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# On the range of convolution operators on non-quasianalytic ultradifferentiable functions

by

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**Abstract.** Let $\mathcal{E}_{(\omega)}(\Omega)$ denote the non-quasianalytic class of Beurling type on an open set $\Omega$ in $\mathbb{R}^n$. For $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ the surjectivity of the convolution operator $T_\mu : \mathcal{E}_{(\omega)}(\Omega_1) \to \mathcal{E}_{(\omega)}(\Omega_2)$ is characterized by various conditions, e.g. in terms of a convexity property of the pair $(\Omega_1, \Omega_2)$ and the existence of a fundamental solution for $\mu$ or equivalently by a slowly decreasing condition for the Fourier–Laplace transform of $\mu$. Similar conditions characterize the surjectivity of a convolution operator $S_\mu : \mathcal{D}'_{\{\omega\}}(\Omega_1) \to \mathcal{D}'_{\{\omega\}}(\Omega_2)$ between ultradistributions of Roumieu type whenever $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$. These results extend classical work of Hörmander on convolution operators between spaces of $C^\infty$-functions and more recent one of Ciorănescu and Braun, Meise and Vogt.

Since the classical work of Ehrenpreis [10] and Hörmander [14], convolution operators on various spaces of infinitely differentiable functions and distributions have been investigated by many authors (see e.g. Berenstein and Dostal [1], Chou [8], Ciorănescu [9], Franken and Meise [11], v. Grudzinski [12], Meise, Taylor and Vogt [20], Braun, Meise and Vogt [7], Meyer [23], Momm [24], [25]). The starting point for the research presented here was a recent result of Bonet and Galbis [3]. They proved that each convolution operator $T_\mu$ acting on the non-quasianalytic class $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ (defined in the sense of Braun, Meise and Taylor [6]) for which $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^n))$ contains some smaller class $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ already acts surjectively on $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.

In the present paper we show that this holds in greater generality and is an immediate corollary to the following extension of results of Hörmander [14] to the non-quasianalytic classes $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ (see 2.7–2.9).

THEOREM A. *Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ and open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ with $\Omega_1 + \operatorname{Supp} \check{\mu} \subset \Omega_2$ be given. Then the following conditions are equivalent:*

(1) *For each $g \in \mathcal{E}_{(\omega)}(\Omega_1)$ there exists $f \in \mathcal{E}_{(\omega)}(\Omega_2)$ with $\mu * f|_{\Omega_1} = g$.*
(2) *For each $g \in \mathcal{E}_{(\omega)}(\Omega_1)$ there exists $f \in \mathcal{D}'_{(\omega)}(\Omega_2)$ with $\mu * f|_{\Omega_1} = g$.*

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(3) $(\Omega_1, \Omega_2)$ *is* $\check{\mu}$*-convex for* $(\omega)$ *and there exists* $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ *satisfying* $\mu * E = \delta$.

Here $(\Omega_1, \Omega_2)$ is *$\mu$-convex for* $(\omega)$ if for each compact subset $K_2$ of $\Omega_2$ there exists a compact subset $K_1$ of $\Omega_1$ such that each $\varphi \in \mathcal{D}_{(\omega)}(\Omega_1)$ satisfying $\operatorname{Supp} \mu * \varphi \subset K_2$ already satisfies $\operatorname{Supp} \varphi \subset K_1$.

Similarly we characterize the surjectivity of convolution operators between ultradistributions of Roumieu type $\{\omega\}$ by the following theorem which extends a result of Braun, Meise and Vogt [7] for the case of $\Omega_1 = \Omega_2 = \mathbb{R}$ (see Thm. 3.5).

THEOREM B. *Let* $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ *and open sets* $\Omega_1, \Omega_2$ *in* $\mathbb{R}^n$ *with* $\Omega_1 + \operatorname{Supp} \check{\mu} \subset \Omega_2$ *be given. Then the following assertions are equivalent*:

(1) *For each* $g \in \mathcal{D}'_{\{\omega\}}(\Omega_1)$ *there exists* $f \in \mathcal{D}'_{\{\omega\}}(\Omega_2)$ *with* $\mu * f|_{\Omega_1} = g$.

(2) *For each* $g \in \mathcal{E}_{\{\omega\}}(\Omega_1)$ *there exists* $f \in \mathcal{D}'_{\{\omega\}}(\Omega_2)$ *with* $\mu * f|_{\Omega_1} = g$.

(3) $(\Omega_1, \Omega_2)$ *is* $\check{\mu}$*-convex for* $\{\omega\}$ *and there exists* $E \in \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$ *satisfying* $\mu * E = \delta$.

To prove Theorem A we modify arguments that were used in Hörmander [14]. In doing this, the main difficulty is to show that (2) implies (1). To overcome it we use a result of Braun [5] which sharpens the second structure theorem of Komatsu [17]. Further we apply a result of Hansen [13] on the projective description of the topology on the space of Fourier–Laplace transforms of $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ to characterize the surjectivity of $T_\mu$ on $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ by a slowly decreasing condition of Ehrenpreis type, in the form due to Momm [24]. Also we apply a surjectivity criterion for continuous linear maps between Fréchet spaces (see Meise and Vogt [22], 26.1) which is better adapted to our applications than classical results of this type.

Earlier versions of Theorem B appear in the literature only in the case $\Omega_1 = \Omega_2 = \mathbb{R}$ in Braun, Meise and Vogt [7]. From this paper it also follows that Theorem A does not extend literally to the Roumieu case because there exists $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$ for which not all equations $\mu * f = g$, $g \in \mathcal{E}_{(\omega)}(\mathbb{R})$, admit a solution $f$ in $\mathcal{E}_{(\omega)}(\mathbb{R})$, though there exists $E \in \mathcal{D}'_{\{\omega\}}(\mathbb{R})$ satisfying $\mu * E = \delta$. The proof of Theorem B is based on the arguments mentioned above and on reductions to the Beurling case which go back to Braun, Meise and Taylor [6].

Note that the above results apply in particular to the Gevrey classes $\Gamma^{(d)}$ and $\Gamma^{\{d\}}$ for $d > 1$ and also to the classes $\mathcal{E}^{(M_p)}$ and $\mathcal{E}^{\{M_p\}}$ whenever the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfies the conditions (M1), (M2) and (M3) of Komatsu [17], because then $\mathcal{E}^{(M_p)}(\Omega) = \mathcal{E}_{(\omega_M)}(\Omega)$ and $\mathcal{E}^{\{M_p\}}(\Omega) = \mathcal{E}_{\{\omega_M\}}(\Omega)$ for $\omega_M(t) := \sup_{p\in\mathbb{N}_0} \log(t^p M_0/M_p)$ for $t > 0$ and $\omega_M(0) := 0$, by Meise and Taylor [19], 3.11.



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## 1. Preliminaries.

In this preliminary section we introduce the non-quasianalytic classes, the spaces of ultradistributions and most of the notation that will be used in the sequel.

**1.1.** DEFINITION. A continuous increasing function $\omega : [0,\infty[ \to [0,\infty[$ is called a *weight function* if it satisfies the following conditions:

($\alpha$) there exists $K \ge 1$ with $\omega(2t) \le K(1+\omega(t))$ for all $t \ge 0$,
($\beta$) $\int_1^\infty (\omega(t)/t^2)\,dt < \infty$,
($\gamma$) $\log t = o(\omega(t))$ as $t \to \infty$,
($\delta$) $\varphi : t \mapsto \omega(e^t)$ is convex.

For a weight function $\omega$ we define $\widetilde{\omega} : \mathbb{C}^n \to [0,\infty[$ by $\widetilde{\omega}(z) = \omega(|z|)$ and again call this function $\omega$, by abuse of notation. The function

$$\varphi^* : [0,\infty[ \to \mathbb{R}, \quad \varphi^*(y) := \sup\{xy - \varphi(x) : x \ge 0\},$$

is called the *Young conjugate* of $\varphi$.

**1.2.** Remark. (a) Each weight function $\omega$ satisfies $\lim_{t\to\infty} \omega(t)/t = 0$ by the remark following 1.3 of [20].

(b) For each weight function $\omega$ there exists a weight function $\sigma$ satisfying $\sigma(t) = \omega(t)$ for all large $t > 0$ and $\sigma|[0,1] \equiv 0$. This implies $\varphi_\sigma(y) = \varphi_\omega(y)$ for all large $y$, $\varphi^*_\sigma([0,\infty[) \subset [0,\infty[$ and $\varphi^{**}_\sigma = \varphi_\sigma$. From this it follows that all subsequent definitions do not change if $\omega$ is replaced by $\sigma$. In fact, they do not change if $\omega$ is replaced by a weight function $\kappa$ which for some $a \ge 1$ and $b > 0$ satisfies

$$(*) \qquad \frac{1}{a}\kappa(t) - b \le \omega(t) \le a\kappa(t) + b, \quad t \ge 0.$$

Note that for each weight function $\omega$ there exist $C > 0$ and a differentiable weight function $\kappa$ which satisfies $(*)$ and

$$\kappa'(t) \le C\kappa(t) + C \quad \text{for all } t \ge 0.$$

**1.3.** DEFINITION. Let $\omega$ be a weight function.

(a) For a set $K \subset \mathbb{R}^n$ and $\lambda > 0$ let

$$\mathcal{E}_\omega(K,\lambda) := \{f \in C^\infty(K) : \|f\|_{K,\lambda} := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^n} |f^{(\alpha)}(x)| \exp(-\lambda\varphi^*(|\alpha|/\lambda)) < \infty\}.$$

(b) For an open set $\Omega \subset \mathbb{R}^n$ define

$$\mathcal{E}_{(\omega)}(\Omega) := \underset{\overleftarrow{K \Subset \Omega}}{\text{proj}}\, \underset{\overleftarrow{m \in \mathbb{N}}}{\text{proj}}\, \mathcal{E}_\omega(K, m)$$
$$= \{f \in C^\infty(\Omega) : \|f\|_{K,m} < \infty \text{ for each } K \Subset \Omega \text{ and each } m \in \mathbb{N}\},$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) := \underset{\overleftarrow{K \Subset \Omega}}{\text{proj}}\, \underset{\overrightarrow{m \in \mathbb{N}}}{\text{ind}}\, \mathcal{E}_\omega(K, 1/m)$$
$$= \{f \in C^\infty(\Omega) : \text{for each } K \Subset \Omega \text{ there is } m \in \mathbb{N} \text{ with } \|f\|_{K,1/m} < \infty\}.$$

The elements of $\mathcal{E}_{(\omega)}(\Omega)$ (resp. $\mathcal{E}_{\{\omega\}}(\Omega)$) are called *$\omega$-ultradifferentiable functions of Beurling* (resp. *Roumieu*) *type* on $\Omega$. We write $\mathcal{E}_*(\Omega)$, where $*$ can be either $(\omega)$ or $\{\omega\}$.

(c) For a compact set $K$ in $\mathbb{R}^n$ we let

$$\mathcal{D}_*(K) := \{f \in \mathcal{E}_*(\mathbb{R}^n) : \operatorname{Supp}(f) \subset K\},$$

endowed with the induced topology. For an open set $\Omega \subset \mathbb{R}^n$ and a fundamental sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of $\Omega$ we let

$$\mathcal{D}_*(\Omega) := \underset{j \to}{\text{ind}}\, \mathcal{D}_*(K_j).$$

For $\lambda > 0$ and $\varphi \in \mathcal{D}_*(\mathbb{R}^n)$ we let $\|\varphi\|_\lambda = \|\varphi\|_{\mathbb{R}^n,\lambda}$. The dual $\mathcal{D}'_*(\Omega)$ of $\mathcal{D}_*(\Omega)$ is endowed with its strong topology. The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ (resp. $\mathcal{D}'_{\{\omega\}}(\Omega)$) are called *$\omega$-ultradistributions of Beurling* (resp. *Roumieu*) *type* on $\Omega$.

**1.4.** Remark. (a) By Meise, Taylor and Vogt [21], 3.3, for each open set $\Omega$ in $\mathbb{R}^n$, the semi-norms

$$\|\ \|_{K,\sigma} : f \mapsto \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^n} |f^{(\alpha)}(x)| \exp(-\varphi^*_\sigma(|\alpha|)),$$

where $K$ is any compact set in $\Omega$ and $\sigma$ is a weight function satisfying $\sigma = o(\omega)$, form a fundamental system of semi-norms for $\mathcal{E}_{\{\omega\}}(\Omega)$.

(b) For each compact set $K$ in $\mathbb{R}^n$, $\mathcal{D}_{\{\omega\}}(K)$ is a (DFN)-space by Braun, Meise and Taylor [6], 3.6. A fundamental system of bounded sets is given by

$$B_m := \Big\{\varphi \in \mathcal{D}_{\{\omega\}}(K) : |\varphi|_{K,m} := \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)| e^{\omega(\xi)/m}\, d\xi \le 1\Big\},$$

where $\widehat{\varphi}(\xi) = \int \varphi(x) e^{-i\langle x,\xi\rangle}\, dx$, $\xi \in \mathbb{R}^n$.

(c) For each compact set $K$ in $\mathbb{R}^n$, $\mathcal{D}_{(\omega)}(K)$ is a nuclear Fréchet space, by Braun, Meise and Taylor [6], 3.6. A fundamental system of semi-norms on $\mathcal{D}_{(\omega)}(K)$ is given by $(\|\cdot\|_{K,m})_{m\in\mathbb{N}}$ defined in 1.3 but also by

$$\|\varphi\|_m := \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)| e^{m\omega(\xi)}\, d\xi, \quad \varphi \in \mathcal{D}_{(\omega)}(K).$$

(d) Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $(K_j)_{j\in\mathbb{N}}$ a fundamental sequence of compact subsets of $\Omega$. Then a fundamental system of semi-norms for $\mathcal{D}_{(\omega)}(\Omega)$ is obtained by $(K_0 := \emptyset)$

$$\|\varphi\|_{\mathcal{L},\mathcal{M}} := \sup_{j\in\mathbb{N}_0} L_j \sup_{x\in\Omega\setminus K_j} \sup_{\alpha\in\mathbb{N}_0^n} |\varphi^{(\alpha)}(x)| \exp(-M_j\varphi^*(|\alpha|/M_j)),$$

where $\mathcal{L} = (L_j)_{j\in\mathbb{N}_0}$ and $\mathcal{M} = (M_j)_{j\in\mathbb{N}_0}$ are increasing sequences in $]0,\infty[$ resp. $\mathbb{N}$. This can be shown similarly to Hörmander [16], 15.4.1.

**1.5.** Example. The following functions $\omega : [0,\infty[ \to [0,\infty[$ are examples of weight functions:

(1) $\omega(t) = t^\alpha$, $0 < \alpha < 1$,
(2) $\omega(t) = (\log(1+t))^\beta$, $\beta > 1$,
(3) $\omega(t) = t(\log(e+t))^{-\beta}$, $\beta > 1$.

Note that for $\omega(t) = t^\alpha$, the classes $\mathcal{E}_{(\omega)}$ resp. $\mathcal{E}_{\{\omega\}}$ coincide with the Gevrey classes $\Gamma^{(d)}$ resp. $\Gamma^{\{d\}}$ for $d := 1/\alpha$.

**1.6.** *Convolution operators.* Let $\mu \in \mathcal{E}'_*(\mathbb{R}^n)$, $\mu \ne 0$, and open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ be given. If $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$ then we define (compare Braun, Meise and Taylor [6], Sect. 6):

(a) $$S^t_\mu : \mathcal{D}_*(\Omega_1) \to \mathcal{D}_*(\Omega_2), \qquad S^t_\mu(\varphi) := \mu * \varphi|_{\Omega_2},$$

where $\mu * \varphi : x \mapsto \mu(\varphi(x - \cdot))$, $x \in \mathbb{R}^n$. Since $S^t_\mu$ is continuous and linear, so is its adjoint operator

$$S_\mu := (S^t_\mu)^t : \mathcal{D}'_*(\Omega_2) \to \mathcal{D}'_*(\Omega_1).$$

(b) $$T^t_\mu : \mathcal{E}'_*(\Omega_1) \to \mathcal{E}'_*(\Omega_2), \qquad T^t_\mu(\nu) := \mu * \nu|_{\Omega_2},$$

where $\mu * \nu(\varphi) := (\mu * (\check{\nu} * \varphi))(0)$ and where $\check{\nu}(\psi) := \nu(\check{\psi})$ and $\check{\psi}(x) := \psi(-x)$, $x \in \mathbb{R}^n$. Again $T^t_\mu$ is continuous and linear, so that its adjoint

$$T_\mu := (T^t_\mu)^t : \mathcal{E}_*(\Omega_2) \to \mathcal{E}_*(\Omega_1)$$

is continuous and linear.

Note that $S_\mu(\nu) = \check{\mu} * \nu$ and $T_\mu(f) = \check{\mu} * f$, so that it is reasonable to call the operators $S_\mu$ and $T_\mu$ *convolution operators*. Note further that $T^t_\mu|_{\mathcal{D}_*(\Omega_1)} = S^t_\mu$ and $S_\mu|_{\mathcal{E}_*(\Omega_2)} = T_\mu$ and that $T^t_\mu$ and $S^t_\mu$ are injective.

**1.7.** *Spaces of entire functions.* Let $A(\mathbb{C}^n)$ denote the space of all entire functions on $\mathbb{C}^n$, endowed with the Fréchet space topology of uniform convergence on all compact subsets of $\mathbb{C}^n$. For an upper semi-continuous function $v : \mathbb{C}^n \to ]0,\infty[$ we define

$$A(v,\mathbb{C}^n) := \{f \in A(\mathbb{C}^n) : \|f\|_v := \sup_{z\in\mathbb{C}^n} |f(z)|v(z) < \infty\}$$

and note that $A(v,\mathbb{C}^n)$ is a Banach space.

**1.8.** *Fourier–Laplace transform.* For $\mu \in \mathcal{E}'_*(\mathbb{R}^n)$ its *Fourier–Laplace transform* $\widehat{\mu} \in A(\mathbb{C}^n)$ is defined as

$$\widehat{\mu}(z) := \mu(\exp(-i\langle\cdot,z\rangle)), \quad z \in \mathbb{C}^n.$$

To characterize its growth behaviour, fix a weight function $\omega$ and define the functions $w_j, w_{j,k}, v_j$ and $v_{j,k}$ by

$$w_j(z) := \exp(-j(|\operatorname{Im} z| + \omega(z))), \qquad w_{j,k}(z) := \exp\left(-j|\operatorname{Im} z| - \frac{1}{k}\omega(z)\right),$$
$$v_j(z) := \exp\left(-j|\operatorname{Im} z| + \frac{1}{j}\omega(z)\right)r, \qquad v_{j,k}(z) := \exp(-j|\operatorname{Im} z| + k\omega(z)).$$

Then the Fourier–Laplace transform $\mathcal{F} : \mu \mapsto \widehat{\mu}$ is an isomorphism between the following spaces (see Braun, Meise and Taylor [6], 3.5 and 7.4):

$$\mathcal{E}'_{(\omega)}(\mathbb{R}^n) \to \operatorname*{ind}_{j\to} A(w_j,\mathbb{C}^n), \qquad \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n) \to \operatorname*{ind}_{j\to}\operatorname*{proj}_{\leftarrow k} A(w_{j,k},\mathbb{C}^n),$$
$$\mathcal{D}_{(\omega)}(\mathbb{R}^n) \to \operatorname*{ind}_{j\to}\operatorname*{proj}_{\leftarrow k} A(v_{j,k},\mathbb{C}^n), \qquad \mathcal{D}_{\{\omega\}}(\mathbb{R}^n) \to \operatorname*{ind}_{j\to} A(v_j,\mathbb{C}^n).$$

Moreover, for $\mu,\nu \in \mathcal{E}'_*(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}_*(\mathbb{R}^n)$ we have

$$\mathcal{F}(S^t_\mu(\nu)) = \mathcal{F}(\mu)\mathcal{F}(\nu) \quad\text{and}\quad \mathcal{F}(T^t_\mu(\varphi)) = \mathcal{F}(\mu)\mathcal{F}(\varphi),$$

hence $\mathcal{F}\circ S^t_\mu \circ \mathcal{F}^{-1}$ (resp. $\mathcal{F}\circ T^t_\mu\circ\mathcal{F}^{-1}$) is the operator of multiplication by $\mathcal{F}(\mu)$.

Note that by Bierstedt, Meise and Summers [2], 1.6, the inductive limits $\operatorname{ind}_{j\to} A(w_j,\mathbb{C}^n)$ and $\operatorname{ind}_{j\to} A(v_j,\mathbb{C}^n)$ can be represented as intersections of weighted Banach spaces. To indicate that this can be done also in the more complicated case $\operatorname{ind}_{j\to}\operatorname{proj}_{\leftarrow k} A(v_{j,k},\mathbb{C}^n)$, let

$$\overline{V} := \{v : \mathbb{C}^n \to [0,\infty[ : v \text{ is upper semi-continuous and for each } j\in\mathbb{N}$$
$$\text{there are } \alpha_j > 0 \text{ and } k = k(j)\in\mathbb{N} \text{ with } v \le \alpha_j v_{j,k}\}$$

and let

$$A\overline{V}(\mathbb{C}^n) := \{f \in A(\mathbb{C}^n) : \|f\|_v < \infty \text{ for each } v\in\overline{V}\},$$

endowed with the locally convex topology of the system $(\|\ \|_v)_{v\in\overline{V}}$ of seminorms. Then one can use 1.2(b) and 1.4(d) to modify the proof of Hörmander [16], 15.4.2 (see also Berenstein and Dostal [1], II, §1, and Hansen [13], 4.6), to show that

$$\operatorname*{ind}_{j\to}\operatorname*{proj}_{\leftarrow k} A(v_{j,k},\mathbb{C}^n) = A\overline{V}(\mathbb{C}^n)$$

as locally convex spaces.



**1.9.** *Ultradifferential operators.* Let $\omega$ be a weight function. If $G \in A(\mathbb{C}^n)$ satisfies $\log|G| = O(\omega)$ (resp. $o(\omega)$) then

$$T_G : \varphi \mapsto \sum_{\alpha\in\mathbb{N}_0^n} (-i)^{|\alpha|}\,\frac{G^{(\alpha)}(0)}{\alpha!}\varphi^{(\alpha)}(0)$$

defines an element $T_G$ of $\mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ (resp. $\mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$). The operator

$$G(D) : \mathcal{D}'_*(\mathbb{R}^n) \to \mathcal{D}'_*(\mathbb{R}^n), \quad G(D)\mu := T_G * \mu,$$

is then called an *ultradifferential operator* of class $*$. From 1.4(b) and (c) it follows that $G(D) : \mathcal{D}_*(K) \to \mathcal{D}_*(K)$ is a continuous linear map for each $K\subset\mathbb{R}^n$ compact. Note that $\operatorname{Supp} G(D)T \subset \operatorname{Supp} T$ for each $T\in\mathcal{D}'_*(\mathbb{R}^n)$.

For later application we note the following extension of Komatsu [17], 10.2: For each $K\subset\mathbb{R}^n$ compact and each $j\in\mathbb{N}$ there exists $G\in A(\mathbb{C}^n)$ with $\log|G| = O(\omega)$ such that

$$(*) \qquad \|\varphi\|_{K,j} \le \sup_{\xi\in\mathbb{R}^n} |G(\xi)\widehat{\varphi}(\xi)|, \quad \varphi\in\mathcal{D}_{(\omega)}(K).$$

To prove this, fix $\lambda > 0$ and use Braun [5], Lemma 6 and the proof of Lemma 7 (for an alternative proof see Langenbruch [18], 1.3 and 1.4), to find $G\in A(\mathbb{C}^n)$ satisfying $\log|G| = O(\omega)$ such that $\log|G(\xi)| \ge (\lambda+1)\omega(\xi)$ for all $\xi\in\mathbb{R}^n$. Then for each $\varphi\in\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ we have

$$\int|\widehat{\varphi}(\xi)|e^{\lambda\omega(\xi)}\,d\xi \le \int|\widehat{\varphi}(\xi)G(\xi)|e^{-\omega(\xi)}\,d\xi \le \left(\int e^{-\omega(\xi)}\,d\xi\right)\sup_{\xi\in\mathbb{R}^n}|G(\xi)\widehat{\varphi}(\xi)|.$$

Since $\omega$ satisfies 1.1($\gamma$) this implies $(*)$ in view of 1.4(c).

## 2. The Beurling case.

In this section we characterize those ultradistributions $\mu\in\mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ for which the convolution operator $T_\mu : \mathcal{E}_{(\omega)}(\Omega_2) \to \mathcal{E}_{(\omega)}(\Omega_1)$ is surjective where $\Omega_1$ and $\Omega_2$ are open subsets of $\mathbb{R}^n$ satisfying $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$. In doing this we extend some of the results of Hörmander [14]. Throughout this section $\omega$ will always denote a fixed weight function.

To formulate our first result we need the following definition.

**2.1.** Definition. For $\mu\in\mathcal{E}'_*(\mathbb{R}^n)$ and open sets $\Omega_1,\Omega_2$ in $\mathbb{R}^n$ satisfying $\Omega_1+\operatorname{Supp}\mu\subset\Omega_2$, the pair $(\Omega_1,\Omega_2)$ is called *$\mu$-convex for* $*$ if the following holds: For each compact set $K_2$ in $\Omega_2$ there exists a compact set $K_1$ in $\Omega_1$ such that the map $S^t_\mu : \mathcal{D}_*(\Omega_1)\to\mathcal{D}_*(\Omega_2)$ satisfies $(S^t_\mu)^{-1}(\mathcal{D}_*(K_2)) \subset \mathcal{D}_*(K_1)$.

Remark. (a) A standard smoothing argument shows that $(\Omega_1,\Omega_2)$ is $\mu$-convex for $*$ if and only if the following holds: For each compact set $K_2$ in $\Omega_2$ there exists a compact set $K_1$ in $\Omega_1$ such that each $\nu\in\mathcal{E}'_*(\Omega_1)$ which satisfies $\operatorname{Supp} T^t_\mu\nu\subset K_2$ already satisfies $\operatorname{Supp}\nu\subset K_1$.

(b) If $\mu \in \mathcal{E}'_*(\mathbb{R}^n)$, $\mu \neq 0$, is given and $\Omega_2$ is a convex open set in $\mathbb{R}^n$ which contains $\operatorname{Supp}\mu$ then the largest open set $\Omega_1$ satisfying $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$ is convex, satisfies even $\Omega_1 + \operatorname{conv}(\operatorname{Supp}\mu) \subset \Omega_2$ and the pair $(\Omega_1, \Omega_2)$ is $\mu$-convex for $*$. This follows by a standard smoothing argument from the theorem of supports (see Hörmander [16], Thm. 4.3.3).

**2.2.** Proposition. *Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ and open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ with $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$ be given. Then the following assertions are equivalent:*

(1) $\mathcal{E}_{(\omega)}(\Omega_1) \subset S_\mu \mathcal{D}'_{(\omega)}(\Omega_2)$.

(2) $(S^t_\mu)^{-1} : S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1)) \to \mathcal{D}_{(\omega)}(\Omega_1)$ *is sequentially continuous.*

(3) $(\Omega_1, \Omega_2)$ *is $\mu$-convex for $(\omega)$ and the following condition is satisfied: For each compact set $K_1$ in $\Omega_1$ and each $j \in \mathbb{N}$ there exist $l \in \mathbb{N}$ and $C > 0$ such that $\|\varphi\|_{K_1,j} \le C\|S^t_\mu \varphi\|_l$ for each $\varphi \in \mathcal{D}_{(\omega)}(K_1)$.*

(4) $T_\mu : \mathcal{E}_{(\omega)}(\Omega_2) \to \mathcal{E}_{(\omega)}(\Omega_1)$ *is surjective.*

Proof. (1)$\Rightarrow$(2). To prove this we claim that (1) implies

(5) For each $K_2$ compact in $\Omega_2$ there exists $K_1$ compact in $\Omega_1$ and $m, l \in \mathbb{N}_0$ and $C > 0$ such that for each $f \in \mathcal{E}_{(\omega)}(\Omega_1)$ and $\varphi \in \mathcal{D}_{(\omega)}(\Omega_1)$ satisfying $S^t_\mu \varphi \in \mathcal{D}_{(\omega)}(K_2)$ we have
$$\Big|\int f\varphi\, d\lambda\Big| \le C\|f\|_{K_1,m}\|S^t_\mu\varphi\|_{K_2,l}.$$

To derive (5) from (1) fix $K_2$ as above, let
$$H := \{v \in \mathcal{D}_{(\omega)}(K_2) : v = S^t_\mu \psi \text{ for some } \psi \in \mathcal{D}_{(\omega)}(\Omega_1)\}$$
and define the bilinear form
$$B : \mathcal{E}_{(\omega)}(\Omega_1) \times H \to \mathbb{C}, \qquad B(f, v) := \int f(S^t_\mu)^{-1} v\, d\lambda.$$
Obviously, $B(\cdot, v)$ is continuous on $\mathcal{E}_{(\omega)}(\Omega_1)$ for each $v \in H$. If $f \in \mathcal{E}_{(\omega)}(\Omega_1)$ is fixed, then the hypothesis implies the existence of some $u \in \mathcal{D}'_{(\omega)}(\Omega_2)$ satisfying $f = S_\mu u$. Hence, for each $v \in H$,
$$B(f, v) = \int f(S^t_\mu)^{-1} v\, d\lambda = S_\mu(u)[(S^t_\mu)^{-1} v] = u[S_\mu(S^t_\mu)^{-1} v] = u(v).$$
Thus, $B(f, \cdot)$ is continuous on $H$. Since $B$ is a separately continuous bilinear form on the product of a Fréchet space with a metrizable locally convex space, $B$ is continuous, which implies (5).

To derive (2) from (5), let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}_{(\omega)}(\Omega_1)$ for which $(S^t_\mu \varphi_k)_{k\in\mathbb{N}}$ is a null-sequence in $\mathcal{D}_{(\omega)}(\Omega_2)$. Since $\mathcal{D}_{(\omega)}(\Omega_2)$ is a strict (LF)-space, there exists a compact $K_2$ in $\Omega_2$ so that $(S^t_\mu\varphi_k)_{k\in\mathbb{N}}$ is a null-sequence in $\mathcal{D}_{(\omega)}(K_2)$. Applying (5) for this $K_2$, we get a compact set $K_1$ in $\Omega_1$ such that $\varphi_k \in \mathcal{D}_{(\omega)}(K_1)$ for each $k$. To show that $\varphi_k$ tends to zero weakly in $\mathcal{D}_{(\omega)}(K_1)$ fix $\nu \in \mathcal{E}'_{(\omega)}(\Omega_1)$ and $m \in \mathbb{N}_0$ according to (5). By Braun [5], Thm. 8, there exist an ultradifferential operator $G(D)$ of class $(\omega)$ and $g \in \mathcal{E}_\omega(\mathbb{R}^n, m)$ so that $\nu = G(D)g$. Next note that by an easy regularization argument condition (5) holds even for all $f \in \operatorname{proj}_{K\Subset\Omega_1} \mathcal{E}_\omega(K, m)$. This implies
$$|\nu(\varphi_k)| = |(G(D)g)(\varphi_k)| = \Big|\int g(G(-D)\varphi_k)\, d\lambda\Big|$$
$$\le C\|g\|_{K_1,m}\|S^t_\mu(G(-D)\varphi_k)\|_{K_2,l} \le C'\|g\|_{K_1,m}\|S^t_\mu\varphi_k\|_{K_2,\tilde{l}}$$
since $G(-D)$ is a continuous linear operator on $\mathcal{D}_{(\omega)}(K_2)$. Hence $(\nu(\varphi_k))_k$ is a null-sequence. Now the fact that $\mathcal{D}_{(\omega)}(K_1)$ is a Fréchet–Montel space implies that the weak null-sequence $(\varphi_k)_{k\in\mathbb{N}}$ is indeed a null-sequence. Hence $(S^t_\mu)^{-1}$ is sequentially continuous.

(2)$\Rightarrow$(3). If we assume that $(\Omega_1, \Omega_2)$ is not $\mu$-convex for $(\omega)$ then there exist a compact set $K_2$ in $\Omega_2$ and a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathcal{D}_{(\omega)}(\Omega_1)$ so that $\bigcup_{k\in\mathbb{N}} \operatorname{Supp}\varphi_k$ is not relatively compact in $\Omega_1$, while $S^t_\mu\varphi_k \in \mathcal{D}_{(\omega)}(K_2)$ for all $k \in \mathbb{N}$. Since $\mathcal{D}_{(\omega)}(K_2)$ is a Fréchet space we can find a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $]0,1]$ so that $(S^t_\mu(\lambda_k\varphi_k))_{k\in\mathbb{N}}$ is a null-sequence in $\mathcal{D}_{(\omega)}(K_2)$. Since $(S^t_\mu)^{-1}$ is sequentially continuous by hypothesis, this implies that $(\lambda_k\varphi_k)_{k\in\mathbb{N}}$ is a null-sequence in $\mathcal{D}_{(\omega)}(\Omega_1)$. Hence there exists a compact set $K_1$ in $\Omega_1$ so that $\operatorname{Supp}\varphi_k \subset K_1$ for all $k \in \mathbb{N}$, contradicting our choice of the sequence $(\varphi_k)_{k\in\mathbb{N}}$. Consequently, $(\Omega_1, \Omega_2)$ is $\mu$-convex for $(\omega)$.

To show that the second condition also holds, fix a compact set $K_1$ in $\Omega_1$. Then $K_2 := K_1 + \operatorname{Supp}\mu$ is compact in $\Omega_2$, by hypothesis. Hence the $\mu$-convexity of $(\Omega_1, \Omega_2)$ implies the existence of a compact set $Q \supset K_1$ so that
$$(S^t_\mu)^{-1}(\mathcal{D}_{(\omega)}(K_2) \cap S^t_\mu(\mathcal{D}_\omega(\Omega_1))) \subset \mathcal{D}_{(\omega)}(Q).$$
Therefore, the restriction of $(S^t_\mu)^{-1}$ to $\mathcal{D}_{(\omega)}(K_2) \cap S^t_\mu(\mathcal{D}_\omega(\Omega_1))$ maps this space into $\mathcal{D}_{(\omega)}(Q) \subset \mathcal{D}_{(\omega)}(\Omega_1)$. By (2) this map is sequentially continuous for the topologies induced by $\mathcal{D}_{(\omega)}(\Omega_2)$ resp. $\mathcal{D}_{(\omega)}(\Omega_1)$ and therefore continuous. Obviously, this implies (3).

(3)$\Rightarrow$(4). By the surjectivity criterion in Meise and Vogt [22], 26.1, condition (4) follows from

(6) If $T^t_\mu(B)$ is bounded in $\mathcal{E}'_{(\omega)}(\Omega_2)$ for some $B \subset \mathcal{E}'_{(\omega)}(\Omega_1)$ then $B$ is bounded in $\mathcal{E}'_{(\omega)}(\Omega_1)$.

To prove that (3) implies (6), fix any set $B$ in $\mathcal{E}'_{(\omega)}(\Omega_1)$ for which $T^t_\mu(B)$ is bounded. Since $\mathcal{E}_{(\omega)}(\Omega_2)$ is a Fréchet space, there exist a compact set $K_2$ in $\Omega_2$, $m \in \mathbb{N}$ and $C > 0$ such that

(7) $\quad |T^t_\mu\nu(f)| \le C\|f\|_{K_2,m}$ for all $f \in \mathcal{E}_{(\omega)}(\Omega_2)$ and $\nu \in B$.

Obviously (7) implies $\operatorname{Supp}(T^t_\mu\nu) \subset K_2$ for each $\nu \in B$. By the remark after 2.1, the $\mu$-convexity of $(\Omega_1, \Omega_2)$ implies the existence of a compact set $K_1$ in $\Omega_1$ so that $\operatorname{Supp}\nu \subset K_1$ for all $\nu \in B$. Note that $B$ is bounded in

$\mathcal{E}'_{(\omega)}(\Omega_1)$ if for each sequence $(\sigma_j)_{j\in\mathbb{N}}$ in $B$ and each null-sequence $(\alpha_j)_{j\in\mathbb{N}}$ the sequence $(\nu_j)_{j\in\mathbb{N}}$, $\nu_j := \alpha_j\sigma_j$, is bounded in $\mathcal{E}'_{(\omega)}(\Omega_1)$. To prove this fix $(\sigma_j)_{j\in\mathbb{N}}$ and $(\alpha_j)_{j\in\mathbb{N}}$. Then (7) implies that $(T^t_\mu\nu_j)_{j\in\mathbb{N}}$ is a null-sequence in $\mathcal{E}'_{(\omega)}(\Omega_2)$. Next choose $\varepsilon > 0$ so that $K_1 + \overline{B_\varepsilon(0)} \subset \Omega_1$ and $K_2 + \overline{B_\varepsilon(0)} \subset \Omega_2$ and note that for each $\chi \in \mathcal{D}_{(\omega)}(B_\varepsilon(0))$ we have

$$(8)\qquad T^t_\mu(\nu_j) * \chi = (\mu * \nu_j) * \chi = \mu * (\nu_j * \chi) = S^t_\mu(\nu_j * \chi).$$

Since $(T^t_\mu(\nu_j))_{j\in\mathbb{N}}$ converges to zero in $\mathcal{E}'_{(\omega)}(\mathbb{R}^n)$, the left hand side in (8) converges to zero in $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ and hence in $\mathcal{D}_{(\omega)}(K_2 + \overline{B_\varepsilon(0)})$. Using (8), it follows from (3) that $(\nu_j * \chi)_{j\in\mathbb{N}}$ converges to zero in $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$. Next fix $f \in \mathcal{E}_{(\omega)}(\Omega_1)$. Then there exists $\varphi \in \mathcal{D}_{(\omega)}(\Omega_1)$ so that $\nu_j(f) = \nu_j(\varphi)$ for all $j \in \mathbb{N}$. Since $\operatorname{Supp}\varphi$ is compact there exist $x_1, \ldots, x_p \in \Omega_1$ and $\varphi_k \in \mathcal{D}_{(\omega)}(B_\varepsilon(x_k))$, $1 \le k \le p$, so that $\varphi = \sum_{k=1}^p \varphi_k$. Then let $\chi_k := \varphi_k(x_k - \cdot)$. Since $\operatorname{Supp}\chi_k \subset B_\varepsilon(0)$ we get from the above

$$\nu_j(f) = \nu_j(\varphi) = \nu_j\Big(\sum_{k=1}^p \varphi_k\Big) = \sum_{k=1}^p \nu_j * \chi_k(x_k) \to 0.$$

Hence $(\nu_j)_{j\in\mathbb{N}}$ converges to zero pointwise. Since $\mathcal{E}_{(\omega)}(\Omega_1)$ is barrelled, $(\nu_j)_{j\in\mathbb{N}}$ is bounded in $\mathcal{E}'_{(\omega)}(\Omega_1)$. Hence (6) holds.

(4)⇒(1). This holds trivially.

To derive further conditions that are equivalent to 2.2(2), we will use the following definition which goes back to Ehrenpreis [10]. The present formulation is due to Momm [24].

**2.3.** Definition. An ultradistribution $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ is called *slowly decreasing for* $(\omega)$ if there exists $C > 0$ such that for each $x \in \mathbb{R}^n$ with $|x| \ge C$ there is $\xi \in \mathbb{C}^n$ with

$$|x - \xi| \le C\omega(x) \quad\text{and}\quad |\widehat{\mu}(\xi)| \ge \exp(-C|\operatorname{Im}\xi| - C\omega(\xi)).$$

From Bonet, Galbis and Momm [4] we recall:

**2.4.** Lemma. *The ultradistribution $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ is slowly decreasing for $(\omega)$ if and only if there exists $k \in \mathbb{N}$ such that for each $j \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$, $R > 0$ such that for each $z \in \mathbb{C}^n$, $|z| \ge R$, there exists $w \in \mathbb{C}^n$ satisfying*

$$|w - z| \le k\omega(z) + \frac{1}{j}|\operatorname{Im} z| \quad\textit{and}\quad |\widehat{\mu}(w)| \ge C\exp(-m(|\operatorname{Im} z| + \omega(z))).$$

Similarly to Ehrenpreis [10], Thm. 2.2, we prove:



**2.5.** Proposition. *If $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ is slowly decreasing for $(\omega)$ then $S_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^n) \to \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ is surjective. In particular, there exists $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ satisfying $S_\mu(E) = \delta$.*

Proof. It suffices to show that

$$(1)\qquad (S^t_\mu)^{-1} : S^t_\mu(\mathcal{D}_{(\omega)}(\mathbb{R}^n)) \to \mathcal{D}_{(\omega)}(\mathbb{R}^n) \quad\text{is continuous}$$

if $S^t_\mu(\mathcal{D}_{(\omega)}(\mathbb{R}^n))$ carries the topology induced by $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$. Namely, (1) implies that for each $\lambda \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ the linear form $\widetilde{\nu} := \lambda \circ (S^t_\mu)^{-1}$ is continuous on $S^t_\mu(\mathcal{D}_{(\omega)}(\mathbb{R}^n))$, hence admits an extension $\nu \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$, by the Hahn–Banach theorem, and $\nu$ satisfies

$$(S_\mu(\nu))(\varphi) = \nu(S^t_\mu\varphi) = \widetilde{\nu}(S^t_\mu\varphi) = \lambda(\varphi), \qquad \varphi \in \mathcal{D}_{(\omega)}(\mathbb{R}^n).$$

Since the Fourier–Laplace transform is an isomorphism between $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ and the space $\operatorname{ind}_{j\to}\operatorname{proj}_{\leftarrow k} A(v_{j,k}, \mathbb{C}^n) = A\overline{V}(\mathbb{C}^n)$, introduced in 1.8, and since $(S_\mu(\varphi))^\wedge = \widehat{\mu}\widehat{\varphi}$, (1) follows from

(2) for each $v \in \overline{V}$ there exists $w \in \overline{V}$ and $\varepsilon > 0$ such that $f \in A\overline{V}(\mathbb{C}^n)$ and $\|\widehat{\mu}f\|_w \le \varepsilon$ imply $\|f\|_v \le 1$.

To prove (2) we note that by Lemma 2.4 there exist $k \in \mathbb{N}$, $m \in \mathbb{N}$, $C > 0$ and $R > 0$ such that for each $z \in \mathbb{C}^n$ with $|z| \ge R$ and $r(z) := k\omega(z) + \frac{1}{8}|\operatorname{Im} z|$ we have

$$(3)\qquad \sup_{|\zeta - z| \le r(z)} |\widehat{\mu}(\zeta)| \ge Cw_m(z).$$

Since $\mu$ is in $\mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ we get from 1.8 and 1.1($\alpha$) the existence of $l \in \mathbb{N}$ and $A > 0$ such that

$$(4)\qquad \sup_{|\zeta - z| \le 4r(z)} |\widehat{\mu}(\zeta)| \le A/w_l(z).$$

Now let $p := l + 2m$, fix $v \in \overline{V}$ and note that without restriction $v(z) > 0$ for each $z \in \mathbb{C}^n$. Then define

$$\widetilde{w}(z) := \sup_{|\zeta - z| \le 4r(\zeta)} v(\zeta)/w_p(\zeta), \quad z \in \mathbb{C}^n.$$

To show that $\widetilde{w}$ belongs to $\overline{V}$, fix $j \in \mathbb{N}$ and let $q := p + 2j$. Since $v$ is in $\overline{V}$ there exist $\alpha_q > 0$ and $k(q) \in \mathbb{N}$ such that $v \le \alpha_q v_{q,k(q)}$. To apply this we need some preparation. First note that by 1.2(a) we can find $L \ge 1$ such that $\omega(t) \le t + L$ for all $t \ge 0$. Then for $\zeta, z \in \mathbb{C}^n$ satisfying $|\zeta - z| \le 4r(\zeta)$ we have

$$|\operatorname{Im}\zeta| \ge |\operatorname{Im} z| - |z - \zeta| \ge |\operatorname{Im} z| - 4k\omega(\zeta) - \tfrac{1}{2}|\operatorname{Im}\zeta|$$

and hence

$$|\mathrm{Im}\,\zeta| \ge \tfrac{2}{3}|\mathrm{Im}\,z| - \tfrac{8}{3}k\omega(\zeta). \tag{5}$$

Also we have

$$|\zeta| \ge |z| - |z-\zeta| \ge |z| - 4k\omega(\zeta) - \tfrac{1}{2}|\mathrm{Im}\,\zeta| \ge |z| - 4k\omega(\zeta) - \tfrac{1}{2}|\zeta|.$$

This implies

$$|z| \le \tfrac{3}{2}|\zeta| + 4k\omega(\zeta) \le \left(\tfrac{3}{2}+4k\right)|\zeta| + 4kL. \tag{6}$$

By 1.2(a) we can find $R_0 > 0$ such that $\omega(t) \le \frac{1}{16k}t$ for $t \ge R_0$. Because of (6) we can choose $R_1 \ge R_0$ such that $|z| \ge R_1$ and $|\zeta - z| \le 4r(\zeta)$ imply $|\zeta| \ge R_0$. Therefore we have, for such $z$ and $\zeta$,

$$|\zeta| \le |\zeta - z| + |z| \le 4k\omega(\zeta) + \tfrac{1}{2}|\zeta| + |z| \le \tfrac{1}{4}|\zeta| + \tfrac{1}{2}|\zeta| + |z| = \tfrac{3}{4}|\zeta| + |z|$$

and hence $|\zeta| \le 4|z|$. Now the choice of $q$ and (5) imply for $|z| \ge R_1$ and $|\zeta - z| \le 4r(\zeta)$,

$$\begin{aligned}\frac{v(\zeta)}{w_p(\zeta)} &\le \alpha_q \frac{v_{q,k(q)}(\zeta)}{w_p(\zeta)} = \alpha_q \exp((-q+p)|\mathrm{Im}\,\zeta| + (k(q)+p)\omega(\zeta))\\ &\le \alpha_q \exp\left(-2j\left(\tfrac{2}{3}|\mathrm{Im}\,z| - \tfrac{8}{3}k\omega(\zeta)\right) + (k(q)+p)\omega(\zeta)\right)\\ &\le \alpha_q \exp\left(-j|\mathrm{Im}\,z| + \left(\tfrac{16}{3}kj + k(q) + p\right)\omega(\zeta)\right).\end{aligned}$$

Since $|\zeta| \le 4|z|$ and since 1.1($\alpha$) implies the existence of $S \in \mathbb{N}$ satisfying $\omega(4t) \le S\omega(t)$ for $t \ge R_1$ (assuming that $R_1$ is sufficiently large), we get from this

$$\frac{v(\zeta)}{w_p(\zeta)} \le \alpha_q \exp(-j|\mathrm{Im}\,z| + \nu(j)\omega(z)), \qquad |z| \ge R_1,\ |\zeta - z| \le 4r(\zeta),$$

if we let $\nu(j) := S\left(\frac{16}{3}kj + k(q) + p\right)$. Since $v_{j,\nu(j)}$ is continuous and since $\widetilde{w}$ is bounded on $\{z \in \mathbb{C}^n : |z| \le R_1\}$, we can find $\beta_j \ge \alpha_q$ such that $\widetilde{w} \le \beta_j v_{j,\nu(j)}$. Since $j \in \mathbb{N}$ was chosen arbitrarily, this proves $\widetilde{w} \in \overline{V}$. Similarly to the proof of Bierstedt, Meise and Summers [2], Prop. 0.2, we can find a continuous function $w \in \overline{V}$ which satisfies $w \ge \widetilde{w}$ and $w(z) > 0$ for each $z \in \mathbb{C}^n$.

Next let

$$M := \left(\sup_{|z|\le R_1} v(z)\right)\left(\sup_{|z|\le R_1} \frac{1}{v(z)}\right) + 1$$

and choose $0 < \varepsilon < C^2/(AM)$, where $C$ (resp. $A$) is the constant from (3) (resp. (4)). Then fix $f \in A\overline{V}(\mathbb{C}^n)$ satisfying $\|\widehat{\mu} f\|_w \le \varepsilon$. To show that $\|f\|_v \le 1$, fix $z \in \mathbb{C}^n$ with $|z| \ge R$, apply Hörmander [14], Lemma 3.2, to $f = \widehat{\mu} f/\widehat{\mu}$, and use (3) and (4) together with the choice of $p$ to get for $z \in \mathbb{C}^n$ with $|z| \ge R_1$,



$$\begin{aligned} v(z)|f(z)| &\le v(z)\Big(\sup_{|\zeta - z|\le 4r(z)} |\widehat{\mu}(\zeta)|\Big)\Big(\sup_{|\zeta - z|\le 4r(z)} |\widehat{\mu}(\zeta) f(\zeta)|\Big)\Big(\sup_{|\zeta - z|\le r(z)} |\widehat{\mu}(\zeta)|\Big)^{-2}\\ &\le v(z)\frac{A}{w_l(z)}\left(\frac{1}{Cw_m(z)}\right)^2 \sup_{|\zeta - z|\le 4r(z)} w(\zeta)|\widehat{\mu}(\zeta) f(\zeta)|/w(\zeta)\\ &\le \frac{Av(z)}{C^2 w_p(z)} \sup_{|\zeta - z|\le 4r(z)} \frac{\varepsilon}{w(\zeta)}.\end{aligned} \tag{7}$$

By the continuity of $w$, the supremum in the last estimate is attained at some $\zeta_0 \in \mathbb{C}^n$ satisfying $|\zeta_0 - z| \le 4r(z)$. Because of the definition of $\widetilde{w}$ and the choice of $w$ this implies

$$w(\zeta_0) \ge \widetilde{w}(\zeta_0) = \sup_{|\zeta - \zeta_0|\le 4r(\zeta)} \frac{v(\zeta)}{w_p(\zeta)} \ge \frac{v(z)}{w_p(z)}.$$

Hence from (7) we get

$$v(z)|f(z)| \le AC^{-2}\varepsilon \frac{v(z)}{w_p(z)} \cdot \frac{1}{w(\zeta_0)} \le AC^{-2}\varepsilon < \frac{1}{M} < 1, \qquad |z| \le R_1. \tag{8}$$

By the maximum principle we conclude from this

$$\sup_{|z|\le R_1} v(z)|f(z)| \le \sup_{|z|\le R_1} v(z) \cdot \frac{1}{\sup_{|z|\le R_1} v(z)} \cdot \frac{1}{M} < 1. \tag{9}$$

Obviously, (8) and (9) imply $\|f\|_v \le 1$, which completes the proof.

**2.6.** Proposition. *Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ and open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ with $\Omega_1 + \mathrm{Supp}\,\mu \subset \Omega_2$ be given. Then the following conditions are equivalent:*

(1) *$T_\mu : \mathcal{E}_{(\omega)}(\Omega_2) \to \mathcal{E}_{(\omega)}(\Omega_1)$ is surjective.*
(2) *$T_\mu^t(\mathcal{E}'_{(\omega)}(\Omega_1))$ is closed in $\mathcal{E}'_{(\omega)}(\Omega_2)$.*
(3) *$(\Omega_1, \Omega_2)$ is $\mu$-convex for $(\omega)$ and $\mu$ is slowly decreasing for $(\omega)$.*
(4) *$(\Omega_1, \Omega_2)$ is $\mu$-convex for $(\omega)$ and there exists $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ with $S_\mu(E) = \delta$.*
(5) *$S_\mu^t(\mathcal{D}_{(\omega)}(\Omega_1))$ is closed in $\mathcal{D}_{(\omega)}(\Omega_2)$.*
(6) *$S_\mu^t(\mathcal{D}_{(\omega)}(\Omega_1))$ is sequentially closed in $\mathcal{D}_{(\omega)}(\Omega_2)$.*
(7) *$S_\mu^t(\mathcal{D}_{(\omega)}(\Omega_1)) \cap G$ is sequentially closed for each Fréchet subspace $G$ of $\mathcal{D}_{(\omega)}(\Omega_2)$.*
(8) *$(\Omega_2, \Omega_2)$ is $\mu$-convex for $(\omega)$ and the following condition holds: For each $K_1$ compact in $\Omega_1$ there exist $m \in \mathbb{N}$ and $C > 0$ such that $\sup_{x\in\mathbb{R}^n} |\varphi(x)| \le C\|S_\mu^t \varphi\|_m$ for all $\varphi \in \mathcal{D}_{(\omega)}(K_1)$.*

Proof. (1)⇒(2). This is well known; see e.g. Meise and Vogt [22], 26.3.

(2)⇒(3). Since convolutions commute with translations, we may assume $0 \in \Omega_1$. Then we choose $\delta > 0$ so that $\overline{B_\delta(0)} \subset \Omega_1$. If we assume that $\mu$ is not slowly decreasing for $\omega$ then also $\nu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ defined by $\widehat{\nu}(z) := \widehat{\mu}(z/\delta)$ is not slowly decreasing for $\omega$. Therefore, it follows from the proof of Bonet and Galbis [3], Thm. 11, that there exists a sequence $(f_j)_{j\in\mathbb{N}}$ in $\operatorname{ind}_{k\to} A(w_k, \mathbb{C}^n)$, where $w_k(z) := \exp(-k(|\operatorname{Im} z| + \omega(z)))$, for which $(\widehat{\nu} f_j)_{j\in\mathbb{N}}$ is bounded in $\operatorname{ind}_{k\to} A(w_k, \mathbb{C}^n)$ while $(f_j)_{j\in\mathbb{N}}$ is not bounded in this space and satisfies

$$|f_j(z)| \le C \exp(|\operatorname{Im} z| + C\omega(z)) \quad \text{for all } z \in \mathbb{C}^n \text{ with } |z| \ge r_j,$$

for some sequence $(r_j)_{j\in\mathbb{N}}$ in $]0,\infty[$. Hence it follows from Braun, Meise and Taylor [6], 7.4, that there is a sequence $(\nu_j)_{j\in\mathbb{N}}$ in $\mathcal{E}'_{(\omega)}(\Omega_1)$ satisfying $\widehat{\nu}_j(z) = f_j(\delta z)$. Since $\widehat{\mu * \nu_j}(z) = \widehat{\mu}(z)\widehat{\nu}_j(z) = \widehat{\nu}(\delta z) f_j(\delta z)$, the sequence $(T^t_\mu(\nu_j))_{j\in\mathbb{N}}$ is bounded in $\mathcal{E}'_{(\omega)}(\Omega_2)$, while $(\nu_j)_{j\in\mathbb{N}}$ is unbounded in $\mathcal{E}'_{(\omega)}(\Omega_1)$. However, this contradicts (2), since the injectivity of $T^t_\mu$ in connection with (2) implies by Meise and Vogt [22], 26.3, that $(T^t_\mu)^{-1} : T^t_\mu(\mathcal{E}'_{(\omega)}(\Omega_1)) \to \mathcal{E}'_{(\omega)}(\Omega_1)$ is continuous.

To show that the continuity of $(T^t_\mu)^{-1}$ also implies the $\mu$-convexity of $(\Omega_1, \Omega_2)$, let $K_2$ be any compact subset of $\Omega_2$. Then the set

$$B := \{\varphi \in \mathcal{D}_{(\omega)}(K_2) \cap S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1)) : \sup_{x\in K_2} |\varphi(x)| \le 1\}$$

is bounded in $\mathcal{E}'_{(\omega)}(\Omega_2)$ and $\operatorname{span}(B) = S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1))$. Hence $(T^t_\mu)^{-1}(B)$ is bounded in $\mathcal{E}'_{(\omega)}(\Omega_1)$. This implies the existence of a compact set $K_1$ in $\Omega_1$ so that $\operatorname{Supp}\psi \subset K_1$ for each $\psi \in (T^t_\mu)^{-1}(B)$ and hence the $\mu$-convexity of $(\Omega_1, \Omega_2)$.

(3)⇒(4). This holds by Proposition 2.5.

(4)⇒(6). Let $(\varphi_j)_{j\in\mathbb{N}}$ be any sequence in $\mathcal{D}_{(\omega)}(\Omega_1)$ for which $(S^t_\mu\varphi_j)_{j\in\mathbb{N}}$ converges to some $\psi$ in $\mathcal{D}_{(\omega)}(\Omega_2)$. Then there exists $K_2$ compact in $\Omega_2$ so that $(S^t_\mu\varphi_j)_{j\in\mathbb{N}}$ converges in $\mathcal{D}_{(\omega)}(K_2)$. Since $(\Omega_1, \Omega_2)$ is $\mu$-convex by hypothesis, there exists $K_1$ compact in $\Omega_1$ so that $\varphi_n \in \mathcal{D}_{(\omega)}(K_1)$ for all $n \in \mathbb{N}$. By hypothesis there exists a fundamental solution $E$ in $\mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ for $\mu$. Hence we have

$$E * (S^t_\mu\varphi_j) = (E * \mu) * \varphi_j = \delta * \varphi_j = \varphi_j \quad \text{for each } j \in \mathbb{N}.$$

Since convolution with $E$ maps $\mathcal{D}_{(\omega)}(\mathbb{R}^n)$ continuously into $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ and since $(\varphi_j)_{j\in\mathbb{N}}$ is in $\mathcal{D}_{(\omega)}(K_1)$, the sequence $(\varphi_j)_{j\in\mathbb{N}}$ converges to some $\varphi \in \mathcal{D}_{(\omega)}(K_1)$ which satisfies $\psi = S^t_\mu(\varphi) \in S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1))$.

(6)⇒(7). This holds trivially.

(7)⇒(8). To show that $(\Omega_1, \Omega_2)$ is $\mu$-convex for $(\omega)$, fix a compact set $K_2$ in $\Omega_2$. Then choose a sequence $(Q_j)_{j\in\mathbb{N}}$ of compact sets in $\Omega_1$ satisfying $Q_j \subset Q_{j+1}$ for all $j \in \mathbb{N}$ and $\Omega_1 = \bigcup_{j\in\mathbb{N}} \mathring{Q}_j$. By (7), $F := S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1)) \cap \mathcal{D}_{(\omega)}(K_2)$ is a Fréchet space in the topology induced by $\mathcal{D}_{(\omega)}(K_2)$ and $F \subset \bigcup_{j=1}^\infty S^t_\mu(\mathcal{D}_{(\omega)}(Q_j))$. Hence Grothendieck's factorization theorem implies the existence of $k \in \mathbb{N}$ and of a continuous linear map $u : F \to \mathcal{D}_{(\omega)}(Q_k)$ so that $F \subset S^t_\mu(\mathcal{D}_{(\omega)}(Q_k))$ and that $S^t_\mu \circ u$ is the inclusion of $F$ into $\mathcal{D}_{(\omega)}(\Omega_2)$. Since $K_2$ was chosen arbitrarily, $F \subset S^t_\mu(\mathcal{D}_{(\omega)}(Q_k))$ implies the $\mu$-convexity of $(\Omega_1, \Omega_2)$, while the continuity of $u$ implies (8).



(8)⇒(1). By Proposition 2.2 it suffices to show that the second condition in (8) implies the second one in 2.2(3). To do this, fix a compact set $K_1$ in $\Omega_1$ and $j \in \mathbb{N}$. By 1.9 there exists $P \in A(\mathbb{C}^n)$ satisfying $\log|P| = O(\omega)$ so that for each $\varphi \in \mathcal{D}_{(\omega)}(K_1)$,

$$\|\varphi\|_{K_1,j} \le \sup_{\xi\in\mathbb{R}^n} |P(\xi)\widehat{\varphi}(\xi)| \le m_n(K_1) \sup_{x\in\mathbb{R}^n} |(P(D)\varphi)(x)|.$$

Then $L := K_1 + \operatorname{Supp}\mu$ is compact in $\Omega_2$ and for $\varphi \in \mathcal{D}_{(\omega)}(K_1)$ we have $S^t_\mu\varphi \in \mathcal{D}_{(\omega)}(L) \cap \mathcal{D}_\omega(\Omega_2)$. Since $P(D) : \mathcal{D}_{(\omega)}(L) \to \mathcal{D}_{(\omega)}(L)$ is linear and continuous, there exist $l \in \mathbb{N}$ and $C' > 0$ so that for $m$ as in (8),

$$\|P(D)\varphi\|_{L,m} \le C'\|\varphi\|_{L,l} \quad \text{for each } \varphi \in \mathcal{D}_{(\omega)}(L).$$

Hence from (8) applied to $P(D)\varphi$ we get

$$\sup_{x\in\mathbb{R}^n} |(P(D)\varphi)(x)| \le C\|S^t_\mu(P(D)\varphi)\|_m = C\|P(D)(S^t_\mu\varphi)\|_{L,m} \le CC'\|S^t_\mu\varphi\|_l$$

and consequently

$$\|\varphi\|_{K_{1,j}} \le m_n(K_1)CC'\|S^t_\mu\varphi\|_l.$$

(5)⇒(6). This holds trivially.

(2)⇒(5). Let $\varphi \in \mathcal{D}_{(\omega)}(\Omega_2)$ be in the closure of $S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1))$ in $\mathcal{D}_{(\omega)}(\Omega_2)$. Since $\mathcal{D}_{(\omega)}(\Omega_2) \hookrightarrow \mathcal{E}'_{(\omega)}(\Omega_2)$ is continuous, $\varphi$ is in the closure of $T^t_\mu(\mathcal{E}'_{(\omega)}(\Omega_1))$ in $\mathcal{E}'_{(\omega)}(\Omega_2)$. Hence (2) implies the existence of $\nu \in \mathcal{E}'_{(\omega)}(\Omega_1)$ satisfying $\mu * \nu = \varphi$. Since we have already shown that (2) and (4) are equivalent, $\mu$ admits a fundamental solution $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$. Hence

$$\nu = (E * \mu) * \nu = E * (\mu * \nu) = E * \varphi.$$

This shows that $\nu \in \mathcal{E}_{(\omega)}(\mathbb{R}^n) \cap \mathcal{E}'_{(\omega)}(\Omega_1) = \mathcal{D}_{(\omega)}(\Omega_1)$, hence $\varphi \in S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1))$.

By Propositions 2.2 and 2.6 we have proved the following theorem.

**2.7.** Theorem. *Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ and open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ with $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$ be given. Then the following assertions are equivalent:*

(1) *$T_\mu : \mathcal{E}_{(\omega)}(\Omega_2) \to \mathcal{E}_{(\omega)}(\Omega_1)$ is surjective.*

(2) *$(S^t_\mu)^{-1} : S^t_\mu(\mathcal{D}_{(\omega)}(\Omega_1)) \to \mathcal{D}_{(\omega)}(\Omega_1)$ is sequentially continuous.*

(3) *$\mathcal{E}_{(\omega)}(\Omega_1) \subset S_\mu(\mathcal{D}'_{(\omega)}(\Omega_2))$.*

(4) *$(\Omega_1, \Omega_2)$ is $\mu$-convex for $(\omega)$ and there exists $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ with $S_\mu(E) = \delta$.*

(5) *$(\Omega_1, \Omega_2)$ is $\mu$-convex for $(\omega)$ and for each $K_1$ compact in $\Omega_1$ there exist $m \in \mathbb{N}$ and $C > 0$ such that $\sup_{x\in\mathbb{R}^n} |\varphi(x)| \le C\|S^t_\mu\varphi\|_m$ for all $\varphi \in \mathcal{D}_{(\omega)}(K_1)$.*

The following corollary is an immediate consequence of Theorem 2.7. It extends and "explains" Bonet and Galbis [3], Thm. 11.

**2.8.** COROLLARY. *Let $\omega$ and $\sigma$ be weight functions satisfying $\omega(t) = O(\sigma(t))$ as $t$ tends to infinity. Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ and open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ with $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$ be given. Then the following conditions are equivalent:*

(1) $\mathcal{E}_{(\sigma)}(\Omega_1) \subset T_\mu(\mathcal{E}_{(\omega)}(\Omega_2))$.

(2) $T_\mu : \mathcal{E}_{(\sigma)}(\Omega_2) \to \mathcal{E}_{(\sigma)}(\Omega_1)$ *is surjective.*

Proof. (1)⇒(2). Since $T_\mu(\mathcal{E}_{(\omega)}(\Omega_2)) \subset S_\mu(\mathcal{D}'_{(\omega)}(\Omega_2))$, (2) follows from Theorem 2.7.

(2)⇒(1). This is an obvious consequence of $\mathcal{E}_{(\omega)}(\Omega_2) \supset \mathcal{E}_{(\sigma)}(\Omega_2)$.

**2.9.** COROLLARY. *For $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ the following conditions are equivalent:*

(1) $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^n) \to \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ *is surjective.*

(2) *There exists $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ satisfying $S_\mu(E) = \delta$.*

(3) *$\mu$ is slowly decreasing for $(\omega)$.*

(4) $S_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^n) \to \mathcal{D}'_{(\omega)}(\mathbb{R}^n)$ *is surjective.*

(5) $\mathcal{E}_{(\omega)}(\mathbb{R}^n) \subset S_\mu(\mathcal{D}'_{(\omega)}(\mathbb{R}^n))$.

Proof. (1) implies (2) by Theorem 2.7; (2) implies (3) by Proposition 2.6 and the $\mu$-convexity of $(\mathbb{R}^n, \mathbb{R}^n)$; (3) implies (4) by Proposition 2.5; (4) trivially implies (5) and (5) implies (1) by Theorem 2.7.

Remark. Note that by Braun, Meise and Taylor [6], 8.6, the equivalences (1)–(4) in Corollary 2.9 extend the main results of Ciorănescu [9] from $\mathbb{R}$ to $\mathbb{R}^n$. For $n = 1$ the equivalence of the conditions 2.9(2)–2.9(4) together with a sequence representation for $\ker S_\mu$ was derived in Franken and Meise [11].

## 3. The Roumieu case.

In this section we characterize those ultradistributions $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ for which the convolution operator $S_\mu : \mathcal{D}'_{\{\omega\}}(\Omega_2) \to \mathcal{D}'_{\{\omega\}}(\Omega_1)$ is surjective, where $\Omega_1$ and $\Omega_2$ are appropriate open sets in $\mathbb{R}^n$. Throughout this section $\omega$ denotes a fixed weight function.

First we treat the case $\Omega_1 = \Omega_2 = \mathbb{R}^n$ and in doing this we will use the following slowly decreasing condition, corresponding to 2.3.

**3.1.** DEFINITION. An ultradistribution $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ is called *slowly decreasing for $\{\omega\}$* if for each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^n$ with $|x| \geq R$ there exists $\xi \in \mathbb{C}^n$ satisfying $|x - \xi| \leq \frac{1}{m}\omega(x)$ such that $|\widehat{\mu}(\xi)| \geq \exp\left(-\frac{1}{m}\omega(\xi)\right)$.



**3.2.** LEMMA. *For $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ the following conditions are equivalent:*

(1) *$\mu$ is slowly decreasing for $\{\omega\}$.*

(2) *For each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^n$, $|x| \geq R$, there exists $\xi \in \mathbb{C}^n$ satisfying $|x - \xi| \leq \frac{1}{m}\omega(x)$ such that $|\widehat{\mu}(\xi)| \geq \exp\left(-\frac{1}{m}\omega(x)\right)$.*

(3) *There exists a weight function $\sigma$ with $\sigma = o(\omega)$ such that $\mu \in \mathcal{E}'_{(\sigma)}(\mathbb{R}^n)$ and $\mu$ is slowly decreasing for $(\sigma)$.*

Proof. (1)⇒(2). In view of 1.1($\alpha$) and 1.2(a) there exist $K \in \mathbb{N}$ and $R_0 > 0$ such that $\omega(2t) \leq K\omega(t)$ and $\omega(t) \leq t$ for $t \geq R_0$. If $m \in \mathbb{N}$ is given, choose $R_1 \geq R_0$ so that 3.1 holds with $m$ replaced by $Km$. This implies that for $x \in \mathbb{R}^n$ with $|x| \geq R_1$ there exists $\xi \in \mathbb{C}^n$ satisfying

$$|x - \xi| \leq \frac{1}{Km}\omega(x) \leq \frac{1}{m}\omega(x) \quad \text{and} \quad |\widehat{\mu}(\xi)| \geq \exp\left(-\frac{1}{Km}\omega(\xi)\right).$$

Now $|\xi| \leq |x| + |x - \xi| \leq |x| + \frac{1}{m}|x| \leq 2|x|$ implies

$$\frac{1}{Km}\omega(\xi) \leq \frac{1}{Km}\omega(2|x|) \leq \frac{1}{m}\omega(x)$$

and hence $|\widehat{\mu}(\xi)| \geq \exp\left(-\frac{1}{m}\omega(x)\right)$.

(2)⇒(3). By Braun, Meise and Taylor [6], 7.6, there exists a weight function $\kappa$ so that $\mu \in \mathcal{E}'_{(\kappa)}(\mathbb{R}^n)$. Applying (2) inductively, we find a strictly increasing sequence $(R_m)_{m\in\mathbb{N}}$ tending to infinity so that the conclusion of (2) holds for $x \in \mathbb{R}^n$ satisfying $|x| \geq R_m$. Then define $g : [0,\infty[ \to [0,\infty[$ by $g(x) = 0$ for $x \in [0, R_1[$ and $g(x) := \frac{1}{m}\omega(x)$ for $x \in [R_m, R_{m+1}[$. Since $g = o(\omega)$, Braun, Meise and Taylor [6], 1.7, gives the existence of a weight function $\sigma$ satisfying $g = o(\sigma)$, $\sigma = o(\omega)$ and $\kappa \leq \sigma$. To show that $\mu$ is slowly decreasing for $(\sigma)$, choose $l$ so that

$$\sigma(t) \leq \omega(t), \quad \sigma(t) \leq t/2 \quad \text{and} \quad g(t) \leq \sigma(t) \quad \text{for } t \geq R_l.$$

Then fix $x \in \mathbb{R}^n$ with $|x| \geq R_l$ and choose $m \geq l$ such that $|x| \in [R_m, R_{m+1}[$. By the choice of $R_m$ there exists $\xi \in \mathbb{C}^n$ satisfying

$$|x - \xi| \leq \frac{1}{m}\omega(x) = g(x) \leq \sigma(x)$$

such that

$$|\widehat{\mu}(\xi)| \geq \exp\left(-\frac{1}{m}\omega(x)\right) \geq \exp(-\sigma(x)).$$

Since $\sigma$ is a weight function, 1.1($\alpha$) implies the existence of some $L \geq 1$ so that $\sigma(2t) \leq L\sigma(t)$ for $t \geq R_l$. Because of

$$|x| \leq |\xi| + \sigma(x) \leq |\xi| + |x|/2$$

we have $\sigma(x) \le L\sigma(\xi)$ and hence
$$|\widehat{\mu}(\xi)| \ge \exp(-L\sigma(\xi)) \ge \exp(-L(|\mathrm{Im}\,\xi| + \sigma(\xi))).$$
Thus, $\mu$ is slowly decreasing for $(\sigma)$.

(3)⇒(1). By hypothesis, $\mu$ satisfies condition 2.3 for some $C \ge 1$ and $\sigma$ instead of $\omega$. Without restriction we can assume that for some $K \ge 1$,
$$\sigma(2t) \le K\sigma(t) \quad \text{for } t \ge C.$$
Since $\sigma = o(\omega)$ and $\omega = o(t)$, by 1.2(a) we can find $C' \ge C$ such that
$$(C + C^2K)\sigma(t) \le \frac{1}{m}\omega(t) \quad \text{and} \quad \sigma(t) \le \frac{1}{2C}t, \quad \text{for } t \ge \frac{C'}{2}.$$
Now fix $x \in \mathbb{R}^n$ with $|x| \ge C' \ge C$. Since $\mu$ is slowly decreasing for $(\sigma)$, there exists $\xi \in \mathbb{C}^n$ with $|x - \xi| \le C\sigma(x) \le \frac{1}{m}\omega(x)$ such that
$$(6) \qquad |\widehat{\mu}(\xi)| \ge \exp(-C|\mathrm{Im}\,\xi| - C\sigma(\xi)).$$
Since $|\xi| \ge |x| - C\sigma(x) \ge |x|/2$, we have $\sigma(x) \le K\sigma(\xi)$ and hence
$$|\mathrm{Im}\,\xi| = |\mathrm{Im}(x - \xi)| \le C\sigma(x) \le CK\sigma(\xi).$$
Therefore, (6) and our choice of $C'$ imply
$$|\widehat{\mu}(\xi)| \ge \exp(-(C^2K + C)\sigma(\xi)) \ge \exp\left(-\frac{1}{m}\omega(\xi)\right).$$
Hence $\mu$ is slowly decreasing for $\{\omega\}$.

To formulate the next proposition in such a way that it completely extends Braun, Meise and Vogt [7], Thm. 2.4, to the case of several variables, we recall the following definition from [7], 2.1.

**3.3.** Definition. For $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ the convolution operator $T_\mu$ is called *locally surjective* on $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ if for each compact set $K$ in $\mathbb{R}^n$ and each $g \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ there exists $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ satisfying $T_\mu(f)|_K = g|_K$.

**3.4.** Proposition. *For $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ the following assertions are equivalent:*

(1) *$S_\mu : \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n) \to \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$ is surjective.*

(2) *There exists $E \in \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$ satisfying $S_\mu(E) = \delta$.*

(3) *$\mu$ is slowly decreasing for $\{\omega\}$.*

(4) *If $B \subset \mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$ and $S^t_\mu(B)$ is bounded in $\mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$ then $B$ is bounded in $\mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$.*

(5) *$T_\mu$ is locally surjective on $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$.*

Proof. (1)⇒(2). This holds trivially.

(2)⇒(3). By Braun, Meise and Taylor [6], 7.6, there exists a weight function $\sigma$ satisfying $\sigma = o(\omega)$ such that $\mu \in \mathcal{E}'_{(\sigma)}(\mathbb{R}^n)$ and $E \in \mathcal{D}'_{(\sigma)}(\mathbb{R}^n)$. Since $(\mathbb{R}^n, \mathbb{R}^n)$ is $\mu$-convex for $(\sigma)$ and $S_\mu(E) = \delta$, also in $\mathcal{D}'_{(\sigma)}(\mathbb{R}^n)$, $\mu$ is slowly decreasing for $(\sigma)$ by Proposition 2.6. Hence (3) follows from Lemma 3.2.

(3)⇒(4). Fix an arbitrary subset $B$ of $\mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$ for which $M := S^t_\mu(B)$ is bounded in $\mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$. Since $\mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$ is a (DFS)-space, there exist $p \in \mathbb{N}$ and $C > 0$ such that
$$(5) \qquad |\widehat{f}(z)| \le C\exp\left(p|\mathrm{Im}\,z| - \frac{1}{p}\omega(z)\right), \quad z \in \mathbb{C}^n,\ f \in M.$$
Since $\mu$ is in $\mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$, there exists $L > 0$ such that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that
$$(6) \qquad |\widehat{\mu}(z)| \le C_\varepsilon \exp(L|\mathrm{Im}\,z| + \varepsilon\omega(z)), \quad z \in \mathbb{C}^n.$$
If $f \in M$ then $f = S^t_\mu(g) = \mu * g$ and hence $\widehat{f} = \widehat{\mu}\widehat{g}$ for some $g \in B$. Consequently, (4) holds if we show the existence of $A > 0$, $m \in \mathbb{N}_0$ and $C_0 > 0$ such that for each $f \in M$ the entire function $\widehat{g} = \widehat{f}/\widehat{\mu}$ satisfies
$$(7) \qquad |\widehat{g}(z)| \le C_0 \exp\left(A|\mathrm{Im}\,z| - \frac{1}{m}\omega(z)\right), \quad z \in \mathbb{C}^n.$$

To prove this, note first that by (3), Lemma 3.2 and Lemma 2.4 there exists a weight function $\sigma$ satisfying $\sigma = o(\omega)$ such that there exist $k \in \mathbb{N}$, $\nu \in \mathbb{N}$, $C_1 > 0$ and $R_0 \ge 1$ such that for each $z \in \mathbb{C}^n$ with $|z| \ge R_0$ there exists $w \in \mathbb{C}^n$ satisfying $|w - z| \le k\sigma(z) + |\mathrm{Im}\,z|$ such that
$$|\widehat{\mu}(w)| \ge C_1\exp(-\nu|\mathrm{Im}\,z| - \nu\sigma(z)).$$
Since $\sigma = o(\omega)$, for each $q \in \mathbb{N}$ there exists $R_q \ge R_0$ such that
$$(\nu + k)\sigma(t) \le \frac{1}{q}\omega(t) \quad \text{for } t \ge R_q$$
and hence for $|z| \ge R_q$ the point $w$ has the properties:
$$(8) \qquad \begin{aligned} &|w - z| \le k\sigma(z) + |\mathrm{Im}\,z| \le \frac{1}{q}\omega(z) + |\mathrm{Im}\,z|, \\ &|\widehat{\mu}(w)| \ge C_1\exp\left(-\nu|\mathrm{Im}\,z| - \frac{1}{q}\omega(z)\right). \end{aligned}$$
Now fix $q \in \mathbb{N}$, let $\varepsilon := 1/q$ and fix an arbitrary $f \in M$. Without restriction we may assume that $R_q$ is so large that
$$\omega(4t) \le \frac{1}{q}t \quad \text{for } t \ge R_q.$$
To prove (7) we want to apply Hörmander [14], 3.2. For that purpose let $r := r(z) := \frac{1}{q}\omega(z) + |\mathrm{Im}\,z|$ and fix $\zeta \in \mathbb{C}^n$ with $|z - \zeta| \le 4r$. Then
$$(9) \qquad |\mathrm{Im}\,\zeta| \le |\mathrm{Im}\,z| + 4r, \quad \omega(\zeta) \ge \frac{1}{K}\omega(z) - \frac{r}{q}$$

because 1.1($\alpha$) for $\omega$ implies the existence of $K \in \mathbb{N}$ so that

$$\omega(z) \le \omega(|\zeta| + 4r) \le K(\omega(\zeta) + \omega(4r)) \le K\left(\omega(\zeta) + \frac{r}{q}\right)$$

provided that $R_q$ is large enough. Hence (5) implies

$$|\widehat{f}(\zeta)| \le C \exp\left(p(|\operatorname{Im} z| + 4r) - \frac{1}{pK}\omega(z) + \frac{r}{pq}\right)$$
$$\le C \exp\left(\left(5p + \frac{1}{pq}\right)|\operatorname{Im} z| + \left(-\frac{1}{pK} + \frac{4p}{q} + \frac{1}{q^2p}\right)\omega(z)\right).$$

Choosing $\varepsilon = 1/q$ also in (6), we get similarly

$$|\widehat{\mu}(\zeta)| \le C' \exp\left(\left(5L + \frac{K}{q^2}\right)|\operatorname{Im} z| + \frac{1}{q}\left(4L + K + \frac{K}{q^2}\right)\omega(z)\right).$$

Using (8), these estimates imply

$$(10)\quad (\sup_{|z-\zeta|\le 4r} |\widehat{f}(\zeta)|)(\sup_{|z-\zeta|\le 4r} |\widehat{\mu}(\zeta)|)(\sup_{|z-\zeta|\le r} |\widehat{\mu}(\zeta)|)^{-2}$$
$$\le \frac{CC'}{C_1^2} \exp(A_q|\operatorname{Im} z| + B_q\omega(z)),$$

where

$$A_q = 2\nu + 5(L+p) + \frac{1}{pq} + \frac{K}{q^2} \le 2\nu + 5(L+p) + K + 1,$$
$$B_q = -\frac{1}{pK} + \frac{1}{q}\left(2 + 4(L+p) + K + \frac{K}{q^2} + \frac{1}{pq}\right).$$

This shows that we can choose $q \in \mathbb{N}$ so large that $B_q \le -1/(2pK)$. Then (10) implies

$$|\widehat{g}(z)| \le C_0' \exp\left(A_q|\operatorname{Im} z| - \frac{1}{2pK}\omega(z)\right), \quad |z| \ge R_q.$$

Since $C_0'$ and $A_q$ do not depend on the particular function $f$, this estimate implies (7), by the maximum principle.

(4)⇒(1). This follows from the surjectivity criterion 26.1 in Meise and Vogt [22].

(2)⇒(5). If a compact set $K$ in $\mathbb{R}^n$ and $g \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ are given, choose $\varphi \in \mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$ so that $\varphi$ is identically 1 in some neighbourhood of $K$. Then $f := E * (\varphi g)$ is in $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ if $E$ is chosen according to (2). It is easy to see that $T_\mu(f)|_K = g|_K$.

(5)⇒(3). Arguing by contradiction, assume that $\mu$ is not slowly decreasing for $\{\omega\}$. Then there exist $m_1 \in \mathbb{N}$ and a sequence $(x_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^n$ for which $(|x_j|)_{j\in\mathbb{N}}$ is increasing and unbounded and such that

$$|\widehat{\mu}(\zeta)| \le \frac{1}{m_1}\omega(\zeta) \quad \text{for all } \zeta \in \mathbb{C}^n \text{ with } |\zeta - x_j| \le \frac{1}{m_1}\omega(x_j).$$



Next choose $D \ge 1$ and $t_0 > 0$ such that $\omega(2t) \le D\omega(t)$ for $t \ge t_0$ and choose $m \in \mathbb{N}$ so large that $2D/m \le 1/m_1$. To localize $T_\mu$ as in Braun, Meise and Vogt [7], 1.8, choose $k > 0$ with $\operatorname{Supp}\mu \subset \overline{B_k(0)}$, and for $r > 0$ let

$$\mathcal{E}_{\{\omega\}}(r) := \{f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n) : f|_{B_r(0)} \equiv 0\}.$$

Then define

$$\mathcal{E}_{\{\omega\}}[r] := \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)/\mathcal{E}_{\{\omega\}}(r)$$

and denote the corresponding quotient map by $q_r$. It is easy to check that the convolution operator $T_\mu$ induces for each $r > 0$ and $R \ge r + k$ a continuous linear map

$$T_\mu(R,r) : \mathcal{E}_{\{\omega\}}[R] \to \mathcal{E}_{\{\omega\}}[r], \quad T_\mu(R,r)[f + \mathcal{E}_{\{\omega\}}(R)] := T_\mu(f) + \mathcal{E}_{\{\omega\}}(r).$$

Obviously, $T_\mu$ is locally surjective if and only if the localized operators $T_\mu(r+k,r)$ are surjective for each $r > 0$. Note that by Braun, Meise and Vogt [7], 1.10, $\mathcal{E}_{\{\omega\}}[r]$ is a (DFN)-space for each $r > 0$ and that by the arguments given in the proof of [7], 2.3, the Fourier–Laplace transform is an isomorphism between $\mathcal{E}_{\{\omega\}}[r]'$ and the Fréchet space

$$A_r(\mathbb{C}^n) :=$$
$$\left\{f \in A(\mathbb{C}^n) : \|f\|_j := \sup_{z\in\mathbb{C}^n} |f(z)| \exp\left(-\left(r + \frac{1}{j}\right)|\operatorname{Im} z| - \frac{1}{j}\omega(z)\right) < \infty \right.$$
$$\left. \text{for each } j \in \mathbb{N}\right\}.$$

Hence (5) implies that $T_\mu(m+k,m)$ is surjective. Since $\mathcal{E}_{\{\omega\}}[r]$ is a (DFN)-space for each $r > 0$, $T_\mu(m+k,m)^t$ is an injective topological homomorphism. As in 1.8 this implies that also the map

$$M_{\widehat{\mu}} : A_m(\mathbb{C}^n) \to A_{m+k}(\mathbb{C}^n), \quad M_{\widehat{\mu}}(f) = \widehat{\mu}f,$$

has this property. Hence we get a contradiction if we show that there exists a sequence $(f_j)_{j\in\mathbb{N}}$ in $A_1(\mathbb{C}^n)$ which is unbounded in $A_m(\mathbb{C}^n)$, while $(\widehat{\mu}f_j)_{j\in\mathbb{N}}$ is bounded in $A_{m+k}(\mathbb{C}^n)$. To construct this sequence, we proceed similarly to Momm [24] (see also [3], Thm. 11): For $R > 0$ let $h_{j,R} : \mathbb{C}^n \to \mathbb{R}$ be defined on $\mathbb{C}^n \setminus B_R(x_j)$ as $|\operatorname{Im} z|$ and on $B_R(x_j)$ as

$$h_{j,R}(z) := \sup\{v(z) : v \text{ is plurisubharmonic on } B_R(x_j),$$
$$\limsup_{\zeta\to\xi} v(\zeta) \le |\operatorname{Im} \xi| \text{ for } \xi \in \partial B_R(x_j)\}.$$

Then let $\varphi_j := 1 + h_{j,s_j}$, where $s_j := 1 + \frac{1}{n}\omega(x_j)$. By Momm [26],

$$\varphi_j(x_j) \geq \frac{2}{\pi\sqrt{n}}\left(1 + \frac{1}{n}\omega(x_j)\right).$$

As in Momm [25], 1.8, we can apply Hörmander's solution of the $\overline{\partial}$-problem [15], 4.4.4, to prove that there exists $f_j \in A(\mathbb{C}^n)$ satisfying

$$|f(x_j)| \geq \exp\big(\inf_{|w-x_j|\leq 1} \varphi_j(w) - c_n \log(1+|x_j|^2)\big) \quad \text{and}$$
$$|f_j(z)| \leq c_n \exp\big(\sup_{|w-z|\leq 1} \varphi_j(w) - c_n \log(1+|z|^2)\big),$$

where $c_n$ is a constant that depends only on the dimension $n$ but not on $j$. Now standard estimates (see [3], Thm. 11) show that $(f_j)_{j\in\mathbb{N}}$ is in $A_1(\mathbb{C}^n)$ but unbounded in $A_m(\mathbb{C}^n)$, while $(\widehat{\mu}f_j)_{j\in\mathbb{N}}$ is bounded in $A_{m+k}(\mathbb{C}^n)$. From this contradiction we conclude that (3) holds.

**3.5. Theorem.** *Let $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ and open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ with $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$ be given. Then the following assertions are equivalent:*

(1) *$S_\mu : \mathcal{D}'_{\{\omega\}}(\Omega_2) \to \mathcal{D}'_{\{\omega\}}(\Omega_1)$ is surjective.*

(2) *$\mathcal{E}_{\{\omega\}}(\Omega_1) \subset S_\mu(\mathcal{D}'_{\{\omega\}}(\Omega_2))$.*

(3) *$(\Omega_1, \Omega_2)$ is $\mu$-convex for $\{\omega\}$ and $\mu$ is slowly decreasing for $\{\omega\}$.*

(4) *$(\Omega_1, \Omega_2)$ is $\mu$-convex for $\{\omega\}$ and there exists $E \in \mathcal{D}'_{\{\omega\}}(\mathbb{R}^n)$ with $S_\mu(E) = \delta$.*

Proof. (1) $\Leftrightarrow$ (2). Obviously, it suffices to show that (2) implies (1). We claim that (2) implies

(5) For each $K_2 \subset \Omega_2$ compact and each $m \in \mathbb{N}_0$ there exist $K_1 \subset \Omega_1$ compact, a weight function $\sigma$ satisfying $\sigma = o(\omega)$ and $C \geq 1$ such that for each $f \in \mathcal{E}_{\{\omega\}}(\Omega_1)$ and each $\varphi \in \mathcal{D}_{\{\omega\}}(\Omega_1)$ with $\operatorname{Supp} S^t_\mu\varphi \subset K_2$,

$$\Big|\int_{\Omega_1} f\varphi\, d\lambda\Big| \leq C\|f\|_{K_1,\sigma}|S^t_\mu\varphi|_{K_2,m}.$$

To prove (5), fix $K_2 \subset \Omega_2$ compact and $m \in \mathbb{N}_0$ and let

$$H := \{v \in \mathcal{D}_{\{\omega\}}(\Omega_2) : \operatorname{Supp} v \subset K_2,\ v \in S^t_\mu(\mathcal{D}_{\{\omega\}}(\Omega_1))\} \subset \mathcal{D}_{\{\omega\}}(K_2),$$

endowed with the induced topology. Then define the bilinear form

$$B : \mathcal{E}_{\{\omega\}}(\Omega_1) \times H \to \mathbb{C}, \qquad B(f,v) := \int f(S^t_\mu)^{-1}v\, d\lambda.$$

Note that $B(f,v) = \langle f, (S^t_\mu)^{-1}v\rangle$ in the dual pairing $\langle \mathcal{D}'_{\{\omega\}}(\Omega_1), \mathcal{D}_{\{\omega\}}(\Omega_1)\rangle$. If $f \in \mathcal{E}_{\{\omega\}}(\Omega_1)$ is fixed then (2) implies the existence of $u \in \mathcal{D}'_{\{\omega\}}(\Omega_2)$ so that $S_\mu(u) = f$ and hence

$$B(f,v) = \langle f, (S^t_\mu)^{-1}v\rangle = \langle S_\mu(u), (S^t_\mu)^{-1}v\rangle = \langle u, v\rangle.$$



Consequently, $v \mapsto B(f,v)$ is continuous on $H$ and we have shown that $B$ is separately continuous. Next note that

$$B_m := \{v \in \mathcal{D}_{\{\omega\}}(\Omega_2) : \operatorname{Supp} v \subset K_2,\ v \in S^t_\mu(\mathcal{D}_{\{\omega\}}(\Omega_1)),\ |v|_{K_2,m} \leq 1\}$$

is bounded in $H$ and that by the separate continuity of $B$ the set

$$T := \{f \in \mathcal{E}_{\{\omega\}}(\Omega_1) : |B(f,v)| \leq 1 \text{ for all } v \in B_m\}$$

is closed and absolutely convex in $\mathcal{E}_{\{\omega\}}(\Omega_1)$. To show that $T$ is absorbing and hence a barrel, fix $f \in \mathcal{E}_{\{\omega\}}(\Omega_1)$. Since $B$ is separately continuous and $B_m$ is bounded in $H$, we have

$$\sup_{v\in B_m} |B(f,v)| \leq \lambda$$

and hence $f \in \lambda T$. Now note that $\mathcal{E}_{\{\omega\}}(\Omega_1)$ is reflexive by Braun, Meise and Taylor [6], 4.9, hence barrelled. Therefore, $T$ is a zero neighbourhood in $\mathcal{E}_{\{\omega\}}(\Omega_1)$. By Meise, Taylor and Vogt [21], 3.2, this implies that there exist $K_1 \subset \Omega_1$ compact, a weight function $\sigma$ with $\sigma = o(\omega)$ and $C \geq 1$ such that

$$\{f \in \mathcal{E}_{\{\omega\}}(\Omega_1) : \|f\|_{K_1,\sigma} \leq 1/C\} \subset T.$$

From this and the definitions of $T$ and $B$ it follows easily that

$$\Big|\int f(S^t_\mu)^{-1}v\, d\lambda\Big| = |B(f,v)| \leq C\|f\|_{K_1,\sigma}|v|_{K_2,m}, \qquad (f,v) \in \mathcal{E}_{\{\omega\}}(\Omega_1) \times H. \tag{6}$$

Obviously (6) gives (5) if we replace $v$ by $S^t_\mu\varphi$, $\varphi \in \mathcal{D}_{\{\omega\}}(\Omega_1)$.

Next note that the surjectivity of $S_\mu : \mathcal{D}'_{\{\omega\}}(\Omega_2) \to \mathcal{D}'_{\{\omega\}}(\Omega_1)$, by the surjectivity criterion 26.1 of Meise and Vogt [22], is equivalent to

(7) If $B \subset \mathcal{D}_{\{\omega\}}(\Omega_1)$ and $S^t_\mu(B)$ is bounded in $\mathcal{D}_{\{\omega\}}(\Omega_2)$ then $B$ is bounded in $\mathcal{D}_{\{\omega\}}(\Omega_1)$,

since $\mathcal{D}'_{\{\omega\}}(\Omega_j)$ is a Fréchet space for $j = 1,2$. To derive (7) from (5), fix any set $B \subset \mathcal{D}_{\{\omega\}}(\Omega_1)$ which satisfies the hypothesis of (7). As $\mathcal{D}_{\{\omega\}}(\Omega_2)$ is a (DFS)-space, there exist $K_2 \subset \Omega_2$ compact, $m \in \mathbb{N}$ and $D > 0$ such that

$$\bigcup_{\varphi\in B} \operatorname{Supp} S^t_\mu(\varphi) \subset K_2 \quad \text{and} \quad \sup\{|S^t_\mu\varphi|_{K_2,m} : \varphi \in B\} \leq D. \tag{8}$$

For $K_2$ and $m$ as above, (5) implies that there exist $K_1 \subset \Omega_1$ compact, a weight function $\sigma$ with $\sigma = o(\omega)$ and $C \geq 1$ such that (5) holds with $m+1$ instead of $m$. To show that $B$ is $\sigma(\mathcal{D}_{\{\omega\}}(K_1), \mathcal{D}'_{\{\omega\}}(K_1))$-bounded, fix $\nu \in \mathcal{E}'_{\{\omega\}}(\Omega_1)$. By Braun [5], Cor. 10, there exist an ultradifferential operator $G$ of class $\{\omega\}$ and $g \in \mathcal{E}_\sigma(\mathbb{R}^n, 1)$ such that $\nu = G(D)g$. Next note that, by a standard smoothing argument, the estimate (5) holds not only for all $f \in \mathcal{E}_{\{\omega\}}(\Omega_1)$ but even for all $f \in C^\infty(\Omega_1)$ satisfying $\|f\|_{K_1,\sigma} < \infty$.

Note further that

$$S_\mu^t(G(-D)\varphi) = \mu * (T_{\check G} * \check\varphi)(0)$$
$$= ((T_{\check G} * \mu) * \check\varphi)(0) = G(-D)(S_\mu^t\varphi), \quad \varphi \in \mathcal{D}_{\{\omega\}}(\mathbb{R}^n),$$

so that $\operatorname{Supp} S_\mu^t(G(-D)\varphi) \subset \operatorname{Supp} S_\mu^t\varphi \subset K_2$ for each $\varphi \in B$. Therefore, (5) in the extended form gives for each $\varphi \in B$,

$$(9) \qquad |\nu(\varphi)| = |G(D)g(\varphi)| = |g(G(-D)\varphi)| = \Big|\int g(G(-D)\varphi)\, d\lambda\Big|$$
$$\le C\|g\|_{K_1,\sigma}|S_\mu^t(G(-D)\varphi)|_{K_2,m+1}.$$

Since $\log|G| = o(\omega)$, we have for each $\psi \in \mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$,

$$\int |(G(-D)\psi)^\wedge(\xi)|e^{\omega(\xi)/(m+1)}\,d\xi \le \int |G(-\xi)\widehat\psi(\xi)|e^{\omega(\xi)/(m+1)}\,d\xi$$
$$\le L_m\int |\widehat\psi(\xi|e^{\omega(\xi)/m}\,d\xi.$$

This together with (8) and (9) implies

$$|\nu(\varphi)| \le CDL_m\|g\|_{K_1,\sigma} \quad \text{for all } \varphi \in B.$$

This proves that $B$ is weakly bounded, hence bounded in $\mathcal{D}_{\{\omega\}}(K_1)$.

(1)⇒(3). To show that $(\Omega_1,\Omega_2)$ is $\mu$-convex for $\{\omega\}$, note that by Braun, Meise and Taylor [6], 7.6, there exists a weight function $\sigma$ satisfying $\sigma = o(\omega)$ such that $\mu \in \mathcal{E}'_{(\sigma)}(\mathbb{R}^n) \subset \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$. Note further that (1) implies

$$\mathcal{E}_{(\omega)}(\Omega_1) \subset \mathcal{D}'_{\{\omega\}}(\Omega_1) \subset S_\mu(\mathcal{D}'_{\{\omega\}}(\Omega_2)) \subset S_\mu(\mathcal{D}'_{(\omega)}(\Omega_2)).$$

Hence $(\Omega_1,\Omega_2)$ is $\mu$-convex for $(\omega)$. By the remark after 2.1 this implies that $(\Omega_1,\Omega_2)$ is $\mu$-convex for $\{\omega\}$.

In the remaining part of the proof we assume without restriction $0 \in \Omega_1$. Then we choose $\delta > 0$ and $k > 0$ such that $\overline{B_\delta(0)} \subset \Omega_1$ and $\operatorname{Supp}\mu \subset \overline{B_k(0)}$. Next we assume that $\mu$ is not slowly decreasing for $\{\omega\}$ and show that this contradicts (1). To do so, note that there exists $\nu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ satisfying $\widehat\nu(z) = \widehat\mu(z/\delta)$ and $\operatorname{Supp}\nu \subset \overline{B_{k/\delta}(0)}$. Obviously, $\nu$ is not slowly decreasing for $\{\omega\}$ since $\mu$ has this property. Using the notation introduced in the proof of Proposition 3.4, (5)⇒(3), we get the existence of $m \in \mathbb{N}$ and of a sequence $(f_j)_{j\in\mathbb{N}}$ in $A_1(\mathbb{C}^n)$ so that $(\widehat\nu f_j)_{j\in\mathbb{N}}$ is bounded in $A_{m+k/\delta}(\mathbb{C}^n)$, while $(f_j)_{j\in\mathbb{N}}$ is unbounded in $A_m(\mathbb{C}^n)$. Next note that for each $s > 0$ the map

$$\Phi : A_s(\mathbb{C}^n) \to A_{\delta s}(\mathbb{C}^n), \quad \Phi(f)(z) := f(\delta z), \quad z \in \mathbb{C}^n,$$

is an isomorphism. Hence, if we let $g_j := \Phi(f_j)$, $j \in \mathbb{N}$, then $g_j$ is in $A_\delta(\mathbb{C}^n)$ and the sequence $(g_j)_{j\in\mathbb{N}}$ is unbounded in $A_{\delta m}(\mathbb{C}^n)$ while $(\widehat\mu g_j)_{j\in\mathbb{N}}$ is bounded in $A_{\delta m+k}$, since $\Phi(\widehat\nu f_j) = \widehat\mu g_j$. Now note that for $0 < r < R$ the inclusion $A_r(\mathbb{C}^n) \hookrightarrow A_R(\mathbb{C}^n)$ is a topological homomorphism since it is—up to the Fourier–Laplace transform—the adjoint of the surjective homomorphism $\mathcal{E}_{\{\omega\}}[R] \to \mathcal{E}_{\{\omega\}}[r]$, which is induced by restriction. Since $g_j \in A_\delta(\mathbb{C}^n)$ and $\widehat\mu g_j \in A_{\delta+k}(\mathbb{C}^n)$ for $j \in \mathbb{N}$, we conclude that $(\widehat\mu g_j)_{j\in\mathbb{N}}$ is bounded in $A_{\delta+k}(\mathbb{C}^n)$, while $(g_j)_{j\in\mathbb{N}}$ is unbounded in $A_\delta(\mathbb{C}^n)$. Hence

$$M\widehat\mu : A_\delta(\mathbb{C}^n) \to A_{\delta+k}(\mathbb{C}^n)$$

is not a topological homomorphism. Consequently,

$$(10) \qquad T_\mu(\delta+k,\delta) : \mathcal{E}_{\{\omega\}}[\delta+k] \to \mathcal{E}_{\{\omega\}}[\delta] \text{ is not surjective.}$$

This contradicts (1) since we will show next that (1) implies

(11) $T_\mu(\delta+k,\delta)^t : (\mathcal{E}_{\{\omega\}}[\delta])' \to (\mathcal{E}_{\{\omega\}}[\delta+k])'$ is an injective topological homomorphism.

And, as we have noted in the proof of Proposition 3.4 that $\mathcal{E}_{\{\omega\}}[r]$ is a (DFS)-space for each $r > 0$, (11) contradicts (10). Hence it suffices to derive (11) from (1). To do this, let $(\nu_j)_{j\in\mathbb{N}}$ be a sequence in $(\mathcal{E}_{\{\omega\}}[\delta])'$ for which $(T_\mu(\delta+k,\delta)^t\nu_j)_{j\in\mathbb{N}}$ converges to zero strongly. In order to show that $(\nu_j)_{j\in\mathbb{N}}$ converges strongly to zero in $(\mathcal{E}_{\{\omega\}}[\delta])'$ we prove $\sigma((\mathcal{E}_{\{\omega\}}[\delta])', \mathcal{E}_{\{\omega\}}[\delta])\text{-}\lim_{j\to\infty}\nu_j = 0$, which is sufficient because $(\mathcal{E}_{\{\omega\}}[\delta])'$ is a Fréchet–Schwartz space. To do so, note that by the definition of the maps $T_\mu(R,r)$, we have

$$T_\mu(\delta+k,\delta)\circ q_{\delta+k} = q_\delta \circ T_\mu,$$

where $q_r : \mathcal{E}_{\{\omega\}}(\mathbb{R}^n) \to \mathcal{E}_{\{\omega\}}[r]$ denotes the quotient map. Consequently, $(T_\mu^t(\nu_j\circ q_\delta))_{j\in\mathbb{N}}$ converges to zero in $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)'$. Now choose $\varepsilon > 0$ such that $\overline{B_{\delta+\varepsilon}(0)} \subset \Omega_1$ and fix $\chi \in \mathcal{D}_{\{\omega\}}(B_\varepsilon(0))$. Then $(\nu_j\circ q_\delta)*\chi \in \mathcal{D}_{\{\omega\}}(B_{\delta+\varepsilon}(0)) \subset \mathcal{D}_{\{\omega\}}(\Omega_1)$ and

$$S_\mu^t((\nu_j\circ q_\delta)*\chi) = \mu * (\nu_j\circ q_\delta)*\chi = T_\mu^t(\nu_j\circ q_\delta)*\chi$$

tends to zero in $\mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$. Hence it tends to zero in $\mathcal{D}_{\{\omega\}}(\Omega_2)$. Since $\mathcal{D}_{\{\omega\}}(\Omega_1)'$ and $\mathcal{D}_{\{\omega\}}(\Omega_2)'$ are Fréchet–Schwartz spaces, (1) implies that $S_\mu^t$ is an injective topological homomorphism. Therefore, we conclude that $((\nu_j\circ q_\delta)*\chi)_{j\in\mathbb{N}}$ is a null-sequence in $\mathcal{D}_{\{\omega\}}(\Omega_1)$ for each $\chi \in \mathcal{D}_{\{\omega\}}(B_\varepsilon(0))$. By the same argument that we used in the proof of Proposition 2.2, (3)⇒(4), this implies that $(\nu_j\circ q_\delta(f))_{j\in\mathbb{N}}$ is a null-sequence for each $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$. Hence $(\nu_j)_{j\in\mathbb{N}}$ is a weak null-sequence, which completes the proof.

(3)⇒(4). This holds by Proposition 3.4.

(4)⇒(1). Since $\mathcal{D}_{\{\omega\}}(\Omega_j)'_b$ is a Fréchet space for $j = 1, 2$, we get (1) from the surjectivity criterion [22], 26.1, if we show:

If $M \subset \mathcal{D}_{\{\omega\}}(\Omega_1)$ and $S_\mu^t(M)$ is bounded in $\mathcal{D}_{\{\omega\}}(\Omega_2)$ then $M$ is bounded in $\mathcal{D}_{\{\omega\}}(\Omega_1)$.

To prove this, fix $M$ as above. Since $\mathcal{D}_{\{\omega\}}(\Omega_2)$ is a (DFN)-space there exists $K_2 \subset \Omega_2$ compact so that $S_\mu^t(M)$ is bounded in $\mathcal{D}_{\{\omega\}}(K_2)$. Choose

$K_1 \subset \Omega_1$ compact according to the $\mu$-convexity of $(\Omega_1, \Omega_2)$ and note that $E* : \mathcal{D}_{\{\omega\}}(K_2) \to \mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$ is continuous and linear. Hence $E * S_\mu^t(M) = M$ is bounded in $\mathcal{D}_{\{\omega\}}(\mathbb{R}^n)$ and has support in $K_1$. This implies that $M$ is bounded in $\mathcal{D}_{\{\omega\}}(\Omega_1)$.

Remark. Note that the conditions in 3.5 in general are not equivalent to the surjectivity of $T_\mu : \mathcal{E}_{\{\omega\}}(\Omega_2) \to \mathcal{E}_{\{\omega\}}(\Omega_1)$. For $\Omega_1 = \Omega_2 = \mathbb{R}^n$ this follows from an easy extension of Braun, Meise and Vogt [7], Ex. 3.11, to the case of several variables. By Meyer [23], Thm. 3.13, this example also shows that there are $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ with $\operatorname{Supp}\mu = \{0\}$ and open sets $\Omega$ in $\mathbb{R}^n$ so that $T_\mu : \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is not surjective. In fact, even linear partial differential operators with constant coefficients in general are not surjective on $\mathcal{E}_{\{\omega\}}(\Omega)$, $\Omega$ open in $\mathbb{R}^n$. For a characterization and references to earlier work on this subject we refer to Langenbruch [18].

**3.6.** COROLLARY. *Let $\omega$ and $\sigma$ be weight functions satisfying $\omega = o(\sigma)$, let $\Omega_1, \Omega_2$ be open subsets of $\mathbb{R}^n$ and let $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ satisfy $\Omega_1 + \operatorname{Supp}\mu \subset \Omega_2$. Then the following conditions are equivalent:*

(1) $\mathcal{E}_{\{\sigma\}}(\Omega_1) \subset T_\mu(\mathcal{E}_{\{\omega\}}(\Omega_2))$.

(2) $S_\mu : \mathcal{D}'_{\{\sigma\}}(\Omega_1) \to \mathcal{D}'_{\{\sigma\}}(\Omega_2)$ *is surjective.*

Proof. (1)⇒(2). Since $\omega = o(\sigma)$, from Braun, Meise and Taylor [6], 3.9, and (1) we get

$$\mathcal{E}_{\{\sigma\}}(\Omega_1) \subset T_\mu(\mathcal{E}_{\{\omega\}}(\Omega_2)) \subset S_\mu(\mathcal{D}'_{\{\omega\}}(\Omega_2)) \subset S_\mu(\mathcal{D}'_{\{\sigma\}}(\Omega_2)).$$

Hence (2) follows from Theorem 3.5.

(2)⇒(1). By Theorem 3.5, there exists $E \in \mathcal{D}'_{\{\sigma\}}(\mathbb{R})$ satisfying $E * \mu = \delta$. By [6], 7.6, there exists a weight function $\kappa$, $\kappa = o(\sigma)$ and $\omega \le \kappa$, so that $\mu \in \mathcal{E}'_{(\kappa)}(\mathbb{R}^n)$ and $E \in \mathcal{D}'_{(\kappa)}(\mathbb{R}^n)$. Again by Theorem 3.5, the pair $(\Omega_1, \Omega_2)$ is $\mu$-convex for $\{\sigma\}$, hence also for $(\kappa)$, by the remark after 2.1. Thus Proposition 2.6 implies $T_\mu(\mathcal{E}_{(\kappa)}(\Omega_2)) = \mathcal{E}_{(\kappa)}(\Omega_1)$ and hence

$$\mathcal{E}_{\{\sigma\}}(\Omega_1) \subset \mathcal{E}_{(\kappa)}(\Omega_1) = T_\mu(\mathcal{E}_{(\kappa)}(\Omega_2)) \subset T_\mu(\mathcal{E}_{\{\omega\}}(\Omega_2)),$$

since $\mathcal{E}_{(\kappa)}(\Omega_2) \subset \mathcal{E}_{\{\omega\}}(\Omega_2)$.

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