ACTA ARITHMETICA
LXXXVII.1 (1998)

Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$

by

Oto Strauch (Bratislava) and János T. Tóth (Nitra)

Dedicated to the memory of Professor Paul Erdős

1. Introduction. Denote by $\mathbb{N}$ the set of all positive integers and if a subset $A \subset \mathbb{N}$ is given, define the *ratio set* by

$$R(A) = \{a/b : a, b \in A\}.$$

The *lower* and *upper asymptotic density* of A, denoted by $\underline{d}(A)$ and $\overline{d}(A)$ respectively, are defined as

$$\underline{d}(A) = \liminf_{x\to\infty} \frac{A(x)}{x}, \quad \overline{d}(A) = \limsup_{x\to\infty} \frac{A(x)}{x},$$

where $A(x) = \#\{a \le x : a \in A\}$.

In the present paper we are concerned with certain relations between the asymptotic densities of a set A as well as with density of $R(A)$ in $[0,\infty)$. T. Šalát [6] showed that $\underline{d}(A) = \overline{d}(A) > 0$ or $\overline{d}(A) = 1$ implies that $R(A)$ is everywhere dense in $[0,\infty)$ and for every sufficiently small $\varepsilon > 0$ there exists a subset $A \subset \mathbb{N}$ such that $\overline{d}(A) = 1 - \varepsilon$ and $R(A)$ is not everywhere dense in $[0,\infty)$. He gave an example of $A \subset \mathbb{N}$ for which $\underline{d}(A) = 1/4$ and $R(A) \cap (5/4, 8/5) = \emptyset$.

We prove that $1/2$ is the lower bound of γ's for which $\underline{d}(A) \ge \gamma$ implies that $R(A)$ is dense in $[0,\infty)$ (Theorem 1). The proof is based on the estimate

$$\underline{d}(A) \le \frac{\alpha}{\beta} \min(1 - \overline{d}(A), \overline{d}(A))$$

where the interval $(\alpha,\beta) \subset [0,\infty)$ is disjoint from $R(A)$ (Theorem 2). To complete our proof we construct an $A \subset \mathbb{N}$ for which the complement of the closure of $R(A)$ is formed by infinitely many pairwise disjoint open intervals

1991 *Mathematics Subject Classification*: Primary 11B05.
Key words and phrases: positive integers, subset, ratio set, asymptotic density.
This research was supported by the Slovak Academy of Sciences Grant 5123.

(α_n, β_n) and $\underline{d}(A) \to 1/2 - 0$ as a limit over some parameters (Example 1). On the other hand, we prove that for every given upper and lower asymptotic density there exists an $A \subset \mathbb{N}$ possessing these densities and having $R(A)$ everywhere dense (Theorem 3). As an application we give a new class of sets $A \subset \mathbb{N}$ having dense ratio set $R(A)$ (Theorem 4). We also prove that the complement of the set $R(A)^l$ of all limit points of $R(A)$ is either empty or contains infinitely many open intervals assuming $\underline{d}(A) > 0$ (Theorem 5). We generalize our results for any open set X disjoint from the set $R(A)^d$ of all accumulation points of $R(A)$ (Theorem 6). The paper concludes with some remarks.

Throughout the paper, without loss of generality, we will use only intervals (α, β) contained in $[0, 1]$.

2. Main results

THEOREM 1. *For every $A \subset \mathbb{N}$, if the lower asymptotic density $\underline{d}(A) \geq 1/2$ then the ratio set $R(A)$ is everywhere dense in $[0, \infty)$. Conversely, if $0 \leq \gamma < 1/2$ then there exists an $A \subset \mathbb{N}$ such that $\underline{d}(A) = \gamma$ and $R(A)$ is not everywhere dense in $[0, \infty)$.*

The proof immediately follows from the following theorem and example.

THEOREM 2. *Let $A \subset \mathbb{N}$ and the interval (α, β), $0 \leq \alpha < \beta \leq 1$, be such that $(\alpha, \beta) \cap R(A) = \emptyset$. Then*

$$\underline{d}(A) \leq \frac{\alpha}{\beta} \min(1 - \overline{d}(A), \overline{d}(A)) \tag{1}$$

and

$$\overline{d}(A) \leq 1 - (\beta - \alpha). \tag{2}$$

Proof of (1). Let $A \subset \mathbb{N}$ be listed in strictly increasing order as $a_1 < a_2 < \ldots < a_n < \ldots$ If $(\alpha, \beta) \cap R(A) = \emptyset$, then the intervals

$$(\alpha a_n, \beta a_n), \quad n = 1, 2, \ldots,$$

cannot intersect A but they may have mutually nonempty intersections. We can select pairwise disjoint subintervals

$$\begin{aligned}(\alpha a_{[\theta n]}, \alpha a_{[\theta n]} + \alpha), (\alpha a_{[\theta n]+1}, \alpha a_{[\theta n]+1} + \alpha), \ldots,& \\ (\alpha a_{n-1}, \alpha a_{n-1} + \alpha), (\alpha a_n, \beta a_n)&\end{aligned} \tag{3}$$

for some $0 \leq \theta \leq 1$ (here we put $a_{[\theta n]} = 0$ if $[\theta n] = 0$). Define $B = \mathbb{N} - A$ and $B(x) = \#\{b \leq x : b \in B\}$. Counting the number of integer points belonging to (3) we obtain

$$B(\beta a_n) \geq (n - [\theta n])(\alpha - 1) + ((\beta - \alpha)a_n - 1) + B(\alpha a_{[\theta n]})$$

for all sufficiently large n. To eliminate 1 in $\alpha - 1$ we replace n with nk and α with $k\alpha$. Then (3) transforms into pairwise disjoint subintervals of the form

$$(4)\qquad (\alpha a_{[\theta n]k}, \alpha a_{[\theta n]k} + k\alpha), (\alpha a_{([\theta n]+1)k}, \alpha a_{([\theta n]+1)k} + k\alpha), \ldots, \\ (\alpha a_{(n-1)k}, \alpha a_{(n-1)k} + k\alpha), (\alpha a_{nk}, \beta a_{nk}).$$

Thus, we have

$$\frac{B(\beta a_{nk})}{\beta a_{nk}} \geq \frac{(n-[\theta n])(k\alpha - 1)}{\beta a_{nk}} + \frac{((\beta-\alpha)a_{nk} - 1)}{\beta a_{nk}} + \frac{B(\alpha a_{[\theta n]k})}{\alpha a_{[\theta n]k}} \cdot \frac{\alpha}{\beta} \cdot \frac{a_{[\theta n]k}}{a_{nk}}.$$

To compute the $\limsup$ of the left and right hand sides, respectively, use the fact that

(i) $\limsup_{n\to\infty} B(\beta a_{nk})/(\beta a_{nk}) \leq \overline{d}(B) = 1 - \underline{d}(A)$,
(ii) $\limsup_{n\to\infty} nk/a_{nk} = \overline{d}(A)$,
(iii) $\liminf_{n\to\infty} B(\alpha a_{[\theta n]k})/(\alpha a_{[\theta n]k}) \geq \underline{d}(B) = 1 - \overline{d}(A)$, and
(iv) by selecting indices n for which $\lim_{n\to\infty} nk/a_{nk} = \overline{d}(A)$ we have (assuming $\overline{d}(A) > 0$)

$$\liminf_{n\to\infty} \frac{a_{[\theta n]k}}{a_{nk}} = \liminf_{n\to\infty} \frac{a_{[\theta n]k}}{[\theta n]k} \lim_{n\to\infty} \frac{[\theta n]k}{a_{nk}} \geq \frac{1}{\overline{d}(A)}\overline{d}(A)\theta.$$

Thus, letting $k \to \infty$ we get

$$1 - \underline{d}(A) \geq (1-\theta)\frac{\alpha}{\beta}\overline{d}(A) + \frac{\beta-\alpha}{\beta} + (1-\overline{d}(A))\frac{\alpha}{\beta}\theta.$$

Computing the maximum of the right hand side for $0 \leq \theta \leq 1$ yields

$$1 - \underline{d}(A) \geq \frac{\beta-\alpha}{\beta} + \frac{\alpha}{\beta}\max(\overline{d}(A), 1 - \overline{d}(A)),$$

which justifies (1).

Proof of (2). Every infinite set $A \subset \mathbb{N}$ with infinite complement $\mathbb{N} - A$ can be expressed as the set of the integer points lying in the intervals

$$(5)\qquad [b_1, c_1], [b_2, c_2], \ldots, [b_n, c_n], \ldots,$$

whose endpoints form two integer sequences ordered as

$$b_1 \leq c_1 < b_2 \leq c_2 < \ldots < b_n \leq c_n < \ldots$$

Clearly

$$(6)\qquad \underline{d}(A) = \liminf_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n-1} (c_i - b_i + 1),$$

$$(7)\qquad \overline{d}(A) = \limsup_{n\to\infty} \frac{1}{c_n} \sum_{i=1}^{n} (c_i - b_i + 1).$$

The points of $A\cap[1,c_n]$ divided by i, $i\in[b_n,c_n]$, form a subset $R_n\subset R(A)$; we obtain the intervals

$$\left[\frac{b_1}{i},\frac{c_1}{i}\right],\left[\frac{b_2}{i},\frac{c_2}{i}\right],\ldots,\left[\frac{b_{n-1}}{i},\frac{c_{n-1}}{i}\right],\left[\frac{b_n}{i},\frac{c_n}{i}\right]$$

which have the following property: the distance of any two neighbouring points of R_n lying in $[b_{n-k}/i,c_{n-k}/i]$ is less than $1/b_n$ and the same holds for the union

$$\bigcup_{i=b_n}^{c_n}\left[\frac{b_{n-k}}{i},\frac{c_{n-k}}{i}\right]=\left[\frac{b_{n-k}}{c_n},\frac{c_{n-k}}{b_n}\right].$$

Thus, for sufficiently large n, every interval $(\alpha,\beta)\subset[0,1]$ satisfying $(\alpha,\beta)\cap R(A)=\emptyset$ must lie in the complement of $[b_{n-k}/c_n,c_{n-k}/b_n]$, $k=0,1,\ldots,n-1$, which is formed by the pairwise disjoint intervals

$$\left(\frac{c_{n-k}}{b_n},\frac{b_{n-k+1}}{c_n}\right),\qquad k=1,\ldots,n-1,\tag{8}$$

some of which may be empty. Hence, a necessary condition for $(\alpha,\beta)\cap R(A)=\emptyset$ is the existence of an integer sequence k_n, $k_n<n$, such that

$$(\alpha,\beta)\subset\left(\frac{c_{n-k_n}}{b_n},\frac{b_{n-k_n+1}}{c_n}\right)\tag{9}$$

for all sufficiently large n. This also gives

$$\frac{b_{n-k_n+1}}{c_n}-\frac{c_{n-k_n}}{c_n}\ge\beta-\alpha.$$

Now we can express the upper asymptotic density as

$$\overline{d}(A)=\limsup_{n\to\infty}\left(\frac{c_n-b_1}{c_n}+\frac{n}{c_n}-\left(\frac{b_2-c_1}{c_n}+\frac{b_3-c_2}{c_n}+\ldots+\frac{b_n-c_{n-1}}{c_n}\right)\right)\tag{10}$$

whence

$$\overline{d}(A)-\overline{d}(C)\le1-(\beta-\alpha),\tag{11}$$

where C is the range of c_n.

For sufficiency of (9) we need the set $R(A)^l$ of all limit points of $R(A)$ (cf. Section 4). By the above reasoning we see that $(\alpha,\beta)\cap R(A)^l=\emptyset$ if and only if there exists $k_n<n$ satisfying (9) for all sufficiently large n. Thus, inequality (11) holds for (α,β) satisfying $(\alpha,\beta)\cap R(A)^l=\emptyset$ as well.

Now, for a positive integer k, transform

$$[b_n,c_n]\to[kb_n,kc_n+k-1]$$

and denote by A_k the set of all integer points lying in $[kb_n,kc_n+k-1]$, $n=1,2,\ldots$ Similarly, C_k is the set of all kc_n+k-1. Evidently

$$\overline{d}(A_k)=\overline{d}(A),\qquad\overline{d}(C_k)=\overline{d}(C)/k,\qquad R(A_k)^l=R(A)^l,$$

which gives $\overline{d}(A)-\overline{d}(C)/k\le1-(\beta-\alpha)$ and (2) follows. ∎

Using (2) and the part $\underline{d}(A) \le (\alpha/\beta)(1 - \overline{d}(A))$ of (1) we have

COROLLARY. *For every subset $A \subset \mathbb{N}$, if $\underline{d}(A) + \overline{d}(A) \ge 1$ then $R(A)$ is everywhere dense in $[0, \infty)$.*

To complete our proof of Theorem 1 consider

EXAMPLE 1. Let γ, δ and a be given positive real numbers satisfying $\gamma < \delta$ and $a > 1$. Let A be the set of all integer points lying in the intervals

$$(\gamma, \delta), (\gamma a, \delta a), (\gamma a^2, \delta a^2), \ldots, (\gamma a^n, \delta a^n), \ldots$$

For this A we see from (5) of A that $b_n = [\gamma a^n] + 1$, $c_n = [\delta a^n]$ and in order that $c_n < b_{n+1}$ we need $\delta/\gamma < a$. In this case, for the intervals in (8) we have

$$\left(\frac{\delta}{\gamma a^k}, \frac{\gamma}{\delta a^{k-1}}\right) \subset \left(\frac{c_{n-k}}{b_n}, \frac{b_{n-k+1}}{c_n}\right), \qquad k = 1, \ldots, n-1;$$

further, $c_{n-k}/b_n \to \delta/(\gamma a^k)$, $b_{n-k+1}/c_n \to \gamma/(\delta a^{k-1})$ as $n \to \infty$. Consequently, the closure of $R(A)$ is $R(A)^l$. Thus, $[0,1] - \overline{R(A)} = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)$, where $(\alpha_i, \beta_i) = (\alpha_1/a^{i-1}, \beta_1/a^{i-1})$ and

$$(\alpha_1, \beta_1) = \left(\frac{\delta}{\gamma a}, \frac{\gamma}{\delta}\right).$$

This implies that

$$[0,1] - \overline{R(A)} \ne \emptyset \Leftrightarrow \delta/\gamma < \sqrt{a}.$$

By (6) and (7) we have

$$\underline{d}(A) = \frac{\delta - \gamma}{\gamma} \cdot \frac{1}{a-1}, \quad \overline{d}(A) = \frac{\delta - \gamma}{\delta} \cdot \frac{a}{a-1}.$$

We can also see that for such A the ratio set $R(A)$ is everywhere dense in $[0, \infty)$ if and only if $\underline{d}(A) + \overline{d}(A) \ge 1$.

Now, if $\delta/\gamma \to \sqrt{a}$ then $\underline{d}(A) \to 1/(\sqrt{a} + 1)$ and if $\sqrt{a} \to 1 + 0$ then $\underline{d}(A) \to 1/2 - 0$. This completes the proof of Theorem 1.

Note that since $\underline{d}(A)/((\alpha_1/\beta_1)\overline{d}(A)) \to 1$ as $\gamma/\delta \to 1$ and $\overline{d}(A)/(1 - (\beta_1 - \alpha_1)) \to 1$ as $a \to \infty$, we cannot extend (1) and (2) to

$$\underline{d}(A) \le c(\alpha/\beta) \min(1 - \overline{d}(A), \overline{d}(A)) \quad \text{and} \quad \overline{d}(A) \le c(1 - (\beta - \alpha))$$

for some positive constant $c < 1$. ∎

In the sequel we demonstrate that (1) and (2) are necessary but not sufficient conditions for $(\alpha, \beta) \cap R(A) = \emptyset$.

THEOREM 3. *For every pair (γ, γ') satisfying $0 \le \gamma \le \gamma' \le 1$ there exists an $A \subset \mathbb{N}$ such that $\underline{d}(A) = \gamma$, $\overline{d}(A) = \gamma'$ and the ratio set $R(A)$ is everywhere dense in $[0, \infty)$.*

P r o o f. For any infinite set $B \subset \mathbb{N}$ and $\lambda \geq 1$ define $[\lambda B]$ as

$$[\lambda B] = \{[\lambda a] : a \in B\}.$$

Clearly,

(i) either both $R(B)$ and $R([\lambda B])$ are everywhere dense in $[0,\infty)$ or neither is;

(ii) $\underline{d}([\lambda B]) = \underline{d}(B)/\lambda$, $\overline{d}([\lambda B]) = \overline{d}(B)/\lambda$.

If $\gamma' > 0$, put $\lambda = 1/\gamma'$ and then use the well-known fact that for every pair (δ, δ') satisfying $0 \leq \delta \leq \delta' \leq 1$ there exists $B \subset \mathbb{N}$ such that $\underline{d}(B) = \delta$ and $\overline{d}(B) = \delta'$. Applying this for $(\delta,\delta') = (\lambda\gamma, \lambda\gamma')$, bearing in mind that $\lambda\gamma' = 1$ and using [6, Th. 1] we find that $R(B)$ is dense in $[0,\infty)$. Accordingly, $A = [\lambda B]$ is the desired set.

If $\gamma' = 0$ we can put $A = \mathbb{P}$, the set of all primes, since by A. Schinzel (cf. [7, p. 155]) $R(\mathbb{P})$ is everywhere dense in $[0,\infty)$. ∎

3. Applications. Applying Theorem 1 we give some new classes of $A \subset \mathbb{N}$ having dense $R(A)$.

THEOREM 4. *Let $f(t)$, $t \geq 1$, be a strictly increasing continuous function with inverse function $f^{-1}(t)$. Assume that*

(i) $\lim_{t\to\infty} f(t) = \infty$,

(ii) $\lim_{n\to\infty}(f^{-1}(n+1) - f^{-1}(n)) = \infty$,

(iii) $\lim_{n\to\infty} \frac{f^{-1}(n+x)-f^{-1}(n)}{f^{-1}(n+1)-f^{-1}(n)} = \psi(x)$ *exists for every* $x \in [0,1]$,

and for $x \in [0,1]$ *put*

(iv) $\liminf_{n\to\infty} f^{-1}(n)/f^{-1}(n+x) = \chi(x)$,

(v) $A_x = \{n \in \mathbb{N} : \{f(n)\} \in [0,x)\}$, *where* $\{f(n)\}$ *is the fractional part of* $f(n)$.

If $\psi(x) + 1 - \chi(x)(1-\psi(x)) \geq 1$, *then* $R(A_x)$ *is everywhere dense in* $[0,\infty)$.

P r o o f. Observe that $\underline{d}(A_x)$ and $\overline{d}(A_x)$ have the same meaning as the lower and upper distribution functions of $f(n)$ mod 1 (cf. [5, Def. 7.1, p. 53]), hence the theorem follows from [5, Th. 7.7, p. 58] and our Corollary. ∎

Applying Theorem 4 to $f(t) = \log t$ we deduce that $x \geq 1/2$ implies the density of $R(A_x)$. Since in this case the set A_x has the form described in Example 1 with $\gamma = 1$, $\delta = e^x$ and $a = e$, it follows that $x \geq 1/2$ is also necessary for the density of $R(A_x)$ to hold.

For another application of Theorem 1 we make use of [4]. Let $a > 1$ be an integer and $\mathcal{A}$ consist of all $A \subset \mathbb{N}$ containing no 3-term progressions of the form k, kq, kq^2, where $k \in \mathbb{N}$ and $q \in \{a, a^2, a^3, a^4\}$. It is proved in [4, Ex. 2] that $\sup_{A\in\mathcal{A}} \underline{d}(A) \geq (1-a^{-1})(1+a^{-1}+a^{-3}+a^{-4})(a^9/(a^9-1))$, which, together with our Theorem 1, implies that $R(A)$ is everywhere dense in $[0,\infty)$ for some $A\in\mathcal{A}$.

4. Complement of the limit points of the ratio set. As before, assume that $A\subset\mathbb{N}$ is ordered into the sequence $a_1<a_2<\ldots$ and consider the ratio set $R(A)$ as a double sequence a_m/a_n, $m,n=1,2,\ldots$ We introduce two further sets:

(i) $R(A)^l$ is the set of all limit points $x=\lim_{i\to\infty}a_{m_i}/a_{n_i}$ of $R(A)$.

(ii) $R(A)^d$ is the set of all accumulation points of $R(A)$, i.e. the points x which can be expressed as a limit $x=\lim_{i\to\infty}a_{m_i}/a_{n_i}$ of a one-to-one sequence a_{m_i}/a_{n_i}.

Clearly, $R(A)^l$ and $R(A)^d$ are closed. It is shown in [1] that for every system of pairwise disjoint open intervals (α_i,β_i), $i\in\mathcal{I}$, there exists $A\subset\mathbb{N}$ such that $[0,1]-R(A)^d=\bigcup_{i\in\mathcal{I}}(\alpha_i,\beta_i)$ and the same proof applies to $R(A)^l$. To extend the above result of [1] we prove

Theorem 5. *If $\underline{d}(A)>0$ and $[0,1]-R(A)^l\neq\emptyset$, then*

$$[0,1]-R(A)^l=\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i),$$

where $\alpha_i<\beta_i$ and $(\alpha_i,\beta_i)\cap(\alpha_j,\beta_j)=\emptyset$ for $i\neq j$.

Proof. We divide the proof into three steps.

1. Let $\gamma>0$ be a limit point of the form

$$\gamma=\lim_{n\to\infty}a_{g(n)}/a_n,\tag{12}$$

where $g(n)$ is a suitable integer sequence. Then

$$(\alpha,\beta)\cap R(A)^l=\emptyset\Rightarrow(\gamma\alpha,\gamma\beta)\cap R(A)^l=\emptyset=(\alpha/\gamma,\beta/\gamma)\cap R(A)^l.\tag{13}$$

Indeed, assuming $\gamma\alpha<\delta<\gamma\beta$ and

$$\delta=\lim_{i\to\infty}a_{m_i}/a_{n_i}$$

we have

$$\frac{\delta}{\gamma}=\frac{\lim_{i\to\infty}a_{m_i}/a_{n_i}}{\lim_{i\to\infty}a_{g(n_i)}/a_{n_i}}=\lim_{i\to\infty}\frac{a_{m_i}}{a_{g(n_i)}},$$

which is a contradiction. Repeating (13) yields $(\gamma^k\alpha,\gamma^k\beta)\cap R(A)^l=\emptyset$ for all $k\in\mathbb{Z}$.

2. Using all points $\gamma,\delta,\eta,\ldots$ of the form (12) we can define a group

$$G(A)=\{\gamma^i\delta^j\eta^k\ldots:i,j,k,\ldots\in\mathbb{Z}\}.$$

Let $[0,1]-R(A)^l=\bigcup_{i\in\mathcal{I}}(\alpha_i,\beta_i)$. Applying (13) for $t\in G(A)\cap[0,1]$ and $i\in\mathcal{I}$, we get some $j,k\in\mathcal{I}$ such that $(t\alpha_i,t\beta_i)\subset(\alpha_j,\beta_j)$ and $(t^{-1}\alpha_j,t^{-1}\beta_j)\subset$

(α_k, β_k). This implies $i = k$ and

$$(t\alpha_i, t\beta_i) = (\alpha_j, \beta_j).$$

For a fixed $(\alpha_{i_0}, \beta_{i_0})$, the intervals $(t\alpha_{i_0}, t\beta_{i_0})$, $t \in G(A) \cap (0,1)$, are nonoverlapping, which implies that $\mathcal{I}$ is infinite. Moreover, $G(A)$ must be discrete and thus cyclic.

3. Assuming $\underline{d}(A) > 0$, we prove that $G(A) \cap (0,1)$ is nonempty. Let $n/a_n > \theta > 0$ for all sufficiently large n. For any u, v satisfying $0 < u < v < \theta$ we have

$$\frac{a_{[un]}}{a_n} \geq \frac{[un]}{a_n} > u\theta, \qquad \frac{a_{[vn]}}{a_n} \leq \frac{a_{[vn]}}{n} = \frac{a_{[vn]}}{vn} v < \frac{v}{\theta}$$

for all sufficiently large n. Thus, we obtain

$$\frac{a_i}{a_n} \in \left(u\theta, \frac{v}{\theta}\right) \quad \text{for } i \in [[un], [vn]],$$

which implies the existence of $t \in G(A)$ satisfying $t \in [u\theta, v/\theta] \subset (0,1)$.

Note that as the proof of (2) shows, $t \in G(A) \cap [0,1]$ if and only if there exists $k_n < n$ such that $t \in [b_{n-k_n}/b_n, c_{n-k_n}/c_n]$ for all sufficiently large n. ∎

In Example 1 the group $G(A)$ is generated by $1/a$ and the complement of $R(A)^l$ has a simple structure. For general A the complement of $R(A)^l$ may be more complicated.

Example 2. In this example we abbreviate $(\gamma a, \delta a)$ as $(\gamma, \delta)a$. Assume that

$$0 < \gamma_1 < \delta_1 < \gamma_2 < \delta_2 < a\gamma_1 < a\delta_1 \quad \text{and} \quad a > 1$$

and let A be the set of all integer points lying in the pairwise disjoint open intervals

$$(\gamma_1, \delta_1), (\gamma_2, \delta_2), (\gamma_1, \delta_1)a, (\gamma_2, \delta_2)a, \ldots, (\gamma_1, \delta_1)a^n, (\gamma_2, \delta_2)a^n, (\gamma_1, \delta_1)a^{n+1}, \ldots$$

Then, in (5) for this A we get two types of intervals $[b_n, c_n]$, which give (asymptotically) two types of intervals in (8) and which form two sequences of pairwise disjoint open intervals

$$I_2 a^{i-1}, I_1 a^{i-1}, \quad i = 1, 2, \ldots, \quad \text{and} \quad J_2 a^{i-1}, J_1 a^{i-1}, \quad i = 1, 2, \ldots,$$

where

$$I_2 = \left(\frac{\delta_2}{\gamma_2 a}, \ \frac{\gamma_1}{\delta_2}\right), \quad I_1 = \left(\frac{\delta_1}{\gamma_2}, \frac{\gamma_2}{\delta_2}\right), \quad J_2 = \left(\frac{\delta_1}{\gamma_1 a}, \frac{\gamma_2}{\delta_1 a}\right), \quad J_1 = \left(\frac{\delta_2}{\gamma_1 a}, \frac{\gamma_1}{\delta_1}\right).$$

Moreover, there are inclusions between the intervals in (8) and the above intervals, respectively. This guarantees $\overline{R(A)} = R(A)^l$ as well as

$$[0,1] - \overline{R(A)} = \bigcup_{i=1}^{\infty} ((I_2 \cup I_1) \cap (J_2 \cup J_1))a^{i-1}.$$

The intersection $(I_2 \cup I_1) \cap (J_2 \cup J_1)$ consists of at most three pairwise disjoint open intervals (α_1, β_1), (α_1', β_1') and (α_1'', β_1'').

In all cases the group $G(A)$ is cyclic with generator $1/a$ or $1/\sqrt{a}$ depending on $(\gamma_2, \delta_2) = (\gamma_1, \delta_1)\sqrt{a}$.

Applying (6) and (7) we have

$$\underline{d}(A) = \min\left(\frac{(\delta_1 - \gamma_1) + (\delta_2 - \gamma_2)}{\gamma_1} \cdot \frac{1}{a-1}, \frac{1}{\gamma_2}\left((\delta_1 - \gamma_1)\frac{a}{a-1} + (\delta_2 - \gamma_2)\frac{1}{a-1}\right)\right),$$

$$\overline{d}(A) = \max\left(\frac{(\delta_1 - \gamma_1) + (\delta_2 - \gamma_2)}{\delta_2} \cdot \frac{a}{a-1}, \frac{1}{\delta_1}\left((\delta_1 - \gamma_1)\frac{a}{a-1} + (\delta_2 - \gamma_2)\frac{1}{a-1}\right)\right).$$

For example, putting $\gamma_1 = 1$, $\delta_1 = 2$, $\gamma_2 = 5$ and $a = 40$, we have

$$(\alpha_1, \beta_1) = (2/5, 1/2), \quad (\alpha_1', \beta_1') = (6/40, 1/6), \quad (\alpha_1'', \beta_1'') = (2/40, 5/80).$$

Further, $\underline{d}(A) = 2/39$, $\overline{d}(A) = 41/78$, $|[0,1] - \overline{R(A)}| = 31/234$ and $G(A)$ is generated by $1/40$. ∎

5. Extension of Theorem 2. In this part we extend (1) and (2) to intervals $(\alpha, \beta) \subset [0,1]$ satisfying

$$(\alpha, \beta) \cap R(A)^d = \emptyset,$$

which does not follow from Theorem 2 directly. Clearly, if $(\alpha, \beta) \cap R(A)^d = \emptyset$ then for every $\varepsilon > 0$ there exist finitely many pairwise disjoint open intervals (α_i, β_i), $i = 1, \ldots, s$, such that

(i) $\bigcup_{i=1}^{s} (\alpha_i, \beta_i) \subset (\alpha, \beta)$,

(ii) $\beta - \alpha - \sum_{i=1}^{s} (\beta_i - \alpha_i) < \varepsilon$,

(iii) $\forall (1 \le i \le s)(\alpha_i, \beta_i) \cap R(A) = \emptyset$.

So, Theorem 2 only implies

$$\underline{d}(A) \le \min_{1 \le i \le s} \frac{\alpha_i}{\beta_i} \min(1 - \overline{d}(A), \overline{d}(A)) \tag{14}$$

and

$$\overline{d}(A) \le \min_{1 \le i \le s} (1 - (\beta_i - \alpha_i)). \tag{15}$$

Finally, in what follows we will replace the open interval (α,β) with an open set $X\subset[0,1]$ and prove estimates better than (14) and (15). Here $|X|$ denotes the Lebesgue measure of X.

THEOREM 6. *Let X be an open set in $[0,1]$ and write $g(x)=|X\cap[0,x)|$. If $X\cap R(A)^d=\emptyset$, then*

$$\underline{d}(A)\le\frac{x}{y}\min(1-\overline{d}(A),\overline{d}(A))+\frac{(y-g(y))-(x-g(x))}{y}\tag{16}$$

for every x,y satisfying

(i) $0\le x<y\le1$,

(ii) *there exist two sequences x_k and $\delta_k>0$ such that $(x_k,x_k+\delta_k)\cap R(A)^d=\emptyset$ for every k and $x_k\to x$ as $k\to\infty$.*

Moreover

$$\overline{d}(A)\le1-|X|.\tag{17}$$

Proof. The proof is similar to the proof of Theorem 2. Instead of (3) we start with the following pairwise disjoint intervals:

$$(xa_{[\theta n]},xa_{[\theta n]}+x),(xa_{[\theta n]+1},xa_{[\theta n]+1}+x),\ldots,\\ (xa_{n-1},xa_{n-1}+x),(xa_n,ya_n).\tag{18}$$

First assume that

(ii)′ $(x,x+\delta)\cap R(A)=\emptyset$ for some $\delta>0$.

Then for sufficiently large i, the interval (xa_i,xa_i+x) cannot intersect A, since $(xa_i,xa_i+\delta a_i)\cap A=\emptyset$. Moreover, for all sufficiently small $\varepsilon>0$, the set $X\cap(x,y)$ can be approximated by a finite sequence of pairwise disjoint open intervals (α_i,β_i), $i=1,\ldots,s$, such that $\bigcup_{i=1}^s(\alpha_i,\beta_i)\subset X\cap(x,y)$, $|X\cap(x,y)-\bigcup_{i=1}^s(\alpha_i,\beta_i)|<\varepsilon$ and $\bigcup_{i=1}^s(\alpha_i,\beta_i)\cap R(A)=\emptyset$. Hence, the number of terms of $B=\mathbb{N}-A$ lying in (xa_n,ya_n) is greater than $a_n(g(y)-g(x)-\varepsilon)-s$ and we have

$$B(ya_n)\ge(n-[\theta n])(x-1)+(a_n(g(y)-g(x)-\varepsilon)-s)+B(xa_{[\theta n]}).$$

Replacing n by nk and x by xk and letting $k\to\infty$ we find (16).

In the general case, since $g(x)$ is continuous, (ii)′ can be replaced by (ii).

To prove (17) note only that (10) can be replaced by

$$\frac{b_2-c_1}{c_n}+\frac{b_3-c_2}{c_n}+\ldots+\frac{b_n-c_{n-1}}{c_n}\ge\sum_{i=1}^s(\beta_i-\alpha_i).\ \blacksquare$$

Observe that in Example 1 we have $\alpha_i/\beta_i=\delta^2/(\gamma^2a)$ and the minimum of the right hand side of (16) is the same as in (14) and (1). In Example 2, for $x = \alpha''$ and $y = \beta'$, the right hand side of (16) equals $0.229\ldots$; further, the right hand side of (14) is $0.379\ldots$

6. Concluding remarks

1. The results of T. Šalát mentioned in the introduction can be proved directly by using (1) and (2):

(i) Assume $(\alpha,\beta)\cap R(A)=\emptyset$. If $0<d(A)=\underline{d}(A)=\overline{d}(A)<1/2$ then by (1) we have $d(A)\le\frac{\alpha}{\beta}d(A)$ which is a contradiction. If $d(A)\ge 1/2$, then in view of (1) we have $d(A)\le\frac{\alpha}{\beta}(1-d(A))\le\frac{\alpha}{\beta}\frac{1}{2}<\frac{1}{2}$ which also gives a contradiction. Thus (cf. [6, Th. 4]) $d(A)>0$ implies that $R(A)$ is everywhere dense.

(ii) Assuming $\overline{d}(A)=1$, (2) implies a contradiction $1\le 1-(\beta-\alpha)$; thus (cf. [6, Th. 1]) $\overline{d}(A)=1$ implies that $R(A)$ is everywhere dense.

2. It is proved in [3, Th. 2] that if $\mathbb{N}=A\cup B$, then at least one of $R(A)$ or $R(B)$ is everywhere dense in $[0,\infty)$. This can also be proved by using our basic relations (1) and (2).

Assume that $\mathbb{N}=A\cup B$, $A\cap B=\emptyset$, $(\alpha,\beta)\cap R(A)=\emptyset$ and $(\alpha',\beta')\cap R(B)=\emptyset$. Since $\underline{d}(A)=1-\overline{d}(B)$ and $\overline{d}(A)=1-\underline{d}(B)$, applying (1) and (2) we get

(i) $(\beta-\alpha)\le\underline{d}(B)$,

(ii) $\underline{d}(B)\le\frac{\alpha'}{\beta'}(1-\overline{d}(B))$,

(iii) $1-\overline{d}(B)\le\frac{\alpha}{\beta}\underline{d}(B)$.

Starting with (i) and then repeatedly applying (ii) and (iii) we get $\beta-\alpha=0$.

3. A related question is studied in [2].

References

[1] J. Bukor and J. T. Tóth, *On accumulation points of ratio sets of positive integers*, Amer. Math. Monthly 103 (1996), 502–504.

[2] J. Bukor, P. Erdős, T. Šalát and J. T. Tóth, *Ratio sets of sets of positive numbers and a class of decomposition of the set* ℕ, Math. Slovaca 47 (1997), 517–526.

[3] J. Bukor, T. Šalát and J. T. Tóth, *Remarks on R-density of sets of numbers*, Tatra Mt. Math. Publ. 11 (1997), 159–165.

[4] W. Klotz, *Generalization of some theorems on sets of multiplies and primitive sequences*, Acta Arith. 32 (1977), 15–26.

[5] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

[6] T. Šalát, *On ratio sets of sets of natural numbers*, Acta Arith. 15 (1969), 273–278; Corrigendum, ibid. 16 (1969), 103.

[7] W. Sierpiński, *Elementary Theory of Numbers*, PWN, Warszawa, 1964.

Mathematical Institute
of the Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava, Slovakia
E-mail: strauch@mau.savba.sk

Department of Mathematics
of the Faculty of Natural Sciences
Constantine the Philosopher University in Nitra
Tr. A. Hlinku 1
SK-949 74 Nitra, Slovakia
E-mail: toth@unitra.sk

Received on 27.1.1998
and in revised form on 31.7.1998 (3325)