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Multiplicative functionals and entire functions, II

by

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Abstract. Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let F be a nonconstant entire function, and let T be a linear functional with $T(e) = 1$ and such that $T \circ F : \mathcal{A} \to \mathbb{C}$ is nonsurjective. Then T is multiplicative.

Introduction. Let T be a nonzero multiplicative functional on a complex Banach algebra $\mathcal{A}$ with a unit e, and let $\mathcal{A}^{-1}$ denote the set of all invertible elements of $\mathcal{A}$. Then $T(e) = 1$, and $T(x) \neq 0$ for any $x \in \mathcal{A}^{-1}$. A. M. Gleason [5] and, independently, J. P. Kahane & W. Żelazko [8], [9] proved that the converse implication also holds.

THEOREM 1 [G-K-Ż]. *If T is a linear functional on a complex unital Banach algebra $\mathcal{A}$ such that $T(e) = 1$ and*

$$T(x) \neq 0 \quad \textit{for } x \in \mathcal{A}^{-1},$$

then T is multiplicative.

In fact, they proved even a stronger result.

THEOREM 2 [G-K-Ż]. *If T is a linear functional on a complex unital Banach algebra $\mathcal{A}$ such that $T(e) = 1$ and*

$$T(x) \neq 0 \quad \textit{for } x \in \exp \mathcal{A}, \tag{1}$$

then T is multiplicative.

Here $\exp \mathcal{A} = \{\exp y : y \in \mathcal{A}\}$. In 1987 R. Arens asked if the exponential function above can be replaced by an arbitrary nonconstant entire function F, that is, whether

$$T(x) \neq 0 \quad \text{for } x \in F(\mathcal{A}) := \{F(y) : y \in \mathcal{A}\}$$

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implies that $T/T(e)$ is multiplicative. In this note we prove the Arens conjecture.

The Gleason–Kahane–Żelazko theorem has been extended in several other directions; a number of problems remain open [6], see also [1], [2] and [7] for another version of the Arens conjecture.

2. The result

THEOREM 3. *Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let F be a nonconstant entire function, and let T be a linear functional on $\mathcal{A}$. Then*

(i) *if F is nonsurjective then $T \circ F : \mathcal{A} \to \mathbb{C}$ is nonsurjective if and only if either $T/T(e)$ is multiplicative or $T = 0$,*

(ii) *if F is surjective then $T \circ F : \mathcal{A} \to \mathbb{C}$ is nonsurjective if and only if $T = 0$.*

Notice that we do not assume that T is continuous.

COROLLARY 4. *Let $\mathcal{A}$ be a complex Banach algebra with a unit e, let F be a nonconstant entire function, and let T be a linear functional on $\mathcal{A}$ such that $T(e) = 1$ and*

$$T(x) \neq 0 \quad \text{for } x \in F(\mathcal{A}). \tag{2}$$

Then T is multiplicative.

COROLLARY 5. *Let $\mathcal{A}$ be a complex Banach algebra with a unit e, and let F be a surjective entire function. Then the linear span of $F(\mathcal{A})$ is $\mathcal{A}$.*

Proof of Theorem 3. Assume first that T is a multiplicative functional and F is a nonsurjective entire function. By the Weierstrass Factorization Theorem [3] any nonsurjective entire function F is of the form $F(z) = a + \exp g(z)$ for some entire function g and a constant a. For any x in $\mathcal{A}$ we have

$$T \circ F(x) = T(ae + \exp g(x)) = a + \exp(g(Tx)) \neq a,$$

so $T \circ F : \mathcal{A} \to \mathbb{C}$ is nonsurjective.

To prove the remaining parts of the theorem we need to show that T is continuous.

LEMMA 6. *If $T \circ F : \mathcal{A} \to \mathbb{C}$ is nonsurjective then T is continuous.*

Proof. Since F is nonconstant there is a point $z_0 \in \mathbb{C}$ such that $F'(z_0) \neq 0$. Replacing $F(z)$ with $F(z + z_0) - F(z_0)$ we may assume that $F'(0) \neq 0$ so that $F(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ is a homeomorphism on a neighborhood U of the origin. Let

$$(F_{|U})^{-1}(w) = \sum_{n=1}^{\infty} \beta_n w^n$$

for all w sufficiently close to 0. We have

$$w = \sum_{k=1}^{\infty} \alpha_k \Big(\sum_{n=1}^{\infty} \beta_n w^n \Big)^k = \alpha_1 \beta_1 w + (\alpha_1 \beta_1^2 + \alpha_2 \beta_1) w^2 + (\ldots) w^3 + \ldots$$

Since the power series representation is unique $\alpha_1 \beta_1 = 1$ and all the coefficients of the higher powers of w are zero. Hence for all sufficiently small $x \in \mathcal{A}$ we have $F \circ (F_{|U})^{-1}(x) = x$, so $(F_{|U})^{-1}$ also defines a local inverse of the map $F : \mathcal{A} \to \mathcal{A}$. It follows that the range of $F : \mathcal{A} \to \mathcal{A}$ has a nonempty interior in $\mathcal{A}$. However, each discontinuous functional on any Banach space maps every open nonempty subset of the space *onto* the set of scalars. So, if $T : \mathcal{A} \to \mathbb{C}$ were discontinuous, then $T \circ F$ would be surjective, contrary to our assumption. ∎

To end the proof of the theorem we assume that $T \circ F : \mathcal{A} \to \mathbb{C}$ is nonsurjective and we consider two cases:

$$Te \neq 0 \quad \text{or} \quad Te = 0.$$

We show that in the first case F is nonsurjective and $T/T(e)$ is multiplicative, and in the second case $T = 0$.

Case $Te \neq 0$. Dividing by Te we may then assume, without loss of generality, that $Te = 1$.

By [9], T is multiplicative on $\mathcal{A}$ if and only if T is multiplicative on every commutative subalgebra of $\mathcal{A}$, so without loss of generality we may assume that our algebra is commutative.

For any x, y in a commutative algebra we have $xy = ((x+y)^2 - (x-y)^2)/4$ so to prove that T is multiplicative it is sufficient to show that it preserves the operation of taking the square. Thus it is sufficient to prove that T is multiplicative on subalgebras with one generator. Consequently, we may assume that $\mathcal{A}$ has a single generator.

Step 1. We now show that without loss of generality we may also assume that $F(z) = \exp g(z)$ for some entire function g, and that the range of $T \circ F : \mathcal{A} \to \mathbb{C}$ is $\mathbb{C} \setminus \{0\}$.

Assume that $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is surjective. Then, since $T(e) = 1$, for any $\lambda \in \mathbb{C}$ we have

$$T \circ F(\mathcal{A}) \supseteq T(F(\lambda e)) = T\Big(\sum_{n=0}^{\infty} a_n (e\lambda)^n \Big) = Te\Big(\sum_{n=0}^{\infty} a_n \lambda^n \Big) = F(\lambda), \tag{3}$$

so the range of $T \circ F : \mathcal{A} \to \mathbb{C}$ contains the range of $F : \mathbb{C} \to \mathbb{C}$. Hence, by our assumption, F is nonsurjective. By the Weierstrass Factorization Theorem F is of the form $F(z) = a + \exp g(z)$. Since T is linear, $T \circ F$ is nonsurjective if and only if $T \circ \exp g$ is nonsurjective. Hence we may assume that $a = 0$,

so that
$$F(z) = \exp g(z).$$

It follows that the value missing from the range of F considered as an entire function on the complex plane is zero. By (3) the number missing from the range of $T \circ F : \mathcal{A} \to \mathbb{C}$ is also zero. ∎

Step 2. We now prove the theorem for the disc algebra $A(\mathbb{D})$, that is, for the algebra of all analytic functions on the open unit disc $\mathbb{D}$ that extend continuously to $\overline{\mathbb{D}}$, and later we show how the general result follows from this special case.

So suppose that $\mathcal{A} = A(\mathbb{D})$. Denote by $\mathbf{Z}$ the identity function on $\mathbb{C}$, and fix a nonzero complex number $\lambda = re^{i\theta}$. Since g is a nonconstant entire function there is a region Ω in $\mathbb{C}$ such that g is one-to-one on the closure $\overline{\Omega}$ of Ω and $g(\overline{\Omega})$ is a closed disc of radius r and center at some point w_0. The existence of such a region Ω is obvious if g is a linear function; if g is a nonlinear entire function then the derivative of g is unbounded and the existence of Ω follows from Bloch's Theorem [3]. By the Riemann Mapping Theorem there is a conformal homeomorphism $\varkappa$ from the unit disc onto Ω, and since the boundary of Ω is homeomorphic to the unit circle, $\varkappa$ can be extended to a homeomorphism between the closed unit disc $\overline{\mathbb{D}}$ and $\overline{\Omega}$ (see [4]). The function
$$f(z) := (g \circ \varkappa(z) - w_0)/r$$
is an analytic homeomorphism of $\overline{\mathbb{D}}$ onto itself. Put
$$\psi(z) = \varkappa(f^{-1}(e^{i\theta} z)).$$
We have $\psi \in A(\mathbb{D})$ and $g \circ \psi = \lambda \mathbf{Z} + w_0$. Hence, by our assumption,

(4) $$T(e^{\lambda \mathbf{Z}}) = T(e^{g\circ\psi - w_0}) = e^{-w_0} T(F(\psi)) \neq 0.$$

The rest of this step runs now exactly as in the original papers by Gleason [5] and Kahane & Żelazko [8].

Put

(5) $$\varphi(\lambda) = T(e^{\lambda \mathbf{Z}}) = T\bigg(\sum_{n=0}^{\infty} \frac{(\lambda \mathbf{Z})^n}{n!}\bigg) = \sum_{n=0}^{\infty} \frac{T(\mathbf{Z}^n)}{n!}\lambda^n.$$

For any λ we have
$$|\varphi(\lambda)| \le \|T\| \cdot \|e^{\lambda \mathbf{Z}}\| = \|T\| e^{|\lambda|}$$
and by (4), $\varphi(\lambda) \neq 0$, so by the Weierstrass Factorization Theorem and by Hadamard's Factorization Theorem [3],
$$\varphi(\lambda) = e^{\alpha\lambda + \beta}.$$

Since $\varphi(0) = T(e^0) = T(e) = 1$, we have $\beta = 0$, hence

(6) $$\varphi(\lambda) = e^{\alpha\lambda} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\lambda^n.$$

Comparing (5) and (6) we get
$$T(\mathbf{Z}^n) = (T(\mathbf{Z}))^n \quad \text{for } n \in \mathbb{N}.$$
Since polynomials are dense in $A(\mathbb{D})$ it follows that T is multiplicative of the form $T(f) = f(\alpha)$ for $f \in A(\mathbb{D})$, where $\alpha = T(\mathbf{Z})$.

Step 3. Let now $\mathcal{A}$ be an arbitrary complex Banach algebra with a single generator x, that is, the polynomials of x are dense in $\mathcal{A}$. The spectrum K of x is simply connected and without loss of generality we may assume that K is a subset of the open unit disc. So for any $a(z) = \sum \gamma_n z^n$ in $A(\mathbb{D})$, $a(x) = \sum \gamma_n x^n$ is a well defined element of $\mathcal{A}$. We define $\widetilde{T} : A(\mathbb{D}) \to \mathbb{C}$ by
$$\widetilde{T}(a) = T(a(x)).$$
Then $\widetilde{T}$ is a linear functional which maps $F(A(\mathbb{D})) = \{e^{g(a)} : a \in A(\mathbb{D})\}$ into $\mathbb{C} \setminus \{0\}$ and such that $\widetilde{T}(e) = 1$. By the previous steps $\widetilde{T}$ is continuous and there is $\alpha \in \overline{\mathbb{D}}$ such that
$$\widetilde{T}(a) = a(\alpha) \quad \text{for } a \in A(\mathbb{D}).$$
So T is multiplicative on the dense subset $\{a(x) : a \in A(\mathbb{D})\}$ of $\mathcal{A}$. Since, by Step 2, T is continuous it follows that T is multiplicative.

Case $Te = 0$. We still assume that $T \circ F : \mathcal{A} \to \mathbb{C}$ is not surjective. Since $Te = 0$, the value missing from the range of $T \circ F$ is not 0, so we can divide T by the missing number and assume without loss of generality that $T \circ F$ does not assume value 1. As before, we first consider the case when $\mathcal{A}$ is the disc algebra $A(\mathbb{D})$.

Fix a nonzero complex number $\lambda = re^{i\theta}$ and a positive integer n. As in Step 2 before, by Bloch's Theorem and the Riemann Mapping Theorem there is a region Ω in $\mathbb{C}$ such that F is injective on $\overline{\Omega}$ and $F(\overline{\Omega})$ is a closed disc of radius r and center at some point w_0. Notice that in contrast to the previous case the region Ω is selected directly for the function F, and not for the function $g = \ln F$ which may not be well defined now. Put
$$f(z) = (F \circ \varkappa(z) - w_0)/r \quad \text{and} \quad \psi(z) = \varkappa(f^{-1}(e^{i\theta} z^n)).$$
We have $\psi \in A(\mathbb{D})$ and $F \circ \psi = \lambda \mathbf{Z}^n + w_0$. Hence, by our assumptions, for arbitrary λ we have
$$\lambda T(\mathbf{Z}^n) = T(\lambda \mathbf{Z}^n) = T(F \circ \psi - w_0 e) = T(F \circ \psi) \neq 1.$$
Thus $T(\mathbf{Z}^n) = 0$ for $n \in \mathbb{N} \cup \{0\}$, so $T = 0$.

Now, if $\mathcal{A}$ is an arbitrary Banach algebra and $\mathcal{A}_0$ its subalgebra with a single generator, it follows from the disc algebra case, exactly as in Step 3 before, that T is zero on $\mathcal{A}_0$, and consequently T is zero on $\mathcal{A}$. ■

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