\, [\mathrm{dom}\, e = V_\gamma \wedge \gamma > \lambda \wedge e(f)(\kappa) = y],$$

and so

$$V_{j(\kappa)} \models \exists e \in \mathcal{E}_\kappa^{\mathrm{ext}}\, \exists\gamma\, [\mathrm{dom}\, e = V_\gamma \wedge \gamma > \lambda \wedge e(f)(\kappa) = y].$$

But this contradicts $(*)$ and therefore shows that f is indeed $\mathcal{E}_\kappa^{\mathrm{ext}}$-Laver at κ. ∎

Let us call a large cardinal property $A(\kappa)$ *Laver-generating* if $A(\kappa)$ implies that there exists an "$A(\kappa)$"-Laver sequence at κ (see [C] for a more precise statement). It is known that the properties of supercompactness and strongness are Laver-generating. The question left open by the present work is the following:

Open question. Is extendibility of κ Laver-generating? In other words, can the hypothesis of Theorem 3.1 be weakened to "κ is an extendible cardinal"?

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