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The Minlos lemma for positive-definite functions on additive subgroups of $\mathbb{R}^n$

by

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Abstract. Let H be a real Hilbert space. It is well known that a positive-definite function φ on H is the Fourier transform of a Radon measure on the dual space if (and only if) φ is continuous in the Sazonov topology (resp. the Gross topology) on H. Let G be an additive subgroup of H and let $G^\wedge_{\mathrm{pc}}$ (resp. $G^\wedge_{\mathrm{b}}$) be the character group endowed with the topology of uniform convergence on precompact (resp. bounded) subsets of G. It is proved that if a positive-definite function φ on G is continuous in the Gross topology, then φ is the Fourier transform of a Radon measure μ on $G^\wedge_{\mathrm{pc}}$; if φ is continuous in the Sazonov topology, μ can be extended to a Radon measure on $G^\wedge_{\mathrm{b}}$.

1. Introduction. Every continuous positive-definite function on an LCA group G is the Fourier transform of a (unique) Radon measure on the character group $G^\wedge$. This fact, known as the Bochner theorem, has been generalized to certain abelian topological groups which are not locally compact; a brief survey can be found in [1, Sec. 11], see also Remark 1.5. In particular, R. A. Minlos [7] proved that the Bochner theorem remains valid if G is a nuclear locally convex space. In what follows, D is an n-dimensional ellipsoid in $\mathbb{R}^n$ with centre at 0 and principal semiaxes of lengths $\lambda_1, \ldots, \lambda_n$. By $x \cdot y$ we denote the euclidean inner product of vectors $x, y \in \mathbb{R}^n$. The proof of the Minlos theorem is based on the following fact (see Lemma 4.1 in [11, Ch. VI]):

LEMMA 1.1 (R. A. Minlos). *Let μ be a probability measure on $\mathbb{R}^n$ and $\widehat{\mu}$ the characteristic functional of μ:*

$$\widehat{\mu}(x) = \int_{\mathbb{R}^n} e^{ix\cdot y}\, d\mu(y), \qquad x \in \mathbb{R}^n.$$

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Let ε and r be arbitrary positive numbers. If $|1-\widehat{\mu}(x)| \le \varepsilon$ for each $x \in D$, then

$$\mu(\{y \in \mathbb{R}^n : y \cdot y > r\}) \le 3\Big(\varepsilon + 2r^{-1}\sum_{k=1}^{n}\lambda_k^{-2}\Big).$$

Actually, the following fact is true:

LEMMA 1.2. *Let μ, $\widehat{\mu}$ and ε be as in Lemma 1.1. Let B_∞^n be the unit cube in $\mathbb{R}^n$ given by $|x_k| \le 1$ for $k = 1, \ldots, n$. If $\operatorname{Re}\widehat{\mu}(x) \ge 1-\varepsilon$ for each $x \in B_\infty^n$, then*

$$\mu(\{y \in \mathbb{R}^n : y \cdot y > 1\}) < 7\varepsilon.$$

The proof can easily be obtained by standard methods. We omit the details because we shall not use this result below.

Let X be a Hausdorff topological space. The family of Borel subsets of X is denoted by $\mathcal{B}(X)$. A positive finite Borel measure μ on X is called a *Radon measure* if

$$\mu(A) = \sup_{Z \subset A} \mu(Z), \qquad A \in \mathcal{B}(X),$$

the supremum being taken over all compact subsets of A.

Let G be an abelian topological group. By a *character* of G we mean a homomorphism of G into the multiplicative group of complex numbers with modulus 1. The group of all continuous characters of G is denoted by $G^\wedge$. Let τ be a topology on $G^\wedge$ such that the mappings $\chi \mapsto \chi(g)$, $g \in G$, are continuous, and let μ be a finite positive Borel measure on $(G^\wedge, \tau)$. The Fourier transform $\widehat{\mu}$ of μ is given by

$$\widehat{\mu}(g) = \int_{G^\wedge} \chi(g)\, d\mu(\chi), \qquad g \in G.$$

Let $\mathfrak{S}$ be a family of subsets of G such that $S \cup T \in \mathfrak{S}$ whenever $S, T \in \mathfrak{S}$. The family of sets of the form

$$\{\chi \in G^\wedge : |1-\chi(g)| < \varepsilon \text{ for each } g \in S\}$$

where $S \in \mathfrak{S}$ and $\varepsilon > 0$, forms a base of neighbourhoods of zero for a unique group topology on $G^\wedge$, called the *topology of uniform convergence on the sets* $S \in \mathfrak{S}$. By $G_{\mathrm{p}}^\wedge$, $G_{\mathrm{c}}^\wedge$ and $G_{\mathrm{pc}}^\wedge$ we denote, respectively, the group $G^\wedge$ endowed with the topology of uniform convergence on finite, compact and precompact subsets of G. If G is an additive subgroup of a Hilbert space H, then $G_{\mathrm{b}}^\wedge$ denotes the group $G^\wedge$ endowed with the topology of uniform convergence on bounded subsets of G. If G is closed in H, then $G_{\mathrm{c}}^\wedge = G_{\mathrm{pc}}^\wedge$ but, in general, $G_{\mathrm{pc}}^\wedge \ne G_{\mathrm{b}}^\wedge$.

By B_2^n we denote the euclidean unit ball in $\mathbb{R}^n$. If K is an additive subgroup of $\mathbb{R}^n$, then we write

$$Z_K = \{\chi \in K^\wedge : \operatorname{Re}\chi(x) \ge 0 \text{ for each } x \in K \cap \tfrac{1}{4}B_2^n\}.$$

Notice that Z_K is a closed subset of $K_{\mathrm{p}}^\wedge$.

It was proved in [1, Sec. 12] that the Bochner theorem remains valid for an arbitrary additive subgroup of a nuclear locally convex space (cf. Remark 1.5 below). The proof was based on the following fact (Lemma (12.2) of [1]):

LEMMA 1.3. *Let K be an additive subgroup of $\mathbb{R}^n$ and μ a Borel probability measure on $K_{\mathrm{p}}^\wedge$ such that $\operatorname{Re}\widehat{\mu}(x) \ge 1-\varepsilon$ for each $x \in K \cap D$. Assume that*

$$\sum_{k=1}^{n} \lambda_k^{-1/2} < \frac{1}{12}. \tag{1.1}$$

Then $\mu(K^\wedge \setminus Z_K) < 2\varepsilon$.

It was clear from the very beginning that the condition (1.1) was too strong. The aim of this paper is to prove the following analogue of Lemma 1.1:

LEMMA 1.4. *Let K be a closed additive subgroup of $\mathbb{R}^n$ and μ a Borel probability measure on $K_{\mathrm{p}}^\wedge$ such that $\operatorname{Re}\widehat{\mu}(x) \ge 1-\varepsilon$ for each $x \in K \cap D$. Then*

$$\mu(K^\wedge \setminus Z_K) \le 2\varepsilon + \sum_{k=1}^{n} \lambda_k^{-2}. \tag{1.2}$$

The proof is given in Section 2. Lemma 1.4 remains true for arbitrary, not necessarily closed, additive subgroups of $\mathbb{R}^n$, but then the proof becomes slightly more complicated. We also obtain analogues of Lemma 1.4 for convex bodies other than ellipsoids; roughly speaking, Lemma 1.4 remains true if D is replaced by a convex body with sufficiently large n-dimensional gaussian measure (see Lemma 4.3). It should be pointed out that this result is a consequence of the Talagrand theorem on the majorizing measure (see Lemmas 2.3 and 4.2). In Sections 3 and 4 we apply the results of Section 2 to prove that if a positive-definite function φ on an additive subgroup G of a Hilbert space H is continuous in the Sazonov topology (resp. the Gross topology) on H, then φ is the Fourier transform of a Radon measure on $G_{\mathrm{b}}^\wedge$ (resp. on $G_{\mathrm{pc}}^\wedge$).

Remark 1.5. In [1, Ch. 3] the author introduced the so-called *nuclear groups*, a class of abelian topological groups which contains all LCA groups and nuclear locally convex spaces, and is closed with respect to the operations of taking subgroups, Hausdorff quotients and arbitrary products. Then it was proved in [1, Sec. 12] that every nuclear group satisfies an analogue of the Bochner theorem.

2. Finite-dimensional inequalities. By a *lattice* in $\mathbb{R}^n$ we mean an additive subgroup of $\mathbb{R}^n$ generated by n linearly independent vectors. The family of all lattices in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$. For $x \in \mathbb{R}^n$, we write x^2 instead of $x \cdot x$.

Let $L \in \mathcal{L}^n$ and $y \in \mathbb{R}^n$. Then $\sum_{x\in L+y} e^{-\pi x^2} < \infty$. To see this, it is enough to estimate in a standard way the number of elements of $L+y$ in the ball rB_2^n, for large r. It is convenient to denote

$$\varrho(A) = \sum_{x\in A} e^{-\pi x^2}, \qquad A \subset L+y;$$

then $\varrho(A) \le \varrho(L+y) < \infty$.

Let $L \in \mathcal{L}^n$. By σ_L we denote the probability measure on L given by

$$\sigma_L(A) = \varrho(A)/\varrho(L), \qquad A \subset L.$$

By $\sigma_L^\wedge$ we denote the Fourier transform of σ_L:

$$\sigma_L^\wedge(\chi) = \int_L \chi(x)\, d\sigma_L(x), \qquad \chi \in L^\wedge.$$

LEMMA 2.1. *Let L be a lattice in $\mathbb{R}^n$. Then*

$$\varrho(\{x \in L+y : |x\cdot z| \ge t\}) < 2e^{-\pi t^2}\varrho(L)$$

for arbitrary $y \in \mathbb{R}^n$, $z \in B_2^n$ and $t > 0$.

This is Lemma 2.4 of [3].

LEMMA 2.2. *If $L \in \mathcal{L}^n$ and $\chi \in L^\wedge \setminus X_L$, then $\sigma_L^\wedge(\chi) < 2e^{-\pi}$.*

Proof. Consider the epimorphism $f : \mathbb{R}^n \to L^\wedge$ given by

$$\langle x, f(y)\rangle = e^{2\pi i x\cdot y}, \qquad x \in L,\ y \in \mathbb{R}^n.$$

Let L^* be the dual lattice:

$$L^* = \{y \in \mathbb{R}^n : x\cdot y \in \mathbb{Z} \text{ for each } x \in L\}.$$

Next, consider the function

$$\psi(y) = \varrho(L^*+y)/\varrho(L^*), \qquad y \in \mathbb{R}^n.$$

Then $\psi = \sigma_L^\wedge f$ (see Corollary (1.2) of [2]). Now, take an arbitrary $\chi \in L^\wedge \setminus X_L$. We can find some $y_0 \in \mathbb{R}^n$ with $f(y_0) = \chi$. The condition $\chi \notin X_L$ means that there exists some $x_0 \in L \cap \frac{1}{4}B_2^n$ such that $\cos 2\pi(x_0\cdot y_0) < 0$. It is clear that if $y \in L^* + y_0$, then $|x_0 \cdot y| > \frac14$. Thus, by Lemma 2.1, we have

$$\begin{aligned}\sigma_L^\wedge(\chi) &= \psi(y_0) = \varrho(L^*+y_0)/\varrho(L^*)\\ &\le \varrho(\{y \in L^*+y_0 : |y\cdot x_0| > \tfrac14\})/\varrho(L^*) < 2e^{-\pi}. \ \blacksquare\end{aligned}$$

Let $\mathcal{C}^n$ denote the family of all convex bodies in $\mathbb{R}^n$ which are symmetric with respect to 0. For $U \in \mathcal{C}^n$, we define

$$\alpha(U) = \sup_{L\in\mathcal{L}_n} \varrho(L\setminus U)/\varrho(L) = \sup_{L\in\mathcal{L}_n} \sigma_L(L\setminus U).$$

LEMMA 2.3. *Let K be a closed subgroup of $\mathbb{R}^n$ and let μ be a Borel probability measure on $K_{\mathrm{p}}^\wedge$. Suppose $U \in \mathcal{C}^n$ and $\varepsilon \in (0,1)$ are such that $\operatorname{Re}\widehat{\mu}(x) \ge 1-\varepsilon$ for each $x \in K\cap U$. Then*

$$\mu(K^\wedge \setminus Z_K) \le (1-2e^{-\pi})^{-1}(\varepsilon + 2\alpha(U)).$$

Proof. Without loss of generality we may assume that K is a lattice in $\mathbb{R}^n$ (cf. the proof of Lemma (12.2) in [1]). We may also assume that μ is symmetric, i.e. that $\widehat{\mu}$ is real-valued. According to our definitions, we may write

$$\begin{aligned}\int_{K^\wedge} \sigma_K^\wedge(\chi)\, d\mu(\chi) &= \int_K \widehat{\mu}(x)\, d\sigma_K(x)\\ &= \int_U + \int_{K\setminus U} \widehat{\mu}(x)\, d\sigma_K(x) \ge (1-\varepsilon)\,\sigma_K(U) - \sigma_K(K\setminus U)\\ &= 1-\varepsilon-(2-\varepsilon)\,\sigma_K(K\setminus U) > 1-\varepsilon-2\alpha(U).\end{aligned}$$

On the other hand, Lemma 2.2 implies that

$$\begin{aligned}\int_{K^\wedge} \sigma_K^\wedge(\chi)\, d\mu(\chi) &= \int_{Z_K} + \int_{\setminus Z_K} \sigma_K^\wedge(\chi)\, d\mu(\chi)\\ &< \mu(Z_K) + 2e^{-\pi}\mu(\setminus Z_K) = 1-(1-2e^{-\pi})\,\mu(\setminus Z_K).\end{aligned}$$

Hence

$$1-\varepsilon-2\alpha(U) < 1-(1-2e^{-\pi})\,\mu(\setminus Z_K). \ \blacksquare$$

To apply Lemma 2.3 to a given convex body U we need upper bounds for $\alpha(U)$. For certain special convex bodies such bounds were found in [3, Sec. 2]; let us recall them here. See also Lemma 4.2 below.

LEMMA 2.4. *Let $a_1, \ldots, a_n$ be positive numbers, let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and let $a^2 = a_1^2 + \ldots + a_n^2$. Define*

$$U_p^a = \Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{k=1}^n |a_k x_k|^p \le 1\Big\}, \qquad 1 \le p < \infty,$$

$$U_\infty^a = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |a_k x_k| \le 1 \text{ for } k = 1, \ldots, n\}.$$

Then

(i) $\alpha(U_2^a) \le a^2/2\pi$;
(ii) $\alpha(U_2^a) \le (2\pi e/a^2)^{1/2}\, e^{-\pi/a^2}$ *for* $a^2 \le 2\pi$;
(iii) $\alpha(U_p^a) \le p\pi^{-p/2}\Gamma(p/2)\sum_{k=1}^n a_k^p$ *for* $1 \le p < \infty$;
(iv) $\alpha(U_\infty^a) \le 2\sum_{k=1}^n e^{-\pi/a_k^2}$.

Proof of Lemma 1.4. From Lemma 2.3 we have

$$\mu(K^\wedge \setminus Z_K) \le (1-2e^{-\pi})^{-1}(\varepsilon + 2\alpha(D)),$$

and Lemma 2.4(i) says that

$$\alpha(D) \le (2\pi)^{-1} \sum_{k=1}^{n} \lambda_k^{-2}.$$

Putting these two inequalities together we obtain (1.2). ■

Remark 2.5. Lemma 1.4 may be treated as an analogue of Lemma 1.1 for subgroups of $\mathbb{R}^n$. The author does not know whether the corresponding analogue of Lemma 1.2 is true. More precisely, set $U = tB_\infty^n$ in Lemma 2.3, where t is a positive coefficient. Then, by Lemma 2.4(iv), $\mu(K^\wedge \setminus Z_K)$ is small if t is of order $(\log n)^{1/2}$. It is not clear if $(\log n)^{1/2}$ may be replaced here by a constant.

3. The Sazonov topology. In the rest of the paper H denotes a real Hilbert space, and B_H is the closed unit ball of H. If p is a seminorm on H, then we write

$$B_p = \{x \in H : p(x) \le 1\}.$$

We say that p is a *pre-Hilbert seminorm* if

$$p^2(x+y) + p^2(x-y) = 2p^2(x) + 2p^2(y), \qquad x, y \in H.$$

Let X, Y be symmetric convex subsets of H with $X \subset tY$ for some $t > 0$. For $k = 1, 2, \ldots$, we define

$$d_k(X,Y) = \inf_M \inf\{t > 0 : X \subset tY + M\},$$

where the first infimum is taken over all linear subspaces M of H with $\dim M < k$. The numbers $d_k(X,Y)$ are called the *Kolmogorov diameters* of X with respect to Y.

By $\mathcal{S}(H)$ we denote the family of continuous pre-Hilbert seminorms p on H such that

$$\sum_{k=1}^{\infty} d_k^2(B_H, B_p) < \infty.$$

The topology on H induced by $\mathcal{S}(H)$ is called the *Sazonov topology* (cf. Sec. 1.1 of [11, Ch. VI]).

LEMMA 3.1. *The family $\{B_p\}_{p\in\mathcal{S}(H)}$ is a basis of neighbourhoods of zero for the Sazonov topology on H.*

Proof. Let U be a neighbourhood of zero in the Sazonov topology. Then we can find some $p_1, \ldots, p_n \in \mathcal{S}(H)$ with $B_{p_1} \cap \ldots \cap B_{p_n} \subset U$. It is not hard to see that $p = (p_1^2 + \ldots + p_n^2)^{1/2} \in \mathcal{S}(H)$ and $B_p \subset B_{p_1} \cap \ldots \cap B_{p_n}$. ■

Let τ be a topology on H such that (H, τ) is an additive topological group. We say that τ is *sufficient* if every τ-continuous positive-definite function on H is the Fourier transform of some Radon measure on $H_{\rm b}^\wedge$. Next, we say that τ is *subgroup-sufficient* if every τ-continuous positive-definite function on an arbitrary additive subgroup G of H is the Fourier transform of some Radon measure on $G_{\rm b}^\wedge$. The Sazonov topology on H is sufficient (see e.g. Theorem 1.1 of [11, Ch. VI]); we shall prove it is subgroup-sufficient as well.

LEMMA 3.2. *A character χ of an abelian topological group G is continuous if and only if there is a neighbourhood U of zero in G such that* $\operatorname{Re}\chi(x) \ge 0$ *for each $x \in U$.*

This simple fact is well known (see e.g. Lemma (1.4) of [1]).

LEMMA 3.3. *Let U be a neighbourhood of zero in an abelian topological group G, and let*

$$Z = \{\chi \in G^\wedge : \operatorname{Re}\chi(x) \ge 0 \text{ for each } x \in U\}.$$

Then Z is a compact subset of $G_{\rm pc}^\wedge$.

This is a standard fact. For the proof see e.g. Proposition (1.5) of [1].

LEMMA 3.4. *Let E be a finite-dimensional real vector space and let p, q be pre-Hilbert seminorms on E with $p \le q$. Let K be a closed subgroup of E and μ a Borel probability measure on $K_{\rm p}^\wedge$ such that* $\operatorname{Re}\widehat{\mu}(x) \ge 1 - \varepsilon$ *for each $x \in K \cap B_p$, where $\varepsilon > 0$. Define*

$$Z = \{\chi \in K^\wedge : \operatorname{Re}\chi(x) \ge 0 \text{ for each } x \in K \cap \tfrac{1}{4}B_q\}.$$

Then

$$\mu(K^\wedge \setminus Z) \le 2\varepsilon + \sum_{k=1}^{\infty} d_k^2(B_q, B_p).$$

Proof. The special case where p, q are norms is nothing but another formulation of Lemma 1.4. The general case follows by an easy approximation argument. ■

Let X, Y be topological spaces, $f : X \to Y$ a continuous mapping and μ a Borel measure on X. By $f(\mu)$ we denote the image of μ, i.e. the Borel measure on Y given by $f(\mu)(A) = \mu(f^{-1}(A))$ for $A \in \mathcal{B}(Y)$.

LEMMA 3.5. *Let $(\mathcal{M}, \subset)$ be a directed set. For each $M \in \mathcal{M}$, let X_M be a Hausdorff topological space and μ_M a regular Borel probability measure on X_M. Suppose that, for each pair $(M,N) \in \mathcal{M}^2$ such that $M \subset N$, a continuous mapping π_{MN} of X_N onto X_M is given, such that*

(i) $\pi_{MM} = \mathrm{id}_{X_M}$ *for each* $M \in \mathcal{M}$;

(ii) $\pi_{MN} \circ \pi_{NP} = \pi_{MP}$ *for each triple* $(M, N, P) \in \mathcal{M}^3$ *such that* $M \subset N \subset P$;

(iii) $\mu_M = \pi_{MN}(\mu_N)$ *for each pair* $(M, N) \in \mathcal{M}^2$ *such that* $M \subset N$.

Suppose further that X *is a Hausdorff topological space, and that, for each* $M \in \mathcal{M}$, *a continuous mapping* π_M *of* X *onto* X_M *is given, such that*

(iv) $\pi_M = \pi_{MN} \circ \pi_N$ *for each pair* $(M, N) \in \mathcal{M}^2$ *such that* $M \subset N$.

Finally, suppose that, for any two distinct points $x, y \in X$, *there exists some* $M \in \mathcal{M}$ *such that* $\pi_M(x) \neq \pi_M(y)$. *Then the following two conditions are equivalent*:

(*) *there exists a unique Radon probability measure* μ *on* X *such that* $\mu_M = \pi_M(\mu)$ *for each* $M \in \mathcal{M}$;

(**) *for each* $\varepsilon > 0$, *there is a compact subset* Z *of* X *such that* $\mu_M(X_M \setminus \pi_M(Z)) \leq \varepsilon$ *for each* $M \in \mathcal{M}$.

This is Theorem 3.2 of Kisyński [5].

THEOREM 3.6. *The Sazonov topology on* H *is subgroup-sufficient.*

Proof. Let G be a subgroup of H and φ a positive-definite function on G continuous in the Sazonov topology. We have to prove that φ is the Fourier transform of some Radon probability measure on $G_{\mathrm{b}}^{\wedge}$. Let $\overline{\varphi}$ be the continuous extension of φ onto the closure $\overline{G}$ of G. Then $\overline{\varphi}$ is a positive-definite function continuous in the Sazonov topology on $\overline{G}$. We may identify $(\overline{G})_{\mathrm{b}}^{\wedge}$ with $G_{\mathrm{b}}^{\wedge}$. Thus, if μ is a Radon measure on $(\overline{G})_{\mathrm{b}}^{\wedge}$ with $\widehat{\mu} = \overline{\varphi}$, it may also be treated as a Radon measure on $G_{\mathrm{b}}^{\wedge}$ with $\widehat{\mu} = \varphi$. Therefore we may assume G to be closed.

Let $\mathcal{M}$ be the directed family of all finite-dimensional subspaces of H. For each $M \in \mathcal{M}$, define $X_M = (G \cap M)_{\mathrm{pc}}^{\wedge}$ and let μ_M be the Radon probability measure on X_M such that $\widehat{\mu}_M = \varphi_{|G\cap M}$, existing due to the Bochner theorem. Define $X = G_{\mathrm{b}}^{\wedge}$ and let $\pi_M : X \to X_M$ and $\pi_{MN} : X_N \to X_M$, $M \subset N$, be the natural homomorphisms given by $\chi \mapsto \chi_{|G\cap M}$. Then all the assumptions of Lemma 3.5 are satisfied. The surjectivity of the homomorphisms π_M and π_{MN} follows easily from standard facts. To prove (*), we shall verify (**).

Fix $\varepsilon > 0$. By Lemma 3.1, there exists a seminorm $p \in \mathcal{S}(H)$ such that

$$\operatorname{Re}\varphi(x) \geq 1 - \varepsilon/4 \quad \text{for each } x \in G \cap B_p. \tag{3.1}$$

A standard argument based on the spectral theorem for compact operators allows one to find a continuous pre-Hilbert seminorm $q \geq p$ on H such that

$$\sum_{k=1}^{\infty} d_k^2(B_q, B_p) \leq \frac{\varepsilon}{2}, \tag{3.2}$$

$$d_k(B_H, B_q) \to 0 \quad \text{as } k \to \infty. \tag{3.3}$$

Let us define

$$Z = \{\chi \in G^{\wedge} : \operatorname{Re}\chi(x) \geq 0 \text{ for each } x \in G \cap \tfrac{1}{4}B_q\}.$$

Let G_q denote the group G endowed with the topology induced by q. By Lemma 3.3, the set Z is compact in $(G_q)_{\mathrm{pc}}^{\wedge}$. Condition (3.3) implies that every bounded subset of G is precompact in G_q (one may assume that q is a norm and then apply e.g. Proposition 9.1.4 of [8]). Hence the identity mapping $(G_q)_{\mathrm{pc}}^{\wedge} \to G_{\mathrm{b}}^{\wedge}$ is continuous. Consequently, Z is compact in $G_{\mathrm{b}}^{\wedge}$.

Fix an arbitrary $M \in \mathcal{M}$. We have to prove that

$$\mu_M(X_M \setminus \pi_M(Z)) \leq \varepsilon. \tag{3.4}$$

Define $\mathcal{N} = \{N \in \mathcal{M} : N \supset M\}$ and, for $N \in \mathcal{N}$,

$$Z_N = \{\chi \in (G \cap N)^{\wedge} : \operatorname{Re}\chi(x) \geq 0 \text{ for each } x \in G \cap N \cap \tfrac{1}{4}B_q\}.$$

To prove (3.4), we cannot apply Lemma 3.4 directly because, in general, $\pi_M(Z)$ is strictly smaller than Z_M. We shall prove that

$$\pi_M(Z) = \bigcap_{N \in \mathcal{N}} \pi_{MN}(Z_N). \tag{3.5}$$

Denote the right-hand side by S. Let $G^{\#}$ be the group of all characters of G (continuous or not) endowed with the topology of pointwise convergence. Suppose $\chi \in S$. Then a standard argument based on the compactness of $G^{\#}$ proves the existence of some $\xi \in G^{\#}$ such that $\xi_{|G\cap M} = \chi$ and $\operatorname{Re}\xi(x) \geq 0$ for each $x \in G \cap \frac{1}{4}B_q$. By Lemma 3.2, the latter condition means that $\xi \in Z$; then $\chi = \pi_M(\xi) \in \pi_M(Z)$. Thus $S \subset \pi_M(Z)$, and the opposite inclusion is trivial.

By (3.5), we have

$$\mu_M(X_M \setminus \pi_M(Z)) = \mu_M\Big(\bigcup_{N \in \mathcal{N}} (X_M \setminus \pi_{MN}(Z_N))\Big).$$

For each $N \in \mathcal{N}$, the set Z_N is compact in X_N due to Lemma 3.3; hence $\pi_{MN}(Z_N)$ is compact and $X_M \setminus \pi_{MN}(Z_N)$ is open in X_M. Being a Radon measure, μ_M is τ-smooth (see Proposition 3.1(c) of [11, Ch. I]), so that

$$\mu_M\Big(\bigcup_{N \in \mathcal{N}} (X_M \setminus \pi_{MN}(Z_N))\Big) = \sup_{N \in \mathcal{N}} \mu_M(X_M \setminus \pi_{MN}(Z_N)).$$

So, to prove (3.4), it is enough to show that, for each $N \in \mathcal{N}$,

$$\mu_M(X_M \setminus \pi_{MN}(Z_N)) \leq \varepsilon. \tag{3.6}$$

Fix N and let p' and q' be the restrictions to N of p and q, respectively. Then $B_{p'} = N \cap B_p$ and $B_{q'} = N \cap B_q$. We have

$$d_k(N \cap B_q, N \cap B_p) \leq d_k(B_q, B_p), \qquad k = 1, 2, \ldots$$

(see e.g. Lemma (2.13) of [1]), whence

$$\sum_{k=1}^{\infty} d_k^2(B_{q'}, B_{p'}) \le \sum_{k=1}^{\infty} d_k^2(B_q, B_p). \tag{3.7}$$

Condition (3.1) implies that $\operatorname{Re}\widehat{\mu}_N(x) \ge 1-\varepsilon/4$ for $x \in G\cap N\cap B_{p'}$. Then Lemma 3.4 yields

$$\mu_N(X_N\setminus Z_N) \le 2\cdot\frac{\varepsilon}{4} + \sum_{k=1}^{\infty} d_k^2(B_{q'}, B_{p'}).$$

Hence, by (3.7) and (3.2), we get $\mu_N(X_N\setminus Z_N)\le\varepsilon$, which clearly implies (3.6). This proves (3.4) and, consequently, (**).

By Lemma 3.5, also condition (*) is satisfied, i.e. there exists a Radon probability measure μ on $G_{\mathrm{b}}^{\wedge}$ with $\mu_M = \pi_M(\mu)$ for every M. The condition $\mu_M = \pi_M(\mu)$ implies that $\widehat{\mu}_M(x) = \widehat{\mu}(x)$ for $x\in G\cap M$, whence $\widehat{\mu}_{|G\cap M} = \widehat{\mu}_M = \varphi_{|G\cap M}$. Since this is true for every M, it follows that $\widehat{\mu}=\varphi$. ∎

4. The Gross topology. Let X be a Banach space and $T: H\to X$ a bounded linear operator. By $\ell(T)$ we denote the ℓ-norm of T (see [9, Ch. 3], [10, (12.2)] or, for finite-dimensional H, [6, (2.3.16) and (2.3.17)]).

Let p be a continuous seminorm on H and let T be the canonical projection of H onto $X = H/p^{-1}(0)$. Let us endow X with the canonical norm given by $\|Tx\| = p(x)$ for $x\in H$, and let $\widetilde{X}$ be the completion of X. By $\ell(p)$ we denote the ℓ-norm of the operator $T: H\to\widetilde{X}$. We say that p is a *bounding seminorm* if $\ell(p)<\infty$. The topology on H induced by the family of all bounding seminorms is called the *Gross topology* (cf. Sec. 2.3 of [11, Ch. VI]).

The author does not know whether the Gross topology is subgroup-sufficient. Below we prove the following weaker fact:

THEOREM 4.1. *Let G be a subgroup of H and φ a positive-definite function on G continuous in the Gross topology. Then φ is the Fourier transform of a Radon measure on $G_{\mathrm{pc}}^{\wedge}$.*

By p_U we denote the Minkowski functional of a convex body $U\in\mathcal{C}^n$. The following fact is an easy consequence of Lemma 2.1 and the Talagrand theorem on the majorizing measure. For the detailed proof we refer the reader to [4, Lemma 2].

LEMMA 4.2. *To each $\sigma>0$ there corresponds some $t>0$ such that if $U\in\mathcal{C}^n$ and $\ell(p_U)<t$, then $\alpha(U)<\sigma$.*

Thus, there exists a continuous non-decreasing function $\sigma:(0,\infty)\to(0,1]$ such that

$$\alpha(U)\le\sigma(\ell(p_U)),\qquad U\in\mathcal{C}^n,\ n=1,2,\ldots, \tag{4.1}$$

$$\sigma(t)\to 0\quad \text{as } t\to 0. \tag{4.2}$$

In the rest of this section, σ is a fixed function with the above properties.

LEMMA 4.3. *Suppose that $\dim H<\infty$ and let p be a seminorm on H. Let G be a closed subgroup of H and μ a Borel probability measure on $G_{\mathrm{p}}^{\wedge}$ such that*

$$\operatorname{Re}\widehat{\mu}(x)\ge 1-\varepsilon\quad \text{for each } x\in G\cap B_p, \tag{4.3}$$

where $\varepsilon>0$. Define

$$Z=\{\chi\in G^{\wedge} : \operatorname{Re}\chi(x)\ge 0 \text{ for each } x\in G\cap \tfrac14 kB_H\}$$

where $k>0$. Then

$$\mu(G^{\wedge}\setminus Z)\le 2\varepsilon+3\sigma(k\ell(p)). \tag{4.4}$$

Proof. We may assume that $k=1$, replacing the norm on H by the new norm $k^{-1}\|x\|$; then kB_H is replaced by B_H and $k\ell(p)$ by $\ell(p)$.

We may identify H with the euclidean space $\mathbb{R}^n$, $n=\dim H$; then $B_H = B_2^n$. If p is a norm on $H=\mathbb{R}^n$, then $B_p\in\mathcal{C}^n$ and, by Lemma 2.3, we have

$$\mu(G^{\wedge}\setminus Z)\le(1-2e^{-\pi})^{-1}(\varepsilon+2\alpha(B_p))<2\varepsilon+3\alpha(B_p).$$

Next, by (4.1), we have $\alpha(B_p)\le\sigma(\ell(p))$, which proves (4.4). If p is not a norm, then we can find a sequence $(p_i)_{i=1}^{\infty}$ of norms on $\mathbb{R}^n$ such that $B_{p_i}\subset B_p$ for every i, and $\ell(p_i)\to\ell(p)$ as $i\to\infty$. When $B_{p_i}\subset B_p$, (4.3) implies that $\operatorname{Re}\widehat{\mu}(x)\ge 1-\varepsilon$ for each $x\in G\cap B_{p_i}$. Applying Lemma 4.3 to the norms p_i, we obtain

$$\mu(G^{\wedge}\setminus Z)\le 2\varepsilon+3\sigma(\ell(p_i))$$

for every i. Passing to the limit with $i\to\infty$, we obtain (4.4). ∎

Proof of Theorem 4.1. The argument is similar to that used in the proof of Theorem 3.6. We may assume G to be closed. Let $\mathcal{M}$, X_M, μ_M and π_{MN} be defined as before. Define $X=G_{\mathrm{pc}}^{\wedge}$ and let $\pi_M: X\to X_M$, $M\in\mathcal{M}$, be the natural homomorphisms. Then all the assumptions of Lemma 3.5 are satisfied. As in the proof of Theorem 3.6, it is enough to verify (**).

Fix $\varepsilon>0$. Since φ is continuous in the Gross topology, there exist some bounding seminorms $p_1,\ldots,p_n$ on H such that

$$\operatorname{Re}\varphi(x)\ge 1-\varepsilon/4\quad \text{for each } x\in G\cap B_{p_1}\cap\ldots\cap B_{p_n}.$$

Then $p=p_1+\ldots+p_n$ is a bounding seminorm again, and

$$\operatorname{Re}\varphi(x)\ge 1-\varepsilon/4\quad \text{for each } x\in G\cap B_p. \tag{4.5}$$

By (4.2), there is a coefficient $k>0$ such that

$$\sigma(k\ell(p))\le\varepsilon/6. \tag{4.6}$$

The set

$$Z = \{\chi \in G^\wedge : \operatorname{Re}\chi(x) \geq 0 \text{ for each } x \in G \cap \tfrac{1}{4}kB_H\}$$

is compact in $G^\wedge_{\mathrm{pc}}$, according to Lemma 3.3.

Fix an arbitrary $M \in \mathcal{M}$. We have to show that

$$\mu_M(X_M \setminus \pi_M(Z)) \leq \varepsilon. \tag{4.7}$$

Define $\mathcal{N} = \{N \in \mathcal{M} : M \subset N\}$ and, for each $N \in \mathcal{N}$,

$$Z_N = \{\chi \in (G \cap N)^\wedge : \operatorname{Re}\chi(x) \geq 0 \text{ for each } x \in G \cap N \cap \tfrac{1}{4}kB_H\}.$$

As in the proof of Theorem 3.6, to prove (4.7), it is enough to show that $\mu_N(X_N \setminus Z_N) \leq \varepsilon$ for every N.

Fix N and let p' be the restriction of p to N. It is clear that $\ell(p') \leq \ell(p)$. Hence, by (4.6), we have

$$\sigma(k\ell(p')) \leq \sigma(k\ell(p)) \leq \varepsilon/6.$$

Then, by (4.5) and Lemma 4.3, we obtain

$$\mu_N(X_N \setminus Z_N) \leq 2 \cdot \frac{\varepsilon}{4} + 3\sigma(k\ell(p')) \leq \frac{\varepsilon}{2} + 3 \cdot \frac{\varepsilon}{6} = \varepsilon.$$

This proves (4.7) and, consequently, (**). ■

Remark 4.4. If a function φ on H is the transform of a Radon measure on the dual space, then φ is continuous in the Sazonov topology and in the Gross topology; we say that these topologies are *necessary*. Of course, they are not *subgroup-necessary* (consider e.g. suitable discrete subgroups of H), and the problem of subgroup-necessary topologies on H does not seem to have much sense.

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