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A corollary to a theorem of Laurent–Mignotte–Nesterenko

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1. Introduction. For any algebraic number α of degree d on $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{i=1}^{d}(X-\alpha^{(i)})$ where the roots $\alpha^{(i)}$ are complex numbers, we define the *absolute logarithmic height* of α by

$$\mathrm{h}(\alpha)=\frac{1}{d}\Big(\log|a|+\sum_{i=1}^{d}\log\max(1,|\alpha^{(i)}|)\Big).$$

Let α_1, α_2 be two non-zero algebraic numbers, and let $\log\alpha_1$ and $\log\alpha_2$ be any values of their logarithms. We consider the linear form

$$\Lambda=b_2\log\alpha_2-b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Without loss of generality, we suppose that $|\alpha_1|$ and $|\alpha_2|$ are ≥ 1. Put

$$D=[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}].$$

The main result of [LMN] is:

Theorem 1. *Let K be an integer ≥ 3, L an integer ≥ 2, and R_1, R_2, S_1, S_2 integers >0. Let ϱ be a real number >1. Put $R=R_1+R_2-1$, $S=S_1+S_2-1$, $N=KL$,*

$$g=\frac{1}{4}-\frac{N}{12RS},\qquad b=\frac{((R-1)b_2+(S-1)b_1)}{2}\Big(\prod_{k=1}^{K-1}k!\Big)^{-2/(K^2-K)}.$$

Let a_1, a_2 be positive real numbers such that

$$a_i\geq\varrho|\log\alpha_i|-\log|\alpha_i|+2D\mathrm{h}(\alpha_i),$$

for $i=1$, 2. Suppose that

$$\begin{aligned}(1)\qquad &\mathrm{Card}\{\alpha_1^r\alpha_2^s:0\leq r<R_1,\ 0\leq s<S_1\}\geq L,\\ &\mathrm{Card}\{rb_2+sb_1:0\leq r<R_2,\ 0\leq s<S_2\}>(K-1)L\end{aligned}$$

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and that

(2) $K(L-1)\log\varrho-(D+1)\log N-D(K-1)\log b-gL\,(Ra_1+Sa_2)>0.$

Then

$$|\Lambda'|\ge\varrho^{-KL+1/2}\quad\textit{with}\quad \Lambda'=\Lambda\max\left\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2},\ \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\right\}.$$

In the case when the numbers α_1 and α_2 are multiplicatively independent we shall deduce from Theorem 1 the following result, which is a variant of Théorème 2 of [LMN].

Theorem 1.5. *Consider the linear form*

$$\Lambda=b_2\log\alpha_2-b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put

$$D=[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}].$$

Let a_1, a_2, h, k be real positive numbers, and ϱ a real number >1. Put $\lambda=\log\varrho$ and suppose that

$$h\ge D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+f(K)\right)+0.023, \tag{3}$$

$$a_i\ge\max\{1,\varrho|\log\alpha_i|-\log|\alpha_i|+2D\mathrm{h}(\alpha_i)\}\qquad(i=1,2), \tag{4}$$

$$a_1a_2\ge\lambda^2, \tag{5}$$

where

$$f(x)=\log\frac{(1+\sqrt{x-1})\sqrt{x}}{x-1}+\frac{\log x}{6x(x-1)}+\frac{3}{2}+\log\frac{3}{4}+\frac{\log\frac{x}{x-1}}{x-1}$$

and

$$L=2+[2h/\lambda],\qquad K=1+[kLa_1a_2].$$

Then we have the lower bound

$$\begin{aligned}\log|\Lambda|\ge{}&-\lambda kL^2a_1a_2\\ &-\max\{\lambda(L-0.5)+\log((L^{3/2}+L^2\sqrt{k})\max\{a_1,a_2\}+L),D\log 2\},\end{aligned}$$

provided that k satisfies

$$kU-V\sqrt{k}-W\ge0$$

with

$$U=(L-1)\lambda-h,\qquad V=L/3,\qquad W=\frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+2\sqrt{\frac{L}{a_1a_2}}\right).$$

REMARK 1. Put $\Delta = V^2 + 4UW$. The condition on k implies $k \ge k_0$ where

$$\sqrt{k_0} = \frac{V+\sqrt{\Delta}}{2U}, \qquad k_0 = \frac{V^2+\Delta+2V\sqrt{\Delta}}{4U^2} = \frac{V^2}{2U^2} + \frac{W}{U} + \frac{V}{2U}\sqrt{\frac{V^2}{U^2}+\frac{4W}{U}}$$

with

$$\frac{V}{U} = \frac{1}{3}\cdot\frac{L}{\lambda L-(h+\lambda)} \ge \frac{1}{3}\cdot\frac{\lambda^{-1}2(h+\lambda)}{2(h+\lambda)-(h+\lambda)} = \frac{2}{3\lambda},$$

since $\partial(V/U)/\partial L < 0$ and $L \le 2(1+h/\lambda)$, and

$$W = \frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+2\sqrt{\frac{L}{a_1a_2}}\right) \ge \frac{2}{3\sqrt{a_1a_2}}(1+\sqrt{L}),$$

so that

$$\frac{W}{U} \ge \frac{2}{3\sqrt{a_1a_2}}\cdot\frac{1+\sqrt{L}}{\lambda L-(h+\lambda)} \ge \frac{4}{3\lambda\sqrt{a_1a_2}}\cdot\frac{1+\sqrt{L}}{L} \ge \frac{4}{3\lambda^2}\cdot\frac{1+\sqrt{L}}{L},$$

since $a_1a_2 \ge \lambda^2$. Hence $k \ge 4/(9\lambda^2)$ and

$$kLa_1a_2 \ge kL\lambda^2 \ge \frac{2L}{9} + \frac{4}{3}(1+\sqrt{L}) + \frac{L}{3}\sqrt{\frac{4}{9}+\frac{16(1+\sqrt{L})}{3L}} = \psi(L) \quad \text{(say)}.$$

Clearly ψ increases with L and computation gives $\psi(2) > 6$.

2. Proof of Theorem 1.5. We suppose that α_1 and α_2 are multiplicatively independent, and we apply Theorem 1 with a suitable choice of the parameters. The proof follows the proof of Théorème 2 of [LMN]. For the convenience of the reader we keep the numbering of formulas of [LMN], except that formula (5.i) in [LMN] is here formula (2.i); moreover, when there is some change the new formula is denoted by (2.i)′.

Put

$$\begin{aligned} L &= 2+[2h/\lambda], & S_1 &= 1+[\sqrt{La_1/a_2}],\\ K &= 1+[kLa_1a_2], & R_2 &= 1+[\sqrt{(K-1)La_2/a_1}],\\ R_1 &= 1+[\sqrt{La_2/a_1}], & S_2 &= 1+[\sqrt{(K-1)La_1/a_2}]. \end{aligned} \tag{2.1}$$

Recall that

$$a_i \ge \varrho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i) \quad \text{for } i = 1, 2.$$

By the Liouville inequality,

$$\begin{aligned} \log|\Lambda| &\ge -D\log 2 - Db_1\mathrm{h}(\alpha_1) - Db_2\mathrm{h}(\alpha_2)\\ &\ge -D\log 2 - \tfrac{1}{2}(b_1a_1+b_2a_2) = -D\log 2 - \tfrac{1}{2}b'a_1a_2, \end{aligned}$$

where

$$b' = \frac{b_1}{a_2} + \frac{b_2}{a_1}.$$

We consider two cases:

$$b' \le 2\lambda k L^2 \quad \text{or} \quad b' > 2\lambda k L^2.$$

In the first case, Liouville's inequality implies

$$\log|\Lambda| \ge -D\log 2 - \lambda k L^2 a_1 a_2$$

and Theorem 1.5 holds.

Suppose now that $b' > 2\lambda k L^2$. Then $\max\{b_1/a_2, b_2/a_1\} > \lambda k L^2$, hence

$$b_1 > \lambda\sqrt{k}L\cdot\sqrt{(K-1)La_2/a_1} \quad \text{or} \quad b_2 > \lambda\sqrt{k}L\cdot\sqrt{(K-1)La_1/a_2}.$$

Since $k \ge 4/(9\lambda^2)$ and $L \ge 2$, we have $\lambda\sqrt{k}L > 1$, which proves that

$$\mathrm{Card}\{rb_2 + sb_1 : 0 \le r < R_2, 0 \le s < S_2\} = R_2 S_2$$

and, by the choice of R_2 and S_2, this is $> (K-1)L$. Moreover, since α_1 and α_2 are multiplicatively independent we have

$$\mathrm{Card}\{\alpha_1^r \alpha_2^s : 0 \le r < R_1, 0 \le s < S_1\} = R_1 S_1 \ge L.$$

This ends the verification of condition (1) of Theorem 1.

Remark 2. The condition $b' > 2k\lambda L^2$ implies

$$\begin{aligned}\lambda L/D \ge 2h/D &\ge 2(\log(2k\lambda L^2) + \log\lambda + f(K))\\ &\ge 2\big(\log(2L\psi(L)) + \tfrac{3}{2} + \log\tfrac{3}{4}\big) > 8.812,\end{aligned}$$

by Remark 1 and $L \ge 2$.

Suppose that (2) holds. Then Theorem 1 implies

$$\log|\Lambda'| \ge -KL\lambda + \lambda/2,$$

where

$$\Lambda' = \Lambda \max\left\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2}, \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\right\}.$$

Notice that

$$\begin{aligned}R = R_1 + R_2 - 1 &\le \sqrt{La_2/a_1} + \sqrt{(K-1)La_2/a_1} + 1\\ &\le 1 + \sqrt{La_2} + \sqrt{k}La_2\\ &\le 1 + (1/\sqrt{L} + \sqrt{k})La_2 \le 1 + (1/\sqrt{L} + \sqrt{k})LA,\end{aligned}$$

where $A = \max\{a_1, a_2\}$ and, in the same way,

$$S = S_1 + S_2 - 1 \le 1 + (1/\sqrt{L} + \sqrt{k})LA.$$

This shows that

$$\max\{LR, LS\} \le L + (1/\sqrt{L} + \sqrt{k})L^2 A.$$

As we may, suppose that $\log|\Lambda| \le -\lambda k L^2 a_1 a_2 - 4$. Then

$$\max\left\{\frac{LR|\Lambda|}{2b_2}, \frac{LS|\Lambda|}{2b_1}\right\} \le \frac{(1.21+\sqrt{k})L^2 a_1 a_2}{2} e^{-\lambda k L^2 a_1 a_2 - 4}$$
$$\le \left(0.61 + \frac{1}{3\lambda}\right) L^2 a_1 a_2 e^{-4L^2 a_1 a_2/(9\lambda) - 4},$$

since $k \ge 4/(9\lambda^2)$ and $\lambda k L^2 a_1 a_2 > 1$. The last term is an increasing function of λ, thus for $\lambda \le 1$,

$$\max\left\{\frac{LR|\Lambda|}{2b_2}, \frac{LS|\Lambda|}{2b_1}\right\} \le \left(0.61 + \frac{1}{3}\right) L^2 a_1 a_2 e^{-4L^2 a_1 a_2/9 - 4} < 0.1$$

since $L^2 a_1 a_2 \ge 4$. For $\lambda \ge 1$,

$$\max\left\{\frac{LR|\Lambda|}{2b_2}, \frac{LS|\Lambda|}{2b_1}\right\} \le \left(0.61 + \frac{1}{3}\right) L^2 a_1 a_2 e^{-4L^2 a_1 a_2/(9\lambda) - 4}$$

and, since $a_1 a_2 \ge \lambda^2$, we get

$$\max\left\{\frac{LR|\Lambda|}{2b_2}, \frac{LS|\Lambda|}{2b_1}\right\} \le \left(0.61 + \frac{1}{3}\right) L^2 \lambda^2 e^{-4L^2\lambda/9 - 4} < L^2 e^{-4L^2/9 - 4} < 0.1.$$

In all cases,

$$|\Lambda'| \le |\Lambda|(L^2(1/\sqrt{L} + \sqrt{k}) \max\{a_1, a_2\} + L),$$

which implies

$$\log|\Lambda| \ge -\lambda k L^2 a_1 a_2 - \lambda(L - 0.5) - \log((L^{3/2} + L^2\sqrt{k}) \max\{a_1, a_2\} + L)$$

and Theorem 1.5 follows.

Now we have to verify that condition (2) is satisfied: we have to prove that

$$\Phi_0 = K(L-1)\log\varrho - (D+1)\log N - D(K-1)\log b - gL\,(Ra_1 + Sa_2) > 0,$$

when $b' > 2\lambda k L^2$.

We replace this condition by the two conditions $\Phi > 0$, $\Theta > 0$, where $\Phi_0 \ge \Phi + \Theta$. The term Φ is the main one, Θ is a sum of residual terms. As indicated in [LMN], the condition $\Phi > 0$ leads to the choice of the parameters (2.1), whereas $\Theta > 0$ is a secondary condition, which leads to assuming some technical hypotheses on h and a_1, a_2. Here, we follow the advice given in [LMN]: for some applications one can modify these technical hypotheses.

As in [LMN] (Lemme 8) we get

$$\text{(2.17)} \qquad \log b \le \log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda - \frac{\log(2\pi K/\sqrt{e})}{K-1} + f(K)$$
$$\le \frac{h}{D} - \frac{0.023}{D} - \frac{\log(2\pi K/\sqrt{e})}{K-1},$$

which follows from the condition

$$h \geq D(\log b' + \log \lambda + f(K)) + 0.023.$$

Lemme 9 of [LMN] gives

$$\text{(2.18)}\quad gL(Ra_1 + Sa_2) \leq \frac{1}{3}L^{3/2}\sqrt{(K-1)a_1a_2} + \frac{2}{3}L^{3/2}\sqrt{a_1a_2} + \frac{1}{3}L(a_1+a_2) - \frac{L^{3/2}\sqrt{a_1a_2}}{6(1+\sqrt{K-1})}.$$

Put

$$\text{(2.21)}\quad \Phi = K(L-1)\lambda - Kh - \frac{L^{3/2}\sqrt{(K-1)a_1a_2}}{3} - \frac{2L^{3/2}\sqrt{a_1a_2}}{3} - \frac{L(a_1+a_2)}{3}$$

and

$$\text{(2.22)}\quad \Theta = 0.023(K-1) + h + \frac{L^{3/2}\sqrt{a_1a_2}}{6(1+\sqrt{K-1})} + D\log\left(\frac{2\pi K}{\sqrt{e}}\right) - (D+1)\log(KL).$$

By (2.17) and (2.18) we see that $\Phi_0 \geq \Phi + \Theta$, where $kLa_1a_2 < K \leq 1 + kLa_1a_2$, hence

$$\Phi > kLa_1a_2((L-1)\lambda - h) - \frac{L^2a_1a_2\sqrt{k}}{3} - \frac{2L^{3/2}\sqrt{a_1a_2}}{3} - \frac{L(a_1+a_2)}{3},$$

which implies

$$\frac{\Phi}{La_1a_2} > kU - V\sqrt{k} - W.$$

This proves that $\Phi > 0$ provided that $kU - V\sqrt{k} - W \geq 0$.

To prove that $\Theta \geq 0$, rewrite (2.22) as $\Theta = \Theta_0(D-1) + \Theta_1$, where

$$\Theta_0 = \log(\lambda b') + f(K) - \log L + \log\left(\frac{2\pi}{\sqrt{e}}\right),$$

$$\Theta_1 = 0.023K - \log K - 2\log L + \log\left(\frac{2\pi}{\sqrt{e}}\right) + \log(\lambda b') + f(K) + \frac{L^{3/2}\sqrt{a_1a_2}}{6(1+\sqrt{K-1})}.$$

We conclude by proving that Θ_0 and Θ_1 are both positive.

Since $b' > 2k\lambda L^2$, by Remark 1 we have $\log(\lambda b') > 2L\psi(L)$, which shows that Θ_0 is positive.

Notice that, by the proof of Remark 2,

$$L^{3/2}\sqrt{a_1a_2} = L\sqrt{La_1a_2} \ge L\sqrt{1+2ha_1a_2/\lambda} \ge L\sqrt{1+2h}$$
$$> 2\sqrt{1+2(\log(2\psi(2)) + f(K) + 0.023)} = \phi(K) \quad \text{(say)}.$$

Thus,

$$\Theta_1 \ge 0.023K - \log K + \log\left(\frac{16\pi}{9\sqrt{e}}\right) + f(K) + \frac{\phi(K)}{3(1+\sqrt{K-1})}$$

and an elementary numerical verification shows that Θ_1 is positive for $K \ge 4$, which holds by Remark 1.

3. A corollary of Theorem 1.5. Now we can apply Theorem 1.5 to get a result closer to Théorème 2 of [LMN].

THEOREM 2. *Consider the linear form*

$$\Lambda = b_2 \log\alpha_2 - b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put

$$D = [\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}].$$

Let a_1, a_2, h, k be real positive numbers, and ϱ a real number > 1. Put $\lambda = \log\varrho$, $\chi = h/\lambda$ and suppose that $\chi \ge \chi_0$ for some number $\chi_0 \ge 0$ and that

$$(3)' \qquad h \ge D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + f(\lceil K_0\rceil)\right) + 0.023,$$

$$(4) \qquad a_i \ge \max\{1, \varrho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\} \quad (i=1,2),$$

$$(5) \qquad a_1a_2 \ge \lambda^2,$$

where

$$f(x) = \log\frac{(1+\sqrt{x-1})\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log\frac{3}{4} + \frac{\log\frac{x}{x-1}}{x-1}$$

and

$$K_0 = \frac{1}{\lambda}\left(\frac{\sqrt{2+2\chi_0}}{3} + \sqrt{\frac{2(1+\chi_0)}{9} + \frac{2\lambda}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) + \frac{4\lambda\sqrt{2+\chi_0}}{3\sqrt{a_1a_2}}}\right)^2 a_1a_2.$$

Put

$$v = 4\chi + 4 + 1/\chi, \qquad m = \max\{2^{5/2}(1+\chi)^{3/2}, (1+2\chi)^{5/2}/\chi\}.$$

Then we have the lower bound

$$\log|\Lambda| \ge -\frac{1}{\lambda}\left(\frac{v}{6}+\frac{1}{2}\sqrt{\frac{v^2}{9}+\frac{4\lambda v}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{8\lambda m}{3\sqrt{a_1a_2}}}\right)^2 a_1a_2$$
$$-\max\{\lambda(1.5+2\chi)$$
$$+\log(((2+2\chi)^{3/2}+(2+2\chi)^2\sqrt{k^*})A+(2+2\chi)), D\log 2\},$$

where

$$A=\max\{a_1,a_2\} \quad \textit{and} \quad k^*=\frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2+\frac{1}{\lambda}\left(\frac{2}{3\chi}+\frac{2}{3}\cdot\frac{(1+2\chi)^{1/2}}{\chi}\right).$$

4. Proof of Theorem 2. We apply Theorem 1.5 with $k=k_0$.

First we estimate certain quantities of the form k_0L^α. The formula

$$\frac{\partial}{\partial x}\frac{x^\alpha}{\lambda x-(\lambda+h)}=\frac{x^{\alpha-1}}{(\lambda x-(\lambda+h))^2}((\alpha-1)\lambda x-\alpha(\lambda+h))$$
$$=\frac{\lambda x^{\alpha-1}}{(\lambda x-(\lambda+h))^2}((\alpha-1)x-\alpha(1+\chi))$$

shows that the functions $L\mapsto L^\alpha/U$ are non-increasing in the interval $I=[1+2\chi,2+2\chi]$ for $\alpha\le 2$.

Hence,

$$\frac{L^2}{U}\le\frac{(1+2\chi)^2}{\lambda\chi}=\frac{4\chi+4+1/\chi}{\lambda}=\frac{v}{\lambda}.$$

Moreover, the previous formula also shows that the function $L\mapsto L^\alpha/U$ is unimodular for all α, which implies

$$\frac{L^{5/2}}{U}\le\frac{1}{\lambda}\max\left\{\frac{(2+2\chi)^{5/2}}{1+\chi},\frac{(1+2\chi)^{5/2}}{\chi}\right\}$$
$$=\frac{1}{\lambda}\max\{2^{5/2}(1+\chi)^{3/2},(1+2\chi)^{5/2}/\chi\}=\frac{m}{\lambda}.$$

These remarks imply

$$\lambda k_0L^2a_1a_2\le\frac{1}{\lambda}\left(\frac{v}{6}+\frac{1}{2}\sqrt{\frac{v^2}{9}+\frac{4v\lambda}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{8\lambda m}{3\sqrt{a_1a_2}}}\right)^2 a_1a_2.$$

Besides,

$$k_0\le\frac{V^2}{U^2}+2\frac{W}{U}$$

thus

$$\begin{aligned} k_0 &\le \frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2 + \frac{1}{\lambda}\left(\frac{1}{3\chi}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) + \frac{2}{3\sqrt{a_1a_2}}\cdot\frac{(1+2\chi)^{1/2}}{\chi}\right) \\ &\le \frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2 + \frac{1}{\lambda}\left(\frac{2}{3\chi} + \frac{2}{3}\cdot\frac{(1+2\chi)^{1/2}}{\chi}\right) = k^*. \end{aligned}$$

Since the function $f(x)$ is decreasing for $x > 1$, the last step is to verify that $K \ge K_0$ (with the notations of Theorem 1.5). We follow the proof of Remark 1. We have

$$\sqrt{k_0L} = \frac{V\sqrt{L}}{2U} + \sqrt{\frac{V^2L}{4U^2} + \frac{WL}{U}}$$

with

$$\frac{V\sqrt{L}}{U} = \frac{1}{3}\cdot\frac{L^{3/2}}{\lambda(L-(1+\chi))} \ge \frac{2\sqrt{2+2\chi}}{3\lambda}$$

and

$$\frac{WL}{U} \ge \frac{2}{3\lambda}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) + \frac{4\sqrt{2+2\chi}}{3\lambda\sqrt{a_1a_2}}$$

so that

$$\sqrt{k_0L} \ge \frac{\sqrt{2+2\chi}}{3\lambda} + \sqrt{\frac{2(1+\chi)}{9\lambda^2} + \frac{2}{3\lambda}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) + \frac{4\sqrt{2+2\chi}}{3\lambda\sqrt{a_1a_2}}}$$

and since $K = 1 + [\lambda k_0 L a_1 a_2]$, we get $K \ge \lceil K_0 \rceil$. [One may verify that $K_0 > 4$.]

Remark 3. The number m satisfies

$$m = \lambda \max_{L\in I}\left\{\frac{L^{5/2}}{U}\right\} \le \lambda \max_{L\in I}\left\{\frac{L^2}{U}\right\}\cdot \max_{L\in I}\sqrt{L} \le \lambda(4\chi+4+1/\chi)\sqrt{2+2\chi}.$$

It is possible to simplify some estimates in Theorem 2 without serious loss. Consider first the term k^* given by

$$\begin{aligned} k^* &= \frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2 + \frac{1}{\lambda}\left(\frac{2}{3\chi} + \frac{2}{3}\cdot\frac{(1+2\chi)^{1/2}}{\chi}\right) \\ &= \frac{1}{9}\left(\frac{2}{\lambda}+\frac{1}{h}\right)^2 + \frac{2}{3h}(1+\sqrt{1+2h/\lambda}). \end{aligned}$$

It is clear that $\partial k^*/\partial\lambda < 0$. Also, $\partial k^*/\partial h < 0$. Indeed,

$$\frac{\partial k^*}{\partial h} = -\frac{2}{3h}\left(\frac{2}{\lambda}+\frac{1}{3h}\right) - \frac{2}{3h^2}(1+\sqrt{1+2h/\lambda}) + \frac{2}{3h}\cdot\frac{1/\lambda}{1+\sqrt{1+2h/\lambda}},$$

which is

$$< \frac{2}{3h}\left(-\frac{\sqrt{1+2h/\lambda}}{h}+\frac{1}{\lambda\sqrt{1+2h/\lambda}}\right)=\frac{2}{3h}\cdot\frac{-\lambda(1+2h/\lambda)+h}{\lambda h\sqrt{1+2h/\lambda}}<0.$$

Thus, for $\lambda \geq \lambda_0$ and $h \geq h_0$, we have

$$k^* \leq \frac{1}{9}\left(\frac{2}{\lambda_0}+\frac{1}{h_0}\right)^2+\frac{2}{3h_0}(1+\sqrt{1+2h_0/\lambda_0}).$$

In particular, when $\lambda \geq \log 4$ and $h \geq 3.5$, we get $k^* \leq 1$.

Now we consider the term $T := \log((x^2+x^{3/2})A+x)/\log(Ax^2)$. Elementary computation shows that $\partial T/\partial A < 0$ and $\partial T/\partial x < 0$. When $x \geq 4$ and $A \geq 4$ we get $T \leq 1.11$.

Concerning Theorem 2, when $\chi \geq 1$, $\varrho \geq 4$, $h \geq 3.5$ and $A \geq 4$, these remarks imply the simplified estimate

$$\log|\Lambda| \geq -(C_0+c_1+c_2)(\lambda+h)^2 a_1 a_2,$$

where

$$C_0=\frac{1}{\lambda^3}\left(\frac{v/3+\sqrt{\frac{v^2}{9}+\frac{4\lambda v}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{8\lambda m}{3\sqrt{a_1a_2}}}}{2(1+\chi)}\right)^2,$$

and

$$c_1=\frac{\lambda(1.5\lambda+2h)}{(\lambda+h)^2a_1a_2},\qquad c_2=\frac{1.11\lambda\log(A(2\lambda+2h)^2)}{(\lambda+h)^2a_1a_2}.$$

When $a_1a_2 \geq 20$, $\varrho \geq 4$ and $h \geq 3.5$, one can prove that $c_2 \leq 0.024$. The formula

$$c_1=\frac{1.5+2\chi}{(1+\chi)^2a_1a_2}$$

shows that c_1 is a decreasing function of χ and, for example, for $\chi \geq 1.5$ and $a_1a_2 \geq 20$, we have $c_1 \leq 0.036$. To summarize, $c_1+c_2 < 0.06$ when $a_1a_2 \geq 20$, $\varrho \geq 4$, $h \geq 3.5$ and $\chi \geq 1.5$. Also notice that for $\chi \geq 1$, one has $m = 2^{5/2}(1+\chi)^{3/2}$.

This leads to the following result.

Corollary. *Consider the linear form*

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Define D, a_1, a_2, ϱ, λ, h, χ as in Theorem 2. Let a_1, a_2, h, k be real positive numbers, and ϱ a real number > 1. Suppose that $\varrho \geq 4$ and that

$$(3)'' \quad h \geq \max\left\{3.5, 1.5\lambda, D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+1.377\right)+0.023\right\},$$

$$(4) \qquad a_i \geq \max\{1, \varrho\,|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\} \qquad (i=1,2),$$

(5)″ $$a_1 a_2 \geq \max\{20, 4\lambda^2\}.$$

Let $v = 4\chi + 4 + 1/\chi$. *Then we have the lower bound*

$$\log |\Lambda| \geq -(C_0 + 0.06)(\lambda + h)^2 a_1 a_2,$$

where

$$C_0 = \frac{1}{\lambda^3}\left\{\left(2+\frac{1}{2\chi(\chi+1)}\right)\left(\frac{1}{3}+\sqrt{\frac{1}{9}+\frac{4\lambda}{3v}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{32\sqrt{2}(1+\chi)^{3/2}}{3v^2\sqrt{a_1 a_2}}}\right)\right\}^2.$$

We apply Theorem 2. After the above preliminaries, we have just to check that the present hypotheses imply $K_0 > 38$ and use the fact that $f(39) < 1.377$.

Remark 4. To get a comparison with the estimates of [LMN], we can consider the Corollaire 2 of [LMN]. Thus we suppose also that α_1 and α_2 are both real. Then we get

$$\log |\Lambda| \geq -22D^4\left(\max\left\{\log\left(\frac{b_1}{D\log A_2}+\frac{b_2}{D\log A_1}\right)+0.06, \frac{21}{D}\right\}\right)^2 \log A_1 \log A_2,$$

where A_1 and A_2 are real numbers > 1 such that

$$\log A_i \geq \max\left\{\mathrm{h}(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}.$$

This result is obtained with the choice $\varrho = 5.58$ in the above Corollary (except that we use the original definitions of c_1 and c_2, not the estimate $c_1 + c_2 < 0.06$). In [LMN], with (very) slightly stronger hypotheses, the constant obtained was 24.34.

References

[LMN] M. Laurent, M. Mignotte et Y. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory 55 (1995), 285–321.

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