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$\mathbb{Q}$-linear relations of special values of the Estermann zeta function

by

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1. Introduction. Let $\sigma_u(n) = \sum_{d|n} d^u$, the sum of uth powers of divisors of n, and let $e(x) = e^{2\pi ix}$. For given integers a and q with $(a,q) = 1$, $q \ge 1$, the *Estermann zeta function* $E_u = E_u(\cdot, a/q)$ is defined by the Dirichlet series

$$E_u\left(s, \frac{a}{q}\right) = \sum_{n=1}^{\infty} \sigma_u(n) e\left(\frac{an}{q}\right) n^{-s}, \qquad \mathrm{Re}(s) > \max\{0, \mathrm{Re}(u)+1\},$$

and has an analytic continuation to the whole s-plane with possible poles at $s = 1$, $u+1$. This function has its origin in Estermann's paper [1] and plays an important role in recent theory of divisor functions and allied problems ([6], [7], [9]).

In this paper we determine the linear relations among the values of E_u at negative integral arguments

$$\{E_u(-j, a/q) : 1 \le a \le q,\ (a,q) = 1\}$$

over the rational number field $\mathbb{Q}$, where $j \ge 1$ and $u \ge 0$ are rational integers, which extends our previous result [4]. Our main result is

THEOREM. *The numbers* $E_u(-j, a/q)$, $1 \le a \le q$, $(a,q) = 1$, *belong to the* q*th cyclotomic field*, *in particular they vanish for any odd number* u, *and if* q *is a prime power*, *we have*

$$\sum_{(a,q)=1} c_a E_u(-j, a/q) = 0 \text{ if and only if } c_a = (-1)^j c_{q-a} \text{ for } c_a \in \mathbb{Q},\ u \text{ even.}$$

From this Theorem, we easily deduce

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COROLLARY. *If q is a prime power and u is even, then the numbers $E_u(-j,a/q)$, $1\le a\le q/2$, $(a,q)=1$, are linearly independent over $\mathbb{Q}$.*

In Section 2, we prove the first part of the Theorem by evaluating these special values in terms of the cotangent function, and in Section 3, $\mathbb{Q}$-linear relations are determined by the method of K. Girstmair [2].

2. Special values. Let B_m and $B_m(x)$ be the mth Bernoulli number and the mth Bernoulli polynomial respectively, and let $\cot^{(m)}(\pi x)$ be the mth derivative of $\cot(\pi x)$.

PROPOSITION 1. *Let $q\ge 2$, $1\le a\le q$, $(a,q)=1$, $i=\sqrt{-1}$. Then*

$$E_u\left(-j,\frac{a}{q}\right)=\frac{q^j}{j+1}\left(-\frac{i}{2}\right)^{j+u+1}\sum_{l=1}^{q-1}B_{j+1}\left(\frac{l}{q}\right)\cot^{(j+u)}\left(\frac{\pi al}{q}\right)+q^j(1+q)^j\frac{B_{j+1}}{j+1}\cdot\frac{B_{j+u+1}}{j+u+1},\tag{1}$$

for $u\ge 0$, $j\ge 1$. For $q=1$,

$$E_u(-j,1)=\frac{B_{j+1}}{j+1}\cdot\frac{B_{j+u+1}}{j+u+1}.$$

In particular, for $q\ge 2$ the right hand side of (1) is 0 for u odd.

Proof. We can express the function $E_u(s,a/q)$ in terms of the Hurwitz zeta function

$$\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s},\qquad 0<x\le 1,$$

with $\zeta(s,1)=\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ the Riemann zeta function, as follows:

$$E_u\left(s,\frac{a}{q}\right)=q^{u-2s}\sum_{k,l=1}^{q}e\left(\frac{akl}{q}\right)\zeta\left(s-u,\frac{k}{q}\right)\zeta\left(s,\frac{l}{q}\right),\qquad \mathrm{Re}(s)>\mathrm{Re}(u)+1$$

(cf. [4]). Since $\zeta(s,x)$ can be analytically continued to a meromorphic function with simple pole at $s=1$, this equation gives an analytic continuation of $E_u(s,a/q)$. To evaluate the values at negative integer points in terms of the cotangent function, we need

$$\zeta(-j,a/q)=-\frac{1}{j+1}B_{j+1}(a/q),\qquad j\ge 0,$$

and

$$(j+1)\left(\frac{i}{2}\right)^{j+1}\cot^{(j)}\left(\frac{\pi a}{q}\right)=q^j\sum_{k=1}^{q}e\left(-\frac{ak}{q}\right)B_{j+1}\left(\frac{k}{q}\right)\qquad\text{(see [2])}.$$

Substituting these formulas for $E_u(-j,a/q)$, we have

$$
\begin{aligned}
E_u\left(-j, \frac{a}{q}\right) &= \frac{q^j}{j+1}\left(-\frac{i}{2}\right)^{j+u+1} \sum_{l=1}^{q-1} B_{j+1}\left(\frac{l}{q}\right) \cot^{(j+u)}\left(\frac{\pi a l}{q}\right) \\
&\quad + \frac{q^{2j+u} B_{j+1}}{(j+1)(j+u+1)} \sum_{k=1}^{q-1} B_{j+u+1}\left(\frac{k}{q}\right) \\
&\quad + \frac{q^{2j+u} B_{j+u+1}}{(j+1)(j+u+1)} \sum_{l=1}^{q} B_{j+1}\left(\frac{l}{q}\right) \\
&= S_1 + S_2 + S_3, \quad \text{say.}
\end{aligned}
$$

Using the Fourier series of $\cot^{(j+u)}(\pi x/q)$ and changing the order of summation, we see that the symmetric terms appearing in the innermost sum over the range from 1 to q will cancel out each other for odd u, so that $S_1 = 0$. $S_2 + S_3$ is evaluated by the distribution relation of the Bernoulli polynomial:

$$
B_k(x) = m^{k-1} \sum_{j=0}^{m-1} B_k\left(\frac{x+j}{m}\right),
$$

and we also have $S_2 + S_3 = 0$ for odd u by the properties of Bernoulli numbers.

For $q = 1$, the formula follows from

$$
E_u(s, 1) = \sum_{n=1}^{\infty} \frac{\sigma_u(n)}{n^s} = \zeta(s)\zeta(s-u).
$$

Thus the proposition is proved.

Since $i^{j+u+1} \cot^{(j+u)}(\pi a l/q)$ belong to the qth cyclotomic field, the above proposition implies the first part of our Theorem.

3. $\mathbb{Q}$-linear relations. Let $\mathbb{Q}_q = \mathbb{Q}(\zeta)$ be the qth cyclotomic field with $\zeta = e(1/q)$ and let $G = \mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q})$ be its Galois group. The $\mathbb{Q}$-linear relations of the conjugate numbers $\{\sigma(b) : b \in \mathbb{Q}_q,\ \sigma \in G\}$ are determined by the annihilator ideal $W_q[b]$ in the group ring $\mathbb{Q}G$ defined by

$$
W_q[b] = \{\alpha \in \mathbb{Q}G : \alpha \circ b = 0\},
$$

where the $\mathbb{Q}G$ action on $\mathbb{Q}_q$ is defined by

$$
\alpha \circ b = \sum_{\sigma \in G} a_\sigma \sigma(b) \quad \text{for } \alpha = \sum_{\sigma \in G} a_\sigma \sigma \in \mathbb{Q}G.
$$

In [2], K. Girstmair proves that $W_q[b]$ is generated by the idempotent element $\varepsilon_X = \sum_{\chi \in X} \varepsilon_\chi$, with $\varepsilon_\chi = |G|^{-1} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma$, attached to a certain subset X of the character group $\widehat{G}$ of G determined by $X = \{\chi \in \widehat{G} : y(\chi|b) = 0\}$. Here, $y(\chi|b)$ are Leopoldt's character coordinates defined by $y(\chi|b)\tau(\overline{\chi}_f|1)=\sum_{\sigma\in G}\chi(\sigma^{-1})\sigma(b)$, where f is the conductor of χ, χ_f is the primitive character modulo f attached to χ and $\tau(\chi|k)$ is the kth Gauss sum.

He also proves, for $q\ge 2$,

$$(2)\qquad y(\chi|i^{j+1}\cot^{(j)}(\pi/q))
=\begin{cases}0, & \chi \text{ principal}, j=0,\\ \left(\frac{2q}{f}\right)^{j+1}\prod_{p|q}\left(1-\frac{\overline{\chi}_f(p)}{p^{j+1}}\right)\frac{B_{j+1,\chi_f}}{j+1}, & \text{otherwise},\end{cases}$$

where B_{j,χ_f} is the generalized Bernoulli number attached to the character χ_f. Thus $W_q[i^{j+1}\cot^{(j)}(\pi/q)]=\langle 1+(-1)^j\sigma_{-1}\rangle$, where $\sigma_k\in G$ are such that $\sigma_k(\zeta)=\zeta^k$, $(k,q)=1$.

In our case $E_u(-j,a/q)=\sigma_a(E_u(-j,1/q))$, and so we also have

Proposition 2. *$W_q[E_u(-j,1/q)]=\langle 1+(-1)^j\sigma_{-1}\rangle$ for prime power q, $j\ge 1$, and u even.*

Proof. Let $l=kd$, $d=(l,q)$ in the formula for $E_u(-j,1/q)$, which gives

$$E_u(-j,1/q)
=\frac{q^j}{j+1}\left(-\frac12\right)^{j+u+1}\sum_{d|q}\sum_{\substack{k=1\\(k,d)=1}}^{d-1}B_{j+1}\left(\frac kd\right)i^{j+u+1}\cot^{(j+u)}\left(\frac{\pi k}{d}\right)
=\frac{q^j}{j+1}\left(-\frac12\right)^{j+u+1}C_{j,u},\quad\text{say.}$$

By (2) and the $\mathbb{Q}G$-linearity of $y(\chi|-)$ with the reduction formula

$$y(\chi|b)=\begin{cases}(\varphi(q)/\varphi(d))\cdot y(\chi_d|b), & f\mid d,\\ 0, & \text{otherwise},\end{cases}$$

for $b\in\mathbb{Q}_d\subset\mathbb{Q}_q$, where χ_d is the character mod d attached to χ (see [8]), we have

$$(3)\qquad y(\chi|C_{j,u})=\frac{\varphi(q)}{j+u+1}\left(\frac2f\right)^{j+u+1}\sum_{\substack{d\\ f|d|q}}\frac{d^{u+1}}{\varphi(d)}
\times\prod_{p|d}\left(1-\frac{\overline{\chi}_f(p)}{p^{j+u+1}}\right)\prod_{p|d}(1-\chi_f(p)p^j)B_{j+1,\chi_f}B_{j+u+1,\chi_f}$$

$$= \begin{cases} \dfrac{\varphi(q)}{j+u+1}\left(\dfrac{2}{f}\right)^{j+u+1} \displaystyle\sum_{\substack{d \\ f|d|q}} \dfrac{d^{u+1}}{\varphi(d)} B_{j+1,\chi_f} B_{j+u+1,\chi_f}, & \chi \neq 1, \\ \dfrac{\varphi(q)}{j+u+1}\left(\dfrac{2}{f}\right)^{j+u+1} \displaystyle\sum_{\substack{d \\ f|d|q}} \dfrac{d^{u+1}}{\varphi(d)} \prod_{p|d}\left(1-\dfrac{1}{p^{j+u+1}}\right) & \\ \qquad\qquad \times \displaystyle\prod_{p|d}(1-p^j) B_{j+1} B_{j+u+1}, & \chi = 1. \end{cases}$$

Here

$$B_{j+1,\chi_d} = d^j \sum_{k=1}^{d} \chi_d(k) B_{j+1}(k/d),$$

and we have the formula

$$B_{j+1,\chi_d} = \prod_{p|d}(1-\chi_f(p)p^j) \cdot B_{j+1,\chi_f},$$

which is a generalization of Hasse's formula [3, p. 18], and can be proved in the same way, or instantly obtained by comparing both sides of the equality

$$L(s,\chi_d) = \prod_{p|d}(1-\chi_f(p)p^{-s}) L(s,\chi_f)$$

at negative integral arguments, where $L(s,\chi_f)$ denotes the Dirichlet L-function.

In the case of a primitive character it is known that

$$\begin{cases} B_{n+1,\chi} \neq 0, & n \not\equiv \delta_\chi \bmod 2, \\ B_{n+1,\chi} = 0, & n \equiv \delta_\chi \bmod 2, \end{cases}$$

for $n \geq 1$, where $\delta_\chi = 0$ for even χ and 1 for odd χ ([5]). Further, for principal χ, we see that $B_{n+1,\chi_f} = B_{n+1} = 0$ for even $n \geq 2$, and $B_{n+1,\chi_d} \neq 0$ for n odd.

Hence we get $X = \{\chi \in \widehat{G} : \chi(\sigma_{-1}) = (-1)^j\}$ from (3), so that $\varepsilon_X = 1 + (-1)^j \sigma_{-1}$ generates $W_q[E_u(-j, 1/q)]$.

This proposition implies the latter half of our Theorem.

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