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On compact spaces carrying Radon measures of uncountable Maharam type

by

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Abstract. If Martin's Axiom is true and the continuum hypothesis is false, and X is a compact Radon measure space with a non-separable L^1 space, then there is a continuous surjection from X onto $[0,1]^{\omega_1}$.

1. Introduction. For any probability space (X,Σ,μ), its *measure algebra* is the quotient Boolean algebra $\Sigma/\mathcal{N}$, where $\mathcal{N}$ is the σ-ideal of sets of measure 0. For more than fifty years we have had a complete description of these Boolean algebras. There is the two-element algebra $\{0,1\}$; for each infinite cardinal κ there is the measure algebra $\mathfrak{B}_\kappa$ of the usual measure on $\{0,1\}^\kappa$; and there are countable products of these of the form $\mathcal{P}J\times\prod_{i\in I}\mathfrak{B}_{\kappa_i}$, where I and J are countable sets and $\langle\kappa_i\rangle_{i\in I}$ is a family of distinct infinite cardinals. And that is all. (See [12] and [5], §3.) Part of the interest of this classification lies in the fact that it completely describes the function spaces $L^p(\mu)$, for $1\le p\le\infty$, up to Banach lattice isomorphism.

Now suppose that $(X,\mathfrak{T})$ is a topological space. In this case a measure on X may or may not be related to the topology in various ways. By far the most important is the idea of *Radon* measure, in which all open sets (and therefore all Borel sets) are measurable and $\mu E=\sup\{\mu K : K\subseteq E$ is compact$\}$ for every measurable set E. It is customary, in this context, to suppose that X is Hausdorff, so that compact sets are closed. There are complications if $\mu X=\infty$; in this paper I will consider probability measures exclusively. Now, given a Hausdorff space $(X,\mathfrak{T})$, we can ask: which Boolean algebras can appear as the measure algebras of Radon probability measures on X? Write $\mathrm{K}(X)$ for the set of infinite cardinals κ for which there is a Radon probability measure μ on X such that the measure algebra of μ is isomorphic to $\mathfrak{B}_\kappa$. It is known that if $\omega\le\lambda\le\kappa\in\mathrm{K}(X)$ then $\lambda\in\mathrm{K}(X)$,

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so that $\mathrm{K}(X)$ must be an initial segment of the class of infinite cardinals. It may or may not contain its supremum, but once we know $\mathrm{K}(X)$ we can determine all the possible measure algebras of Radon probability measures on X, as follows. If $X = \emptyset$ there are none. If X is finite and not empty, we get algebras of the form $\mathcal{P}J$ where $1 \le \#(J) \le \#(X)$. If X is infinite, we get algebras of the form $\mathcal{P}J \times \prod_{i\in I} \mathfrak{B}_{\kappa_i}$, where I and J are countable sets and $\langle\kappa_i\rangle_{i\in I}$ is a family of distinct members of $\mathrm{K}(X)$; and these lists are complete.

So we turn to the determination of $\mathrm{K}(X)$. Because Radon measures are defined by their behaviour on compact sets, $\mathrm{K}(X) = \bigcup\{\mathrm{K}(K) : K \subseteq X$ is compact$\}$, so we begin by investigating compact spaces X. Now if X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous surjection, then $\mathrm{K}(Y) \subseteq \mathrm{K}(X)$ ([7], Prop. 2.1); if Z is a closed subset of X, then $\mathrm{K}(Z) \subseteq \mathrm{K}(X)$; further, $\sup \mathrm{K}(X) \le w(X)$, the topological weight of X; and $\kappa \in \mathrm{K}([0,1]^\kappa)$ for any infinite cardinal κ. What this means is that $\mathrm{K}([0,1]^\kappa)$ must be just $\{\lambda : \omega \le \lambda \le \kappa\}$, and if there is a continuous function from X onto $[0,1]^\kappa$ then $\kappa \in \mathrm{K}(X)$.

The question now arises: is there a converse to this result? If we know that $\kappa \in \mathrm{K}(X)$, when, if ever, can we deduce that there is a continuous surjection from X onto $[0,1]^\kappa$? I will say that an infinite cardinal κ has *Haydon's property* if whenever X is a compact Hausdorff space and $\kappa \in \mathrm{K}(X)$, then there is a continuous surjection from X onto $[0,1]^\kappa$. Of course, ω has Haydon's property, since for any compact Hausdorff space X with a non-empty perfect subset there is a continuous surjection from X onto $[0,1]^\omega$. The first investigation of the question was by R. G. Haydon, who showed that if κ is regular and $\lambda^\omega < \kappa$ for every $\lambda < \kappa$ (for instance, if $\kappa = \mathfrak{c}^+$), then κ has Haydon's property ([7], Theorem 2.4). Recently, G. Plebanek extended Haydon's result by showing that it is true for any cardinal κ such that $\mathrm{cf}(\kappa) \ge \omega_2$ and κ is a precaliber of $\mathfrak{B}_\kappa$ ([14], Theorem 4.1).

These results leave open the basic case $\kappa = \omega_1$. In [8], Haydon gave an example to show that if the continuum hypothesis is true then ω_1 does not have Haydon's property. This example has been refined and adapted in various ways ([10], [2]). In the present context, the best results are due to K. Kunen and J. van Mill, who showed that if $[0,1]^{\omega_1}$ can be covered by ω_1 negligible sets then ω_1 does not have Haydon's property ([11]), and to Plebanek, who showed that if κ is a cardinal of uncountable cofinality which is not a precaliber of $\mathfrak{B}_\kappa$ then κ does not have Haydon's property ([14], Theorem 4.2). M. R. Burke has pointed out that if we add ω_2 random reals to a model of ZFC+CH then ω_1 becomes a precaliber of $\mathfrak{B}_{\omega_1}$, but ω_1 does not have Haydon's property because the conditions of Theorem 6.2 of [14] are satisfied.

All these examples have served to concentrate attention on Martin's Axiom: what happens if Martin's Axiom is true and the continuum hypothesis is false? In this paper I show that under these circumstances ω_1 does have Haydon's property; and in fact the same is true of any infinite cardinal κ such that $\mathrm{MA}(\kappa)$ is true (Theorem 9 below).

2. Notation. I follow [3] in writing $\mathfrak{m}$ for the least cardinal such that $\mathrm{MA}(\mathfrak{m})$ is false, so that Martin's Axiom becomes "$\mathfrak{m} = \mathfrak{c}$". If I is a set, κ a cardinal then $[I]^\kappa$ is the set of subsets of I of cardinal κ, and $[I]^{<\omega} = \bigcup_{n\in\mathbb{N}}[I]^n$ is the set of finite subsets of I. If $\mathfrak{A}$ is a Boolean algebra, a cardinal κ is a *precaliber* of $\mathfrak{A}$ if for every family $\langle a_\alpha\rangle_{\alpha<\kappa}$ of non-zero elements of $\mathfrak{A}$, there is a set $A \in [\kappa]^\kappa$ such that $\inf_{\xi\in I} a_\xi \neq 0$ for every non-empty finite $I \subseteq A$. Note that if $\kappa < \mathfrak{m}$ has uncountable cofinality then κ is a precaliber of every ccc Boolean algebra ([3], 41Ca), and that ω_1 is a precaliber of $\mathfrak{B}_{\omega_1}$ iff $[0,1]^{\omega_1}$ is not the union of ω_1 negligible sets (use [6], A2U).

In a Boolean algebra $\mathfrak{A}$, I will use the symbol $\cap$ to represent "intersection" (the "product" when $\mathfrak{A}$ is regarded as a ring); $1 \setminus a$ will be the "complement" of a, and $\subseteq$ will denote the usual partial order of "inclusion".

3. I start by recalling a well-known fact about uncountable families of sets in probability spaces.

Lemma. *Let (X, Σ, ν) be a probability space and $\langle F_\xi\rangle_{\xi<\omega_1}$ a family of measurable sets of non-zero measure. Then there is an uncountable set $A \subseteq \omega_1$ such that* $\inf_{\xi,\eta\in A} \nu(E_\xi \cap E_\eta) > 0$ (*cf.* [1], *Theorem* 6.15; *see also* [4]).

4. Notation. The core of this proof is an investigation of certain properties of the algebras $\mathfrak{B}_\kappa$.

(a) Much of the argument will be based on the following straightforward idea. Let I be a set. If $E \subseteq \{0,1\}^I$, $J \subseteq I$ then I will say that E is *determined by coordinates in* J if $x \in E$ whenever $x \in \{0,1\}^I$ and there is a $y \in E$ such that $x\restriction J = y\restriction J$; equivalently, if there is a set $F \subseteq \{0,1\}^J$ such that $E = \pi_J^{-1}[F]$, where $\pi_J(x) = x\restriction J$ for $x \in \{0,1\}^I$; equivalently, if $E = \pi_J^{-1}[\pi_J[E]]$. Note that

(i) the family of sets determined by coordinates in J is closed under complements and arbitrary intersections and unions;

(ii) if E is determined by coordinates in J, and $J \subseteq K \subseteq I$, then E is determined by coordinates in K;

(iii) if E is determined by coordinates in J, and also determined by coordinates in K, then it is determined by coordinates in $J \cap K$.

(For if $x \in \{0,1\}^I$, $y \in E$ and $x\restriction J\cap K = y\restriction J\cap K$, define $z \in \{0,1\}^I$ by setting $z(i) = x(i)$ if $i \in J$, and $y(i)$ if $i \in I \setminus J$; then $z\restriction K = y\restriction K$, so $z \in E$, and $x\restriction J = z\restriction J$, so $x \in E$.)

(b) If $E \subseteq \{0,1\}^I$ and $J \subseteq I$, set

$$S_J(E) = \{x : x \in \{0,1\}^I,\ x + z \in E \text{ whenever } z \in \{0,1\}^I,\\ z(i) = 0 \text{ for every } i \in I \setminus J\},$$

writing $+$ for the usual group operation on $\{0,1\}^I$ derived from identifying it with $\mathbb{Z}_2^I$. Observe that

(i) $S_J(E)$ is the largest subset of E determined by coordinates in $I \setminus J$;

(ii) $S_J S_K(E) = S_{J\cup K}(E) \subseteq S_J(E)$;

(iii) if $K \subseteq I$ is such that E is determined by coordinates in K (i.e., $E = S_{I\setminus K}(E)$), then $S_J(E)$ is determined by coordinates in $K \setminus J$;

(iv) if J is finite and E is measurable (for the usual measure on $\{0,1\}^I$) then $S_J(E)$ is measurable;

(v) if E is closed then $S_J(E)$ is closed;

(vi) if E is a zero set (that is, in this context, E is a closed set which is determined by coordinates in some countable set), and J is countable, then $S_J(E)$ is a zero set.

5. Lemma. *Let I be a set and μ the usual measure on $X = \{0,1\}^I$. Let $E \subseteq X$ be a measurable set and $\langle I_k\rangle_{k\in\mathbb{N}}$ a disjoint sequence of subsets of I all of size at most n. Then $\lim_{k\to\infty} \mu S_{I_k}(E) = \mu E$.*

Proof. Let $\varepsilon > 0$. There is a set $F \subseteq X$, determined by a finite set K of coordinates, such that $\mu(E \triangle F) \le \varepsilon$. Take k_0 such that $I_k \cap K = \emptyset$ for $k \ge k_0$. Then for any $k \ge k_0$,

$$E \setminus S_{I_k}(E) \subseteq (E \setminus F) \cup \bigcup \{x : \exists z,\ z(i) = 0\ \forall i \in I \setminus I_k,\ x + z \in F \setminus E\}$$

has measure at most $2^n\varepsilon$. As ε is arbitrary, we have the result.

6. Without further ado, I proceed to the main result.

Theorem. *Suppose that $\kappa < \mathfrak{m}$, and let μ be the usual measure on $X = \{0,1\}^\kappa$. Let $\langle (E_\alpha, E'_\alpha)\rangle_{\alpha<\kappa}$ be a family of pairs of measurable subsets of X such that*

$$\{\alpha : \mu E_\alpha + \mu E'_\alpha \ge 1 - \varepsilon, x(\alpha) = 0 \textit{ for every } x \in E_\alpha,\\ x(\alpha) = 1 \textit{ for every } x \in E'_\alpha\}$$

has cardinal κ for every $\varepsilon > 0$. Then there is a set $D \subseteq \kappa$, of cardinal κ, such that

$$\mu\Big(X \cap \bigcap_{\alpha\in I} E_\alpha \cap \bigcap_{\beta\in J} E'_\beta\Big) > 0$$

for any disjoint finite sets $I, J \subseteq D$.

Proof. *Part A.* I first give the argument for the case in which κ has uncountable cofinality, as this is easier, and then turn to the modifications required if $\mathrm{cf}(\kappa) = \omega$.

(a) Write C for

$$\{\alpha : \mu E_\alpha + \mu E'_\alpha > 1/2, x(\alpha) = 0 \text{ for every } x \in E_\alpha,\\ x(\alpha) = 1 \text{ for every } x \in E'_\alpha\},$$

so that $\#(C) = \kappa$. For each $\alpha \in C$ there is a zero set Z_α, determined by coordinates in $\kappa \setminus \{\alpha\}$, such that $\mu Z_\alpha > 0$ and

$$\{x : x \in Z_\alpha,\ x(\alpha) = 0\} \subseteq E_\alpha, \qquad \{x : x \in Z_\alpha,\ x(\alpha) = 1\} \subseteq E'_\alpha.$$

(Take zero sets $F \subseteq E_\alpha$, $F' \subseteq E'_\alpha$ such that $\mu F + \mu F' > 1/2$. Set $Z_\alpha = S_{\{\alpha\}}(F \cup F')$; this works.)

(b) (i) Let P be the set of all pairs (I, F) where $I \in [C]^{<\omega}$, $F \subseteq \bigcap_{\beta\in I} Z_\beta$ is a zero set determined by coordinates in $\kappa \setminus I$ and $\mu F > 0$. Order P by saying that $(I, F) \le (I', F')$ if $I \subseteq I'$ and $F' \subseteq F$; then P is a partially ordered set. Note that $q_\alpha = (\{\alpha\}, Z_\alpha)$ belongs to P for every $\alpha \in C$.

(ii) P is upwards-ccc. To see this, let $\langle (I_\xi, F_\xi)\rangle_{\xi<\omega_1}$ be a family in P. Then we can find an uncountable set $A \subseteq \omega_1$ such that

(α) $\langle I_\xi\rangle_{\xi\in A}$ is a constant-size Δ-system with root I say;

(β) whenever $\xi < \eta$ in A, $\alpha \in I_\xi$ and $\beta \in I_\eta \setminus I$ then $\alpha < \beta$;

(γ) whenever $\xi < \eta$ in A, then F_ξ is determined by coordinates in $\kappa \setminus (I_\eta \setminus I)$;

(δ) there is a $\delta > 0$ such that $\mu(F_\xi \cap F_\eta) \ge \delta$ for every $\xi, \eta \in A$ (using Lemma 3).

Let $\langle \xi_k\rangle_{k\in\mathbb{N}}$ be a strictly increasing sequence in A and η a member of A greater than any ξ_k. Because $\#(I_{\xi_k} \setminus I)$ is the same for all k, Lemma 5 tells us that there is a k such that

$$\mu S_{I_{\xi_k}\setminus I}(F_\eta) > \mu F_\eta - \delta.$$

Set $F'_\eta = S_{I_{\xi_k}\setminus I}(F_\eta)$. Then

$$\mu(F_\eta \setminus F'_\eta) < \delta, \qquad \mu(F_{\xi_k} \cap F'_\eta) > 0.$$

Now we know that F_{ξ_k} is determined by coordinates in $\kappa \setminus I_{\xi_k}$ and also by coordinates in $\kappa \setminus (I_\eta \setminus I)$, and is therefore determined by coordinates in $\kappa \setminus J$, where $J = I_{\xi_k} \cup I_\eta$. Similarly, F'_η is determined by coordinates in $(\kappa \setminus I_\eta) \setminus (I_{\xi_k} \setminus I) = \kappa \setminus J$, so $F = F_{\xi_k} \cap F'_\eta$ is also determined by coordinates in $\kappa \setminus J$, while $\mu F > 0$. Finally,

$$F \subseteq F_{\xi_k} \cap F_\eta \subseteq \bigcap_{\beta\in J} Z_\beta.$$

This means that $(J, F) \in P$, and evidently it is a common upper bound for (I_{ξ_k}, F_{ξ_k}) and (I_η, F_η).

Thus $\langle (I_\xi, F_\xi)\rangle_{\xi<\kappa}$ is not an up-antichain. As $\langle (I_\xi, F_\xi)\rangle_{\xi<\kappa}$ is arbitrary, P is upwards-ccc.

(c) Because $\mathfrak{m} > \kappa$, there is a sequence $\langle R_n\rangle_{n\in\mathbb{N}}$ of upwards-directed subsets of P such that $q_\alpha \in \bigcup_{n\in\mathbb{N}} R_n$ for every $\alpha \in C$ (see [3], 41Ca). Because $\#(C) = \kappa$ and $\mathrm{cf}(\kappa) \ge \omega_1$, there is an n such that $D = \{\alpha : q_\alpha \in R_n\}$ has cardinal κ. If $I, J \in [D]^{<\omega}$ are disjoint there is a $(K, F) \in R_n$ which is an upper bound for $\{q_\alpha : \alpha \in I \cup J\}$, so that $I \cup J \subseteq K$; now F is determined by coordinates in $\kappa \setminus (I \cup J)$, and

$$F \cap E_\alpha = F \cap Z_\alpha \cap E_\alpha = \{x : x \in F,\ x(\alpha) = 0\} \quad \text{for every } \alpha \in I,$$
$$F \cap E'_\beta = F \cap Z_\beta \cap E'_\beta = \{x : x \in F,\ x(\beta) = 1\} \quad \text{for every } \beta \in J,$$

so

$$\begin{aligned}\mu\Big(X \cap \bigcap_{\alpha\in I} E_\alpha \cap \bigcap_{\beta\in J} E'_\beta\Big)&\\ &\ge \mu\{x : x \in F,\ x(\alpha) = 0\ \forall\alpha \in I,\ x(\beta) = 1\ \forall\beta \in J\}\\ &= 2^{-\#(I\cup J)}\mu F > 0.\end{aligned}$$

Thus D has the declared property.

Part B. I now turn to the adaptations required if $\mathrm{cf}(\kappa) \le \omega$. If $\kappa \le \omega$ the result is easy and of no importance, so I leave it as an exercise for any reader who wishes to check her understanding of the hypotheses. For the case $\omega = \mathrm{cf}(\kappa) < \kappa$, start by expressing κ as the union of a strictly increasing sequence $\langle\kappa_j\rangle_{j\in\mathbb{N}}$ of regular uncountable cardinals. Set

$$\begin{aligned}C_m = \Big\{\alpha : \mu E_\alpha + \mu E'_\alpha > \frac{m+1}{m+2},&\\ &x(\alpha) = 0 \text{ for every } x \in E_\alpha,\ x(\alpha) = 1 \text{ for every } x \in E'_\alpha\Big\},\end{aligned}$$

so that C_m has cardinal κ for each $m \in \mathbb{N}$, by hypothesis. Let $\langle C_{mj}\rangle_{m,j\in\mathbb{N}}$ be a partition of $C = C_0$ such that $C_{mj} \subseteq C_m$ and $\#(C_{mj}) = \kappa_j$ for all $m, j \in \mathbb{N}$. Write $C'_j = \bigcup_{m\in\mathbb{N}} C_{mj}$ for each j, so that $\langle C'_j\rangle_{j\in\mathbb{N}}$ is disjoint and $\#(C'_j \cap C_m) = \kappa_j$ for every j, $m \in \mathbb{N}$.

(a) For j, $m \in \mathbb{N}$ and $\alpha \in C_{mj}$ there is a zero set Z_α, determined by coordinates in $\kappa\setminus\{\alpha\}$, such that $\mu Z_\alpha > m/(m+2)$ and $\{x : x \in Z_\alpha,\ x(\alpha) = 0\} \subseteq E_\alpha$, $\{x : x \in Z_\alpha,\ x(\alpha) = 1\} \subseteq E'_\alpha$. (In (A-a) above, take F, F' such that $\mu F + \mu F' > (m+1)/(m+2)$ and continue as before.)

(b) Define the partially ordered set P as in (A-b-i); as in (A-b-ii), P is ccc. Once again, set $q_\alpha = (\{\alpha\}, Z_\alpha)$ for $\alpha \in C$.

(c) Now for the new idea.

(i) For any $m, j \in \mathbb{N}$ there is a finite set $I^*_{mj} \subseteq \kappa$ such that whenever $(I, F) \in P$ and $I \cap I^*_{mj} = \emptyset$, $\#(I) \le m$, $\mu F \ge 2/(m+2)$ then

$$A_{mj}(I, F) = \{\alpha : \alpha \in C_{mj},\ (I, F) \text{ and } q_\alpha \text{ are compatible in } P\}$$

has cardinal κ_j. For suppose, if possible, otherwise. Then we can find a sequence $\langle(I_k,F_k)\rangle_{k\in\mathbb{N}}$ in P such that $\#(I_k)\le m$, $\mu F_k\ge 2/(m+2)$, $\#(A_{mj}(I_k,F_k))<\kappa_j$ for every k and $\langle I_k\rangle_{k\in\mathbb{N}}$ is disjoint. For each k, let $J_k\subseteq\kappa$ be a countable set such that F_k is determined by coordinates in J_k. Because κ_j has uncountable cofinality, there is an $\alpha\in C_{mj}\setminus\bigcup_{k\in\mathbb{N}}(A_{mj}(I_k,F_k)\cup J_k)$. But now Lemma 5 tells us that

$$\lim_{k\to\infty}\mu S_{I_k}(Z_\alpha)=\mu Z_\alpha>\frac{m}{m+2},$$

so there is some k such that $\mu S_{I_k}(Z_\alpha)>m/(m+2)$. Set $F'=F_k\cap S_{I_k}(Z_\alpha)$. Then

$$\mu F'\ge\mu F_k+\mu S_{I_k}(Z_\alpha)-1>0,$$

and F' is determined by coordinates in $\kappa\setminus(I_k\cup\{\alpha\})$, so $(I_k\cup\{\alpha\},F')$ witnesses that (I_k,F_k) and q_α are compatible, which is supposed to be impossible.

(ii) Set $I^*=\bigcup_{m,j\in\mathbb{N}}I^*_{mj}$, so that I^* is countable. Set $P^*=\{(I,F):(I,F)\in P,\ I\cap I^*=\emptyset\}$. Then two members of P^* are compatible in P^* iff they are compatible in P (if (I_1,F_1), (I_2,F_2) belong to P^* and have a common upper bound $(I,F)\in P$, then $(I_1\cup I_2,F)\in P^*$ is still an upper bound), so P^* is also ccc.

(iii) Enumerate each C'_j as $\langle\gamma_\xi\rangle_{\xi<\kappa_j}$. For $\beta<\kappa_j$, set

$$Q_{j\beta}=\{(I,F):(I,F)\in P^*,\ \exists\xi,\ \beta\le\xi<\kappa_j,\ \gamma_\xi\in I\}.$$

Then $Q_{j\beta}$ is cofinal with P^*. For if $(I,F)\in P^*$, there is some $m\in\mathbb{N}$ such that $\#(I)\le m$ and $\mu F\ge 2/(m+2)$. Because $I\cap I^*_{mj}=\emptyset$, $A_{mj}(I,F)\subseteq C'_j$ has cardinal κ_j and there must be some $\xi\ge\beta$ such that $\gamma_\xi\in A_{mj}(I,F)\setminus I^*$. But now (I,F), q_{γ_ξ} have a common upper bound in P and therefore a common upper bound in P^*, which is a member of $Q_{j\beta}$ greater than or equal to (I,F).

(iv) Because $\kappa<\mathfrak{m}$, there is an upwards-directed set $R\subseteq P^*$ meeting every $Q_{j\beta}$. Set $D=\bigcup\{I:(I,F)\in R\}$. Because R meets every $Q_{j\beta}$, we have $\#(D\cap C'_j)=\kappa_j$ for every j and $\#(D)=\kappa$. And D has the property required by the theorem, just as in (A-c) above.

7. Corollary. *Suppose that $\omega\le\kappa<\mathfrak{m}$. Let $(\mathfrak{A},\overline{\mu})$ be a probability algebra. Suppose that $\langle a_\alpha\rangle_{\alpha<\kappa}$ is a stochastically independent family of elements of measure $1/2$, and that for each $\alpha<\kappa$ we are given elements $e_\alpha\subseteq a_\alpha$, $e'_\alpha\subseteq 1\setminus a_\alpha$ such that $\{\alpha:\overline{\mu}e_\alpha+\overline{\mu}e'_\alpha\ge 1-\varepsilon\}$ has cardinal κ for every $\varepsilon>0$. Then there is a set $D\subseteq\kappa$, of cardinal κ, such that $\inf_{\alpha\in I}e_\alpha\cap\inf_{\beta\in J}e'_\beta\ne 0$ for all disjoint finite $I,J\subseteq D$.*

Proof. Let $\mathfrak{A}_0$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\alpha:\alpha<\kappa\}$, so that $\mathfrak{A}_0$ is isomorphic (as measure algebra) to $\mathfrak{B}_\kappa$. Let $C\subseteq\kappa$ be a set such that C and $\kappa\setminus C$ both have cardinal κ; let $h:\kappa\to C$ be a bijection.

Then we have a measure-preserving Boolean homomorphism $\pi_0 : \mathfrak{A}_0 \to \mathfrak{B}_\kappa$ defined by saying that $\pi a_\alpha = b_{h(\alpha)}$, where $b_\beta \in \mathfrak{B}_\kappa$ is the equivalence class of $G_\beta = \{x : x(\beta) = 0\}$. We can identify $\mathfrak{B}_\kappa$ with the probability algebra free product $\pi_0[\mathfrak{A}] \mathbin{\widehat{\oplus}} \mathfrak{C}$, where $\mathfrak{C}$ is the closed subalgebra generated by $\{b_\beta : \beta \in \kappa \setminus C\}$ (cf. [5], 2.25b). Let $\mathfrak{A}_1$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\alpha : \alpha < \kappa\} \cup \{e_\alpha : \alpha < \kappa\} \cup \{e'_\alpha : \alpha < \kappa\}$. Then π_0 has an extension to a measure-preserving Boolean homomorphism $\pi_1 : \mathfrak{A}_1 \to \mathfrak{B}_\kappa$ (see [5], 3.11a).

Let μ be the usual measure on $\{0,1\}^\kappa$. For $\beta \in C$, take $E_\beta \subseteq G_\beta$, $E'_\beta \subseteq \{0,1\}^\kappa \setminus G_\beta$ such that $E^\bullet_\beta = \pi_1 e_\alpha$, $(E'_\beta)^\bullet = \pi_1 e'_\alpha$, where $\alpha = h^{-1}(\beta)$. For $\beta \in \kappa \setminus C$, set $E_\beta = E'_\beta = \emptyset$. Then

$$\#(\{\beta : \mu E_\beta + \mu E'_\beta \geq 1 - \varepsilon\}) = \#(\{\alpha : \overline{\mu} e_\alpha + \overline{\mu} e'_\alpha \geq 1 - \varepsilon\}) = \kappa$$

for every $\varepsilon > 0$.

Applying Theorem 6, there is a set $D_0 \in [\kappa]^\kappa$ such that $\mu(\bigcap_{\alpha\in I} E_\alpha \cap \bigcap_{\beta\in J} E'_\beta) > 0$ for all disjoint finite $I, J \subseteq D$, that is, $\overline{\mu}(\inf_{\alpha\in I} e_\alpha \cap \inf_{\beta\in J} e'_\beta) > 0$ for all disjoint finite $I, J \subseteq h^{-1}[D_0]$. Of course, $D_0 \subseteq C$. So we take $D = h^{-1}[D_0]$.

8. Corollary. *Suppose that $\omega \leq \kappa < \mathfrak{m}$. Let (X, Σ, μ) be a probability space. Suppose that $\langle G_\alpha\rangle_{\alpha<\kappa}$ is a stochastically independent family of elements of measure $1/2$, and that for each $\alpha < \kappa$ we have measurable sets $E_\alpha \subseteq G_\alpha$, $E'_\alpha \subseteq X \setminus G_\alpha$ such that $\{\alpha : \mu E_\alpha + \mu E'_\alpha \geq 1 - \varepsilon\}$ has cardinal κ for every $\varepsilon > 0$. Then there is a set $D \subseteq \kappa$, of cardinal κ, such that $\mu(X \cap \bigcap_{\alpha\in I} E_\alpha \cap \bigcap_{\beta\in J} E'_\beta) > 0$ for all disjoint finite I, $J \subseteq D$.*

Proof. Apply Corollary 7 with $(\mathfrak{A}, \overline{\mu})$ the measure algebra of μ, $a_\alpha = G^\bullet_\alpha$, $e_\alpha = E^\bullet_\alpha$ and $e'_\alpha = (E'_\alpha)^\bullet$ for each α.

9. Theorem. *Suppose that $\omega \leq \kappa < \mathfrak{m}$. Let $(X, \mathfrak{T})$ be a compact Hausdorff space. Then the following are equivalent*:

(i) *there is a Radon probability measure μ on X with measure algebra isomorphic to $\mathfrak{B}_\kappa$*;

(ii) *there is a continuous surjection from X onto $[0,1]^\kappa$.*

Proof. The implication (ii)⇒(i) is discussed in §1 above (and does not depend on the assumption $\mathfrak{m} > \kappa$); this proof will therefore address (i)⇒(ii). Let $\langle G_\alpha\rangle_{\alpha<\kappa}$ be a stochastically independent family of sets of measure $1/2$. Let $\langle C_m\rangle_{m\in\mathbb{N}}$ be a partition of κ into sets of size κ. For each α, we can find compact sets $E_\alpha \subseteq G_\alpha$, $E'_\alpha \subseteq X \setminus G_\alpha$ such that $\mu E_\alpha + \mu E'_\alpha > m/(m+1)$ if $\alpha \in C_m$. By Corollary 8, there is a set $D \in [\kappa]^\kappa$ such that $\mu(X \cap \bigcap_{\alpha\in I} E_\alpha \cap \bigcap_{\beta\in J} E'_\beta) > 0$ for all disjoint finite $I, J \subseteq D$. In particular, $\bigcap_{\alpha\in I} E_\alpha \cap \bigcap_{\beta\in J} E'_\beta$ is non-empty for all disjoint finite $I, J \subseteq D$; that is, $\langle(E_\alpha, E'_\alpha)\rangle_{\alpha\in D}$

is independent in the sense of [7]. By Lemma 2.1 of [7], $[0,1]^\kappa$ is a continuous image of X.

10. Remarks. (a) If you look back at the demonstration that the partially ordered set P is ccc, in (A-b-ii) of the proof of Theorem 6, you will see that what I show there is that, given a family $\langle p_\xi\rangle_{\xi<\omega_1}$ in P, there is an uncountable set $A\subseteq\omega_1$ such that whenever $\langle\xi_k\rangle_{k\in\mathbb{N}}$ is a strictly increasing sequence in A, and $\eta\in A$ is greater than any ξ_k, then there is some k such that p_{ξ_k}, p_η are compatible in P. But this means that there must be an uncountable $A'\subseteq A$ such that p_ξ, p_η are compatible for all $\xi,\eta\in A'$, by the Erdős–Dushnik–Miller theorem ([3], A2K). Accordingly, P satisfies Knaster's condition ([3], 11A). In all the results of this paper, therefore, we may replace "$\mathfrak{m}>\kappa$" by "$\mathfrak{m}_{\mathrm{K}}>\kappa$", where $\mathfrak{m}_{\mathrm{K}}\geq\mathfrak{m}$ is the cardinal associated with "Martin's axiom for partially ordered sets with Knaster's condition" ([3], 11D). (In part (A-c) of the proof of Theorem 6, we must now use 31B of [3] in place of 41C.)

(b) I note that when $\mathrm{cf}(\kappa)\geq\omega_1$ then the hypotheses of §§6–8 are unnecessarily elaborate. Part A of the proof of Theorem 6 demands only that $\mu E_\alpha+\mu E'_\alpha>1/2$ for κ indices α. So in Corollary 7, for instance, we have:

Suppose that $\kappa<\mathfrak{m}_{\mathrm{K}}$ *is a cardinal of uncountable cofinality. Let* $(\mathfrak{A},\overline{\mu})$ *be a probability algebra. Suppose that* $\langle a_\alpha\rangle_{\alpha<\kappa}$ *is a stochastically independent family of elements of measure* $1/2$ *in* $\mathfrak{A}$, *and that for each* $\alpha<\kappa$ *we are given elements* $e_\alpha\subseteq a_\alpha$, $e'_\alpha\subseteq 1\setminus a_\alpha$ *such that* $\overline{\mu}e_\alpha+\overline{\mu}e'_\alpha>1/2$ *for every* α. *Then there is a set* $D\subseteq\kappa$, *of cardinal* κ, *such that* $\inf_{\alpha\in I}e_\alpha\cap\inf_{\beta\in J}e'_\beta\neq 0$ *for all disjoint finite* $I,J\subseteq D$.

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