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ON APPROXIMATION BY LAGRANGE INTERPOLATING POLYNOMIALS FOR A SUBSET OF THE SPACE OF CONTINUOUS FUNCTIONS

BY

S. P. ZHOU (HANGZHOU)

We construct a C^k piecewise differentiable function that is not C^k piecewise analytic and satisfies a Jackson type estimate for approximation by Lagrange interpolating polynomials associated with the extremal points of the Chebyshev polynomials.

1. Introduction. Let $C^k_{[-1,1]}$ be the class of functions which have k continuous derivatives on $[-1,1]$, in particular, $C^0_{[-1,1]} = C_{[-1,1]}$ be the class of continuous functions on $[-1,1]$. For a function $f \in C_{[-1,1]}$, let $L_n(f,x)$ be the nth Lagrange interpolating polynomial of f associated with the extremal points $\{x_j\} = \{\cos(j-1)\pi/n\}_{j=1}^{n+1}$ of the Chebyshev polynomials in $[-1,1]$. An explicit formula for $L_n(f,x)$ is

$$L_n(f,x) = \frac{\omega_n(x)}{n}\left(\frac{-f(1)}{2(x-1)} + \sum_{l=2}^{n}\frac{(-1)^l f(x_l)}{x-x_l} + \frac{(-1)^{n+1}f(-1)}{2(x+1)}\right),$$

where $\omega_n(x) = \sqrt{1-x^2}\sin(n\arccos x)$.

It is well known that $L_n(f,x)$ does not converge for all $f \in C_{[-1,1]}$. However, Mastroianni and Szabados [MS] considered a subset of $C_{[-1,1]}$ and proved

THEOREM 1. *If $f \in C_{[-1,1]}$ and if there is a partition of $[-1,1]$, $-1 = a_{s+1} < a_s < \ldots < a_0 = 1$, such that each $f|_{[a_{j+1},a_j]}$ $(j = 0,1,\ldots,s)$ is a polynomial, then, for $|x| \le 1$, as $n \to \infty$,*

$$|f(x) - L_n(f,x)| = O\left(\frac{\sqrt{1-x^2}}{n}\right)\min\left\{1, \frac{1}{n\min_{1\le j\le s}|x-a_j|}\right\}.$$

Recently, [Li] improved and generalized the above result to

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THEOREM 2. *If f is C^k piecewise analytic on $[-1,1]$ with singular points $a_1,\ldots,a_s$, then, as $n\to\infty$,*

$$|f(x)-L_n(f,x)| = O\left(\frac{|\omega_n(x)|}{n^{k+1}}\right)\min\left\{1, \frac{1}{n\min_{1\le j\le s}|x-a_j|}\right\} \tag{1}$$

holds uniformly for $x\in[-1,1]$.

[Li] uses the following definition: A function f defined on $[-1,1]$ is called C^k *piecewise analytic* on $[-1,1]$ if $f\in C^k_{[-1,1]}$ and if there is a partition of $[-1,1]$, $-1=a_{s+1}<a_s<\ldots<a_0=1$, such that each $f|_{[a_{j+1},a_j]}$ $(j=0,1,\ldots,s)$ has an analytic continuation to $[-1,1]$. The points $a_1,\ldots,a_s$ are called the *singular points* of f.

At the end of the paper, Li wrote: "*We … note that our term 'piecewise analytic' … has a different meaning from the usual one: We require that each analytic piece has an analytic continuation to $[-1,1]$. The technical requirement is needed because of our method. We do not know if there exists a function that is not C^k piecewise analytic but satisfies the estimation* (1)."

In the present paper we construct a C^k piecewise differentiable function that is not C^k piecewise analytic and satisfies (1). Finally, based on some observations, we raise an open question.

2. Result

DEFINITION. Let $k\ge 0$. A function f defined on $[-1,1]$ is called C^k *piecewise differentiable* on $[-1,1]$ with *singular points* $a_1,\ldots,a_s$ if $f\in C^k_{[-1,1]}$ and if there is a partition of $[-1,1]$, $-1=a_{s+1}<a_s<\ldots<a_0=1$, such that each $f|_{[a_{j+1},a_j]}$ $(j=0,1,\cdots,s)$ has $k+1$ continuous derivatives on $[a_{j+1},a_j]$, but f does not have the $(k+1)$th derivative at the singular points $a_1,\ldots,a_s$.

THEOREM 3. *Let $k\ge 0$. There is a C^k piecewise differentiable function $f(x)$ on $[-1,1]$ with singular point zero which does not have the $(k+2)$th derivative at the endpoints ±1 and for which*

$$|f(x)-L_n(f,x)| = O\left(\frac{|\omega_n(x)|}{n^{k+1}}\right)\min\left\{1, \frac{1}{nx}\right\} \tag{2}$$

holds uniformly for $x\in[-1,1]$.

Proof. Let $T_n(x)=\cos(n\arccos x)$ be the Chebyshev polynomial of degree n, and

$$S(x)=\begin{cases} x^{k+1}, & x\ge 0,\\ 0, & x<0.\end{cases}$$

Set $n_1 = 2$, and $n_{j+1} = n_j^2$ for $j = 1, 2, \ldots$ Define

$$f(x) = T(x) + S(x) := \sum_{j=1}^{\infty} n_j^{-2k-4} T_{n_j}(x) + S(x).$$

We show this is the desired function.

Obviously $f \in C^k_{[-1,1]}$. The argument that $f|_{[-1,0]}$ and $f|_{[0,1]}$ have $k+1$ continuous derivatives on $[-1,0]$ and $[0,1]$ respectively is also trivial. Also, we note that $T(x)$ evidently has $2k+3$ continuous derivatives at zero and $S(x)$ does not have the $(k+1)$th derivative there; consequently, $f(x)$ does not have the $(k+1)$th derivative at zero. All those facts mean that f is C^k piecewise differentiable on $[-1,1]$ with singular point zero.

Let $n_j \le n < n_{j+1}$, $j = 1, 2, \ldots$ For any $x \in [-1,1]$ we have

$$T(x) - L_n(T, x) = J(x) - L_n(J, x),$$

where $J(x) = \sum_{l=j+1}^{\infty} n_l^{-2k-4} T_{n_l}(x)$. Now for $l \ge j+1$,

$$\begin{aligned}
&T_{n_l}(x) - L_n(T_{n_l}, x)\\
&\qquad = \frac{\omega_n(x)}{n}\biggl(\frac{T_{n_l}(1) - T_{n_l}(x)}{2(x-1)} + \sum_{m=2}^{n} \frac{(-1)^m (T_{n_l}(x) - T_{n_l}(x_m))}{x - x_m}\\
&\qquad\quad + \frac{(-1)^{n+1}(T_{n_l}(x) - T_{n_l}(-1))}{2(x+1)}\biggr).
\end{aligned}$$

Hence there are ξ_m between x and x_m and an absolute constant $C > 0$ such that

$$\begin{aligned}
&|T_{n_l}(x) - L_n(T_{n_l}, x)|\\
&\qquad \le \frac{|\omega_n(x)|}{n}\left|\frac{-T'_{n_l}(\xi_1)}{2} + \sum_{m=2}^{n}(-1)^m T'_{n_l}(\xi_m) + \frac{(-1)^{n+1}T'_{n_l}(\xi_{n+1})}{2}\right|\\
&\qquad \le C n_l^2 |\omega_n(x)| \qquad \text{(by the Markov inequality)}.
\end{aligned}$$

Then

$$\begin{aligned}
|J(x) - L_n(J, x)| &\le C|\omega_n(x)| \sum_{l=j+1}^{\infty} n_l^{-2k-2}\\
&\le C n_{j+1}^{-2k-2} |\omega_n(x)| \le C n^{-k-2} |\omega_n(x)|.
\end{aligned}$$

Together with the known result (see [Li]) that

$$|S(x) - L_n(S, x)| = O\left(\frac{|\omega_n(x)|}{n^{k+1}}\right) \min\left\{1, \frac{1}{nx}\right\}$$

holds uniformly for $x \in [-1,1]$, we have thus finished the proof of (2).

Denote by $t_k^{(j)}$ the largest zero of $T_{n_j}^{(k+1)}(x)$. We know that

$$n_j^{-2} \le 1 - t_k^{(j)} \le 1 - \cos\frac{(k+2)\pi}{n_l} \le \frac{(k+2)^2\pi^2}{2n_l^2} \tag{3}$$

since the zeros of $T_{n_j}^{(l+1)}(x)$ interlace those of $T_{n_j}^{(l)}(x)$ for $l = 0, 1, \ldots$ We also notice that $(T_{n_j}^{(k+1)}(1) - T_{n_j}^{(k+1)}(x))/(1-x) = T_{n_j}^{(k+2)}(\xi) > 0$, $\xi \in [x, 1]$, for $x \in [t_k^{(j)}, 1]$, so that by (3),

$$\begin{aligned}\frac{T_{n_j}^{(k+1)}(1) - T_{n_j}^{(k+1)}(x)}{1-x} &> \frac{T_{n_j}^{(k+1)}(1) - T_{n_j}^{(k+1)}(t_k^{(j)})}{1 - t_k^{(j)}} \\ &\ge \frac{2n_j^2}{(k+2)^2\pi^2} T_{n_j}^{(k+1)}(1) \ge C_k n_j^{2k+4}\end{aligned} \tag{4}$$

for $x \in [t_k^{(j)}, 1]$, where C_k is a positive constant only depending upon k. Write

$$\begin{aligned}&\frac{T^{(k+1)}(1) - T^{(k+1)}(t_k^{(j)})}{1 - t_k^{(j)}} \\ &\quad = \frac{1}{1 - t_k^{(j)}} \sum_{l=1}^{j} n_l^{-2k-4}(T_{n_l}^{(k+1)}(1) - T_{n_l}^{(k+1)}(t_k^{(j)})) \\ &\qquad + \frac{1}{1 - t_k^{(j)}} \sum_{l=j+1}^{\infty} n_l^{-2k-4}(T_{n_l}^{(k+1)}(1) - T_{n_l}^{(k+1)}(t_k^{(j)})) =: I_1 + I_2.\end{aligned}$$

We check that, by (4),

$$I_1 \ge C_k j,$$

while by inequality (3), the Markov inequality and the definition of $\{n_j\}$,

$$|I_2| = O(1)n_j^2 \sum_{l=j+1}^{\infty} n_l^{-2k-4} \|T_{n_l}^{(k+1)}\| = O(1)n_j^2 n_{j+1}^{-2} = O(1)n_{j+1}^{-1}.$$

Therefore

$$\frac{T^{(k+1)}(1) - T^{(k+1)}(t_k^{(j)})}{1 - t_k^{(j)}} \ge C_k j.$$

As $\lim_{j\to\infty} t_k^{(j)} = 1$, we conclude that $T(x)$ does not have the $(k+2)$th derivative at the endpoint 1. The same conclusion holds for the other endpoint -1. By noting that $S(x)$ has derivatives of any order at ± 1, we have thus proved that $f(x)$ does not have the $(k+2)$th derivative at ± 1. The proof of Theorem 3 is complete. ∎

3. Remark. Based on observations and calculations, we have reasons to believe that C^k piecewise differentiable functions might achieve the required Jackson type estimate (1). Precisely, we raise the following question:

PROBLEM. *Let $k \geq 0$. If f is a C^k piecewise differentiable function on $[-1,1]$ with singular points $a_1, \ldots, a_s$, does it hold that, for $|x| \leq 1$, as $n \to \infty$,*

$$|f(x) - L_n(f,x)| = O\left(\frac{|\omega_n(x)|}{n^{k+1}}\right) \min\left\{1, \frac{1}{n \min_{1\leq j\leq s} |x - a_j|}\right\}?$$

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Department of Mathematics
Hangzhou University
Hangzhou, Zhejiang 310028
China

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