if, and only if,

$$n^{-1}\sum_{j=1}^{n}\lambda_p^{k_j}\to 0\quad (n\to\infty)$$

for each λ_p *in* $\sigma(T_1)$ *which is not equal to one.*

A similar result holds for an operator T which generates a w-compact monothetic group $G(T)$ of invertible operators in $B(X)$ using the representation of $G(T)$ in terms of the unimodular eigenvalues of T given in [4], Theorem 3.3.

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Cyclic space isomorphism of unitary operators

by

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Abstract. We introduce a new equivalence relation between unitary operators on separable Hilbert spaces and discuss a possibility to have in each equivalence class a measure-preserving transformation.

Introduction. Let U be a unitary operator on a separable Hilbert space H. For any $x\in H$ we define the *cyclic space* generated by x as $Z(x)=\operatorname{span}\{U^n x : n\in\mathbb{Z}\}$. By the *spectral measure* μ_x of x we mean the Borel measure on the circle determined by the equalities

$$\widehat{\mu}_x(n)=\int_{\mathbb{T}} z^n\,d\mu_x(z)=(U^n x,x)$$

for every $n\in\mathbb{Z}$.

THEOREM 0.1 (spectral theorem, see [9]). *There exists in H a sequence $x_1,x_2,\ldots$ such that*

$$H=\bigoplus_{n=1}^{\infty}Z(x_n)\quad\text{and}\quad \mu_{x_1}\gg\mu_{x_2}\gg\ldots \tag{1}$$

Moreover, for any sequence $y_1,y_2,\ldots$ in H satisfying (1) *we have* $\mu_{x_1}\equiv\mu_{y_1}$, $\mu_{x_2}\equiv\mu_{y_2},\ldots$

One of the most important (still open) problems in ergodic theory is a classification of ergodic dynamical systems with respect to spectral equivalence, i.e. given a sequence

$$\mu_1\gg\mu_2\gg\ldots \tag{2}$$

of positive finite measures on the circle we ask if there exists an ergodic dynamical system $T:(X,\mathcal{B},\varrho)\to(X,\mathcal{B},\varrho)$ such that some spectral sequence (1) for $U=U_T$ ($U_T: L^2(X,\varrho)\to L^2(X,\varrho)$, $U_Tf=f\circ T$) coincides with (2).

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The spectral type of μ_{x_1} (the equivalence class of measures) is called the *maximal spectral type* of U. By the *multiplicity function* M_U of U we mean the function $M_U : \mathbb{T} \to \mathbb{N} \cup \{+\infty\}$ given by

$$M_U(z) = \sum_{n=1}^{\infty} \chi_{A_n}(z)$$

where $A_1 = \mathbb{T}$ and $A_n = A_n(U) = \{z \in \mathbb{T} : \frac{d\mu_{x_n}}{d\mu_{x_1}}(z) > 0\}$ (it is well defined up to a μ_{x_1}-null set). Then

$$\mathbb{T} = A_1 \supset A_2 \supset \ldots$$

The set

$$E(U) = \{n \in \mathbb{N} \cup \{+\infty\} : \mu_{x_1}\{z \in \mathbb{T} : M_U(z) = n\} > 0\}$$

is called the set of *essential values* of the multiplicity function M_U.

For the background on spectral theory we refer to [3].

In the last few years, problems concerning spectral multiplicity have become of a renewed interest (see [1], [2], [4], [6]–[8], [10], [11]). In [5], M. Lemańczyk and J. Kwiatkowski Jr. construct, for an arbitrary set $A \subseteq \mathbb{N}^+$ containing 1, an ergodic automorphism T whose set of essential values of the multiplicity function is equal to A. The aim of this paper is to present a new viewpoint on spectral classification suggested to me by Professor Lemańczyk.

Every measure μ can be uniquely decomposed into a sum $\mu = \mu^{\mathrm{c}} + \mu^{\mathrm{d}}$ where μ^{c} is continuous and μ^{d} is discrete. For a spectral sequence $\mu_{x_1} \gg \mu_{x_2} \gg \ldots$ we have $\mu_{x_1}^{\mathrm{c}} \gg \mu_{x_2}^{\mathrm{c}} \gg \ldots$ By the *c-multiplicity function* M_U^{c} we mean the function $M_U^{\mathrm{c}} : \mathbb{T} \to \mathbb{N} \cup \{+\infty\}$ given by

$$M_U^{\mathrm{c}}(z) = \sum_{n=1}^{\infty} \chi_{C_n}(z)$$

where $C_1 = \mathbb{T}$ and $C_n = \{z \in \mathbb{T} : \frac{d\mu_{x_n}^{\mathrm{c}}}{d\mu_{x_1}^{\mathrm{c}}}(z) > 0\}$. The set

$$E^{\mathrm{c}}(U) = \{n \in \mathbb{N} \cup \{+\infty\} : \mu_{x_1}\{z \in \mathbb{T} : M_U^{\mathrm{c}}(z) = n\} > 0\}$$

is called the set of *essential values* of the c-multiplicity function M_U^{c}.

Let $D(U) : \mathbb{N} \cup \{+\infty\} \to \mathbb{N} \cup \{+\infty\}$ be the function given by $D(U)(n) = \operatorname{card} D_n$ where

$$D_n = \begin{cases} \{z \in A_n \setminus A_{n+1} : \mu_{x_1}(\{z\}) > 0\} & \text{for } n = 1, 2, \ldots, \\ \{z \in \bigcap_{n=1}^{\infty} A_n : \mu_{x_1}(\{z\}) > 0\} & \text{for } n = +\infty. \end{cases}$$

In Section 1 we define a cyclic space (s.c.) isomorphism of unitary operators on a separable Hilbert space and we try to find a complete set of invariants for such an isomorphism. Using the results from Section 1 and those from [5], we show that in the c.s. equivalence class of every operator $U : H \to H$ whose maximal spectral type is continuous and $1 \in E^{\mathrm{c}}(U)$ we can find a weakly mixing automorphism.

The author would like to thank Professor Lemańczyk for some valuable discussions.

1. Cyclic space isomorphism and its invariants

LEMMA 1.1. *Let $U_1 : H_1 \to H_1$ and $U_2 : H_2 \to H_2$ be unitary operators. Then for every unitary operator $V : H_1 \to H_2$ the following conditions are equivalent.*

(i) *For every $x \in H_1$, $Z(Vx) = VZ(x)$.*

(ii) *If H is a U_1-invariant closed subspace of H_1, then VH is U_2-invariant, and if H is a U_2-invariant closed subspace of H_2, then $V^{-1}H$ is U_1-invariant.*

Proof. (i)⇒(ii). Suppose that H is a U_1-invariant closed subspace of H_1 and $y \in VH$. There exists $x \in H$ such that $y = Vx$. Since $Z(y) = VZ(x)$,

$$U_2^{-1}y, U_2 y \in Z(y) = VZ(x) \subset VH$$

and so VH is U_2-invariant. Similarly, we can get the remaining part of (ii).

(ii)⇒(i). Let $x \in H_1$. Since $Z(x)$ is U_1-invariant, $VZ(x)$ is U_2-invariant. Since $Vx \in VZ(x)$, $Z(Vx) \subset VZ(x)$. Similarly, if $y = Vx$ then $Z(x) = Z(V^{-1}y) \subset V^{-1}Z(y) = V^{-1}Z(Vx)$. This gives $VZ(x) \subset Z(Vx)$ and finally $Z(Vx) = VZ(x)$. ∎

DEFINITION 1.1. We call a unitary operator $V : H_1 \to H_2$ a *cyclic space isomorphism* of U_1 and U_2 if it satisfies (i) or equivalently (ii) of Lemma 1.1.

LEMMA 1.2. *Let μ and ν be positive finite Borel measures on the circle. Assume $U_1 : L^2(\mathbb{T}, \mu) \to L^2(\mathbb{T}, \mu)$ and $U_2 : L^2(\mathbb{T}, \nu) \to L^2(\mathbb{T}, \nu)$ are unitary operators given by*

$$U_1 f(z) = U_2 f(z) = z f(z).$$

If $V : L^2(\mathbb{T}, \mu) \to L^2(\mathbb{T}, \nu)$ is a c.s. isomorphism of U_1 and U_2 then there exists a nonsingular invertible map $S : (\mathbb{T}, \mathcal{B}, \nu) \to (\mathbb{T}, \mathcal{B}, \mu)$ and $h \in L^2(\mathbb{T}, \nu)$ such that

$$Vf = h \cdot f \circ S$$

for every $f \in L^2(\mathbb{T}, \mu)$.

Proof. For $A \in \mathcal{B}$ put $H = \chi_A L^2(\mathbb{T}, \mu)$. Then H is a U_1-invariant subspace of $L^2(\mathbb{T}, \mu)$. By the Wiener Lemma (e.g. [9], Appendix) there exists a Borel set $\Phi(A)$ such that $VH = \chi_{\Phi(A)} L^2(\mathbb{T}, \nu)$. From $V(\{0\}) = \{0\}$ and $V^{-1}(\{0\}) = \{0\}$ we see that $\mu(A) = 0$ iff $\nu(\Phi(A)) = 0$. If $A \cap B = \emptyset$ then $\chi_A L^2(\mathbb{T}, \mu) \perp \chi_B L^2(\mathbb{T}, \mu)$ hence

$$\chi_{\Phi(A)} L^2(\mathbb{T}, \nu) \perp \chi_{\Phi(B)} L^2(\mathbb{T}, \nu)$$

and finally $\Phi(A)\cap\Phi(B)=\emptyset$. If $A=\bigcup_{n=1}^{\infty}A_n$ with $\{A_n\}$ pairwise disjoint then

$$\begin{aligned}\chi_{\Phi(A)}L^2(\mathbb{T},\nu)&=V(\chi_{\bigcup_{n=1}^{\infty}A_n}L^2(\mathbb{T},\mu))=V\Big(\bigoplus_{n=1}^{\infty}\chi_{A_n}L^2(\mathbb{T},\mu)\Big)\\&=\bigoplus_{n=1}^{\infty}V(\chi_{A_n}L^2(\mathbb{T},\mu))=\bigoplus_{n=1}^{\infty}\chi_{\Phi(A_n)}L^2(\mathbb{T},\nu)\\&=\chi_{\bigcup_{n=1}^{\infty}\Phi(A_n)}L^2(\mathbb{T},\nu),\end{aligned}$$

hence $\Phi(A)=\bigcup_{n=1}^{\infty}\Phi(A_n)$ and by a standard argument the equality also holds if $\{A_n\}$ are not pairwise disjoint. Since $V(L^2(\mathbb{T},\mu))=L^2(\mathbb{T},\nu)$ we have $\Phi(\mathbb{T})=\mathbb{T}$. Hence $\mathbb{T}=\Phi(A)\cup\Phi(A^c)$ and therefore $\Phi(A)^c=\Phi(A^c)$.

Consequently, $\Phi:(\mathcal{B},\mu)\to(\mathcal{B},\nu)$ is a σ-Boolean isomorphism. Therefore there exists a nonsingular invertible map $S:(\mathbb{T},\mathcal{B},\nu)\to(\mathbb{T},\mathcal{B},\mu)$ such that $\Phi(A)=S^{-1}(A)$ for every $A\in\mathcal{B}$.

Set $h=V(1)$. For $A\in\mathcal{B}$ we have $1=\chi_A+\chi_{A^c}$, hence $h=V(\chi_A)+V(\chi_{A^c})$. But the functions $V(\chi_A)$ and $V(\chi_{A^c})$ have disjoint supports, so $V(\chi_A)$ must be equal to h on its support and similarly for $V(\chi_{A^c})$; hence

$$V(\chi_A)=h\cdot\chi_{\Phi(A)}=h\cdot\chi_A\circ S.$$

Since this is true for any characteristic function, it is also true for linear combinations of such functions and finally for all $f\in L^2(\mathbb{T},\mu)$. Since V is unitary, for every $A\in\mathcal{B}$ we have

$$\begin{aligned}\mu(SA)&=\int_{\mathbb{T}}|\chi_AS^{-1}|^2\,d\mu=\|\chi_AS^{-1}\|^2_{L^2(\mu)}\\&=\|V(\chi_AS^{-1})\|^2_{L^2(\nu)}=\|h\cdot\chi_A\|^2_{L^2(\nu)}=\int_A|h|^2d\nu.\end{aligned}$$

Hence $|h|^2=d\mu\circ S/d\nu$. ∎

LEMMA 1.3. *Assume that $U_1:H_1\to H_1$ and $U_2:H_2\to H_2$ are unitary operators and $V:H_1\to H_2$ a c.s. isomorphism of U_1 and U_2. Let*

$$H_1=\bigoplus_{n=1}^{\infty}Z(x_n)\quad\text{and}\quad\mu_{x_1}\gg\mu_{x_2}\gg\ldots$$

be a spectral decomposition of U_1. Then

$$H_2=\bigoplus_{n=1}^{\infty}Z(Vx_n)\quad\text{and}\quad\mu_{Vx_1}\gg\mu_{Vx_2}\gg\ldots$$

Moreover, $\mu_{x_n}\equiv\mu_{x_{n+1}}$ iff $\mu_{Vx_n}\equiv\mu_{Vx_{n+1}}$ and hence $E(U_1)=E(U_2)$.

Proof. Since V is a unitary operator,

$$H_2=V(H_1)=V\Big(\bigoplus_{n=1}^{\infty}Z(x_n)\Big)=\bigoplus_{n=1}^{\infty}VZ(x_n)=\bigoplus_{n=1}^{\infty}Z(Vx_n).$$

We first show that $Z(Vx_1)$ is a maximal cyclic space. Suppose there exists $y\in H_2$ such that $Z(Vx_1)\subsetneq Z(y)$. Then $Z(x_1)\subsetneq Z(V^{-1}y)$. This contradicts the fact that $Z(x_1)$ is maximal. Thus μ_{Vx_1} is the maximal spectral type of U_2.

Similarly, since $V|_{Z(x_1)^\perp}$ is a c.s. isomorphism, μ_{Vx_2} is the maximal spectral type of $U_2|_{Z(Vx_2)^\perp}$. In this way we conclude that μ_{Vx_n} is the maximal spectral type of U_2 restricted to $Z(Vx_n)\oplus Z(Vx_{n+1})\oplus\ldots$ for every $n\ge1$ and finally that $\mu_{Vx_1}\gg\mu_{Vx_2}\gg\ldots$

If $\mu_{x_n}\gg\mu_{x_{n+1}}$ but they are not equivalent then we can write

$$Z(x_n)\oplus Z(x_{n+1})=Z(x'_n)\oplus Z(x''_n)\oplus Z(x_{n+1})$$

where $\mu_{x''_n}\perp\mu_{x_{n+1}}$ and $\mu_{x'_n}\ll\mu_{x_{n+1}}$ (in fact, the latter two measures are equivalent). Now

$$V(Z(x_n)\oplus Z(x_{n+1}))=Z(Vx'_n)\oplus Z(Vx''_n)\oplus Z(Vx_{n+1});$$

but $Z(x''_n)\oplus Z(x_{n+1})$ is a cyclic space, hence necessarily

$$V(Z(x''_n)\oplus Z(x_{n+1}))=Z(Vx''_n)\oplus Z(Vx_{n+1}).$$

This shows that the spectral measures $\mu_{Vx''_n}$ and $\mu_{Vx_{n+1}}$ are orthogonal so $\mu_{Vx_n}\gg\mu_{Vx_{n+1}}$ and they are not equivalent. ∎

Remark. It follows from this lemma that $E(U)$ is an invariant of c.s. isomorphism. Notice that if x is an eigenvector of U_1, then $Z(x)$ is a one-dimensional space. Therefore its image via a c.s. isomorphism V must also be one-dimensional, hence Vx is also an eigenvector (though possibly corresponding to a different eigenvalue). This gives rise to a second invariant of c.s. isomorphism. The theorem below explains how a combination of these two invariants gives rise to a complete set of invariants for c.s. isomorphism.

THEOREM 1.4. *Let $U_i:H_i\to H_i$ be a unitary operator on a separable Hilbert space, $i=1,2$. Then the following conditions are equivalent.*

(i) *U_1 and U_2 are cyclic space equivalent.*

(ii) *There are spectral sequences $\mu_1\gg\mu_2\gg\ldots$ of U_1 and $\nu_1\gg\nu_2\gg\ldots$ of U_2 and a measure space isomorphism $S:(\mathbb{T},\nu_1)\to(\mathbb{T},\mu_1)$ such that*

$$\nu_n=\mu_n\circ S\quad\text{for all } n\ge1.$$

(iii) *$E^c(U_1)=E^c(U_2)$ and $D(U_1)=D(U_2)$.*

Proof. (i)⇒(ii). Suppose $V:H_1\to H_2$ is a c.s. isomorphism of U_1 and U_2. Fix a spectral decomposition $H_1=\bigoplus_{n=1}^{\infty}Z(x_n)$ of U_1 and put $\mu_n:=\mu_{x_n}$ for each $n\ge1$. By Lemma 1.3 we have a spectral decomposition

$H_2 = \bigoplus_{n=1}^{\infty} Z(Vx_n)$ of U_2 and $\nu_n := \mu_{Vx_n}$ for each $n \geq 1$. There exists a unitary isomorphism $V_1 : \bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \mu_n) \to H_1$ of the operators U and U_1 and a unitary isomorphism $V_2 : H_2 \to \bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \nu_n)$ of the operators U_2 and U such that $V_1(L^2(\mathbb{T}, \mu_n)) = Z(x_n)$ and $V_2 Z(x_n) = L^2(\mathbb{T}, \nu_n))$ for $n \geq 1$, where

$$U\Big(\sum_{n=1}^{\infty} f_n(z_n)\Big) = \sum_{n=1}^{\infty} z_n f_n(z_n).$$

Hence the operator $V' = V_2 V V_1$ is a c.s. isomorphism of the operator U on $\bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \mu_n)$ and U on $\bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \nu_n)$, and $V'(L^2(\mathbb{T}, \mu_n)) = L^2(\mathbb{T}, \nu_n)$ (so V' restricted establishes a c.s. isomorphism) for $n \geq 1$.

By Lemma 1.2 there exist nonsingular invertible maps $S_n : (\mathbb{T}, \mathcal{B}, \nu_n) \to (\mathbb{T}, \mathcal{B}, \mu_n)$ and $h_n \in L^2(\mathbb{T}, \nu_n)$ such that $V'|_{L^2(\mathbb{T},\mu_n)} f = h_n \cdot f \circ S_n$ for every $n \geq 1$. Hence we have

$$V'\Big(\sum_{n=1}^{\infty} f_n(z_n)\Big) = \sum_{n=1}^{\infty} h_n(z_n) \cdot f_n(S_n z_n)$$

for $\sum_{n=1}^{\infty} f_n \in \bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \mu_n)$.

For every $n \neq m$, consider

$$H = \{f(z_n) + f(z_m) : f \in L^2(\mathbb{T}, \mu_1)\}.$$

This is a closed U-invariant subspace of $\bigoplus_{k=1}^{\infty} L^2(\mathbb{T}, \mu_k)$. Without loss of generality, we can assume that $\mu_n = \mu_1|_{A_n}$ (i.e. $d\mu_n/d\mu_1 = \chi_{A_n}$). Then

$$V'H = \{h_n(z_n) f(S_n z_n) + h_m(z_m) f(S_m z_m) : f \in L^2(\mathbb{T}, \mu_1)\}.$$

Since $V'H$ is U-invariant, for every $f \in L^2(\mathbb{T}, \mu_1)$ there exists $g \in L^2(\mathbb{T}, \mu_1)$ such that

$$z_n h_n(z_n) f(S_n z_n) + z_m h_m(z_m) f(S_m z_m) = h_n(z_n) g(S_n z_n) + h_m(z_m) g(S_m z_m).$$

By the orthogonality of the natural embedding of $L^2(\mathbb{T}, \mu_n)$ and $L^2(\mathbb{T}, \mu_m)$ in the space under consideration,

$$z h_n(z) f(S_n z) = h_n(z) g(S_n z), \qquad z \in \mathbb{T}\ \mu_n\text{-a.e.},$$
$$z h_m(z) f(S_m z) = h_m(z) g(S_m z), \qquad z \in \mathbb{T}\ \mu_m\text{-a.e.},$$

hence $S_n^{-1}(z) f(z) = g(z)$ and $S_m^{-1}(z) f(z) = g(z)$ a.e., because $h_n \neq 0$ μ_n-a.e. and $h_m \neq 0$ μ_m-a.e. by Lemma 1.2. If $f = 1$ then $S_n^{-1}(z) = g(z) = S_m^{-1}(z)$, hence $S = S_n = S_m$ for every $n \neq m$ and we get $\nu_n \equiv \mu_n \circ S$ so by replacing ν_n by $\mu_n \circ S$ the result follows.

(ii)⇒(i). Suppose there are spectral sequences $\mu_1 \gg \mu_2 \gg \ldots$ of U_1 and $\nu_1 \gg \nu_2 \gg \ldots$ of U_2 and an isomorphism $S : (\mathbb{T}, \nu_1) \to (\mathbb{T}, \mu_1)$ such that $\nu_n = \mu_n \circ S$ for all $n \geq 1$. We will consider the unitary operator $V' : \bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \mu_n) \to \bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \nu_n)$ given by

$$V'\Big(\sum_{n=1}^{\infty} f_n(z_n)\Big) = \sum_{n=1}^{\infty} f_n(S z_n).$$

We first prove that V' is a cyclic space isomorphism of U on $\bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \mu_n)$ and U on $\bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \nu_n)$. Let H be a closed U-invariant subspace of $\bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \mu_n)$. We show that $V'H$ is U-invariant. We know that H is $\psi(U)$-invariant for every $\psi \in L^\infty(\mathbb{T}, \mu_1)$. Hence if $\sum_{n=1}^{\infty} f_n(z_n) \in H$ then $\sum_{n=1}^{\infty} \psi(z_n) f_n(z_n) \in H$.

Let $\sum_{n=1}^{\infty} g_n(z_n) \in V'H$. There exists $\sum_{n=1}^{\infty} f_n(z_n) \in H$ such that $g_n = f_n \circ S$. From $|S^{-1}(z)| = 1$, it follows that

$$\sum_{n=1}^{\infty} S^{-1}(z_n) f_n(z_n) \in H$$

and hence

$$U\Big(\sum_{n=1}^{\infty} g_n(z_n)\Big) = \sum_{n=1}^{\infty} z_n f_n(S z_n) \in V'H.$$

In the same manner we can see that if H is a U-invariant subspace of $\bigoplus_{n=1}^{\infty} L^2(\mathbb{T}, \nu_n)$ then $V'^{-1}H$ is U-invariant. Consequently, the operator $V = V_2^{-1} V' V_1^{-1}$ is a c.s. isomorphism of U_1 and U_2.

(ii)⇒(iii). If there are spectral sequences $\mu_1 \gg \mu_2 \gg \ldots$ of U_1 and $\nu_1 \gg \nu_2 \gg \ldots$ of U_2 and an isomorphism $S : (\mathbb{T}, \nu_1) \to (\mathbb{T}, \mu_1)$ such that $\nu_n = \mu_n \circ S$ for all $n \geq 1$ then

$$A_n(U_2) = S^{-1} A_n(U_1), \quad C_n(U_2) = S^{-1} C_n(U_1), \quad \nu_1^{\mathrm{d}} = \mu_1^{\mathrm{d}} \circ S.$$

Hence

$$C_n(U_2) \setminus C_{n+1}(U_2) = S^{-1}(C_n(U_1) \setminus C_{n+1}(U_1)),$$
$$\nu_1^{\mathrm{d}}|_{A_n(U_2)\setminus A_{n+1}(U_2)} = \mu_1^{\mathrm{d}}|_{A_n(U_1)\setminus A_{n+1}(U_1)} \circ S$$

for $n \geq 1$ and

$$\bigcap_{n=1}^{\infty} C_n(U_2) = S^{-1}\Big(\bigcap_{n=1}^{\infty} C_n(U_1)\Big), \qquad \nu_1^{\mathrm{d}}|_{\bigcap_{n=1}^{\infty} A_n(U_2)} = \mu_1^{\mathrm{d}}|_{\bigcap_{n=1}^{\infty} A_n(U_1)} \circ S$$

and finally $E^{\mathrm{c}}(U_1) = E^{\mathrm{c}}(U_2)$ and $D(U_1) = D(U_2)$.

(iii)⇒(ii). Let μ and ν be the maximal spectral type of U_1 and U_2. If $E^{\mathrm{c}}(U_1) = E^{\mathrm{c}}(U_2)$ and $D(U_1) = D(U_2)$ then

$$\nu(C_n(U_2) \setminus C_{n+1}(U_2)) > 0 \quad \text{iff} \quad \mu(C_n(U_1) \setminus C_{n+1}(U_1)) > 0$$

for $n \geq 1$ and

$$\nu\Big(\bigcap_{n=1}^{\infty} C_n(U_2)\Big) > 0 \quad \text{iff} \quad \mu\Big(\bigcap_{n=1}^{\infty} C_n(U_1)\Big) > 0$$

and $\operatorname{card} D_n(U_1) = \operatorname{card} D_n(U_2)$ for $n \in \mathbb{N} \cup \{+\infty\}$.

Since $A_n \setminus A_{n+1} = (C_n \setminus C_{n+1}) \cup D_n$ and $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} C_n \cup D_\infty$, there exist nonsingular invertible maps $S_n : (A_n(U_2) \setminus A_{n+1}(U_2), \nu) \to (A_n(U_1) \setminus A_{n+1}(U_1), \mu)$ for $n \geq 1$ and $S_\infty : (\bigcap_{n=1}^{\infty} A_n(U_2), \nu) \to (\bigcap_{n=1}^{\infty} A_n(U_1), \mu)$. We define a nonsingular invertible map $S : (\mathbb{T}, \nu) \to (\mathbb{T}, \mu)$ by

$$S(x) = \begin{cases} S_n(x) & \text{for } x \in A_n(U_2) \setminus A_{n+1}(U_2), \\ S_\infty(x) & \text{for } x \in \bigcap_{n=1}^{\infty} A_n(U_2). \end{cases}$$

Then we have

$$\nu|_{A_n(U_2)} \equiv \mu|_{A_n(U_1)} \circ S.$$

Let $\mu_n := \mu|_{A_n(U_1)}$ and $\nu_n := \mu|_{A_n(U_1)} \circ S$. Then $\mu_1 \gg \mu_2 \gg \ldots$ and $\nu_1 \gg \nu_2 \gg \ldots$ are spectral sequences of U_1 and U_2 respectively, and $\nu_n = \mu_n \circ S$ for all $n \geq 1$. ■

2. Cyclic space isomorphism of unitary operators in the case where an operator corresponds to an ergodic dynamical system. Given a dynamical system $T : (X, \mathcal{B}, \varrho) \to (X, \mathcal{B}, \varrho)$, set $\operatorname{Sp}(T) = \{\lambda \in \mathbb{C} : \exists_{f \in L^2(X,\varrho)} fT = \lambda f\}$.

COROLLARY 2.1. *Let $(X_1, \mathcal{B}_1, \varrho_1, T_1)$ and $(X_2, \mathcal{B}_2, \varrho_2, T_2)$ be invertible, ergodic dynamical systems. Then U_{T_1} and U_{T_2} are cyclic space equivalent if and only if $E^c(U_{T_1}) = E^c(U_{T_2})$ and* $\operatorname{card} \operatorname{Sp}(T_1) = \operatorname{card} \operatorname{Sp}(T_2)$.

Proof. By ergodicity, for every spectral sequence $\mu_1^{(i)} \gg \mu_2^{(i)} \gg \ldots$ corresponding to U_{T_i}, $i = 1, 2$, only the maximal spectral type $\mu_1^{(i)}$ may not be a continuous measure. ■

Without ergodicity the above corollary is not valid as the following example shows.

EXAMPLE. Let $Tx = x + \alpha$ be an irrational rotation. Then T and $T \times T$ are not s.c. equivalent (because $D(T)(1) = \infty$ and $D(T \times T)(1) = 0$), though $\operatorname{card} \operatorname{Sp}(T) = \operatorname{card} \operatorname{Sp}(T \times T)$.

COROLLARY 2.2. *Let T_1 and T_2 be weakly mixing. Then U_{T_1} and U_{T_2} are cyclic space equivalent if and only if $E^c(U_{T_1}) = E^c(U_{T_2})$.*

In [5] M. Lemańczyk and J. Kwiatkowski Jr. proved

PROPOSITION 1. *Given a set $A \subseteq \mathbb{N}^+$, $1 \in A$, there exists an ergodic T such that $E(U_T) = A$. Moreover, T can be constructed to be weakly mixing.*

From the proof of Proposition 1 in [5] it follows that for a set $A \subseteq \mathbb{N}^+$, $1 \in A$, there exists a weakly mixing T such that $E^c(U_T) = A$. Since all their

examples have singular spectra, by taking the direct product of an example T realizing $A \subset \mathbb{N}^+$ with a τ having countable Lebesgue spectrum we reach

$$E(T \times \tau) = A \cup \{+\infty\}.$$

Hence

COROLLARY 2.3. *Let $\mathcal{M}_{1,C} = \{U : U$ has continuous spectrum and $1 \in E^c(U)\}$. Partition $\mathcal{M}_{1,C}$ into cyclic space equivalence classes. Then in every class there exists a unitary operator $U_T : L_0^2(X, \varrho) \to L_0^2(X, \varrho)$, where T is weakly mixing and $L_0^2(X, \varrho) = \{f \in L^2(X, \varrho) : \int f \, d\varrho = 0\}$.*

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