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ON A THEOREM OF MIERCZYŃSKI

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We prove that the initial value problem $x'(t) = f(t, x(t))$, $x(0) = x_1$ is uniquely solvable in certain ordered Banach spaces if f is quasimonotone increasing with respect to x and f satisfies a one-sided Lipschitz condition with respect to a certain convex functional.

1. Introduction. Let $(E, \|\cdot\|)$ be a real Banach space and E^* its topological dual space. We consider a partial ordering $\leq$ on E induced by a cone K. A *cone* K is a closed convex subset of E with $\lambda K \subseteq K$, $\lambda \geq 0$, and $K \cap (-K) = \{0\}$. In the sequel we will always assume that K is *solid* (i.e. $\operatorname{Int} K \neq \emptyset$). We define $x \leq y \Leftrightarrow y - x \in K$, and we use the notations $x \ll y$ for $y - x \in \operatorname{Int} K$ and K^* for the *dual cone*, i.e., the set of all functionals $\varphi \in E^*$ with $\varphi(x) \geq 0$, $x \geq 0$. Thus E^* is ordered by $\varphi \leq \psi \Leftrightarrow \psi - \varphi \in K^*$. The cone K is *normal* if there is a $\gamma \geq 1$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq \gamma \|y\|$. For $x, y \in E$ with $x \leq y$, we define the order interval $[x, y] = \{z \in E : x \leq z \leq y\}$. By $K(x, r)$ we will always denote the open ball $\{y \in E : \|y - x\| < r\}$.

Now fix $p \gg 0$. In the sequel we will assume that $\|\cdot\|$ is the Minkowski functional of $[-p, p]$. This is an equivalent renorming of E (see e.g. [7]). Then for $x, y \in E$ we have $0 \leq x \leq y \Rightarrow \|x\| \leq \|y\|$, and $\|x\| \leq c \Leftrightarrow -cp \leq x \leq cp$.

Let $f : [0, T] \times E \to E$ be continuous and let $x_1 \in E$. We consider the initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_1. \tag{1}$$

Let $D \subset E$. A function $f : [0, T] \times D \to E$ is called *quasimonotone increasing* (in the sense of Volkmann [12]) if

$$x, y \in D,\ t \in [0, T],\ x \leq y,\ \varphi \in K^*,\ \varphi(x) = \varphi(y) \Rightarrow \varphi(f(t, x)) \leq \varphi(f(t, y)).$$

In [10] Mierczyński proved the following theorem (for a more general result see Mierczyński [11]):

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Theorem 1. *Let* $E = \mathbb{R}^n$, $K = \{(x_1, \ldots, x_n) : x_k \geq 0, k = 1, \ldots, n\}$ *and let* $f : [0,T] \times E \to E$ *be continuous and quasimonotone increasing with*

$$\sum_{k=1}^{n} f_k(t,x) = 0, \qquad (t,x) \in [0,T] \times E.$$

Then there exists precisely one solution of problem (1).

References [1], [3], [4], [5], [6], [7] and especially [13] give a survey on quasimonotonicity as applied to problem (1). For example, if f is continuous, bounded and quasimonotone increasing and if the cone K is regular then problem (1) is solvable on $[0,T]$; see [7]. A cone is called *regular* if every monotone increasing sequence in E which is order bounded, is convergent. If K is only supposed to be normal, even monotonicity of f does not imply existence of a solution; see [4]. So in this case additional assumptions on f are needed to obtain existence of a solution of problem (1).

In Theorem 1 we have $E = E^*$, $K = K^*$, and with $\psi(x) = (1, \ldots, 1)x$ condition (2) says: $\psi(f(t,x)) = 0$, $(t,x) \in [0,T] \times E$.

Conditions of this type are considered in several papers pertaining to limit sets of autonomous differential equations in $\mathbb{R}^n$ with the natural cone (see e.g. [9], [10] and the references given there). We will study conditions of this type which imply both uniqueness and existence of a solution for problem (1). To this end we consider the set W of all continuous functions $\psi : E \to \mathbb{R}$ with the following properties:

1. $\psi(x) \geq 0$, $x \in K$.
2. $\psi(x+y) \leq \psi(x) + \psi(y)$, $x, y \in E$.
3. $\psi(\lambda x) = \lambda \psi(x)$, $x \in E$, $\lambda \geq 0$.
4. Every monotone decreasing sequence $(x_n)_{n=1}^{\infty}$ in K with $\lim_{n\to\infty} \psi(x_n) = 0$ tends to zero with respect to the norm.

For $\psi \in W$ we consider the one-sided derivative

$$m_{\psi-}[x,y] = \lim_{h \to 0-} (\psi(x + hy) - \psi(x))/h, \qquad x, y \in E.$$

For $x, y, z \in E$ we have

$$m_{\psi-}[x,y] \leq \psi(y), \qquad m_{\psi-}[x, y+z] \leq m_{\psi-}[x,y] + \psi(z),$$

and if $u : [0,T] \to E$ is left-differentiable on $(0,T]$, then

$$(\psi(u))'_{-}(t) = m_{\psi-}[u(t), u'(t)], \qquad t \in (0,T].$$

For this and further properties of the function $m_{\psi-}$ see [8]. Note that if ψ is linear, then $m_{\psi-}[x,y] = \psi(y)$, $x, y \in E$.

We will prove the following theorem:

Theorem 2. *Let E be a Banach space ordered by a normal*, *solid cone K. Let $x_0 \in E$, and let $f : [0,T] \times E \to E$ be a continuous function with the* *following properties*:

1. *f is quasimonotone increasing.*
2. *There exist $\psi \in W$ and $L \in \mathbb{R}$ such that*

$$m_{\psi-}[y-x, f(t,y)-f(t,x)] \le L\psi(y-x) \quad \textit{for } (t,x),(t,y) \in [0,T]\times E,\ x \ll y.$$

Then there exist $r > 0$ and $\tau \in (0,T]$ such that problem (1) *is uniquely solvable on $[0,\tau]$ for every $x_1 \in K(x_0,r)$, and the solution depends continuously on $x_1 \in K(x_0,r)$.*

Let f have in addition the following properties:

3. *For every bounded set $M \subset E$ the set $f([0,T]\times M)$ is bounded.*
4. *There exists a function $q \in C^1([0,T], \operatorname{Int} K)$ and $A, B \ge 0$ such that*

$$\|f(t, sq(t))\| \le A|s| + B, \qquad t \in [0,T],\ s \in \mathbb{R}.$$

Then problem (1) *is uniquely solvable on $[0,T]$.*

Remarks. 1. For the case $\psi = \|\cdot\|$ Theorem 2 is related to Martin's Theorem [8], p. 232.

2. Condition 2 holds if there exists $L \in \mathbb{R}$ such that

$$\psi(f(t,y) - f(t,x)) \le L\psi(y-x), \qquad (t,x),(t,y) \in [0,T]\times E,\ x \le y.$$

3. Suppose $\psi \in E^*$ and $K = \{x \in E : \psi(x) \ge \alpha\|x\|\}$ with $0 < \alpha < \|\psi\|$. Then K is a regular cone with $\operatorname{Int} K \ne \emptyset$, and if $(x_n)_{n=1}^\infty$ is a sequence in K (not necessarily decreasing) such that $\lim_{n\to\infty}\psi(x_n) = 0$, then $\lim_{n\to\infty} x_n = 0$. Hence $\psi \in W$.

4. Consider the Banach space c of all convergent sequences $x = (x_k)_{k=1}^\infty$ with norm $\|x\| = \sup_{k\in\mathbb{N}}|x_k|$, and let $K = \{x \in c : x_k \ge 0, k \in \mathbb{N}\}$. Then K is normal and $\operatorname{Int} K \ne \emptyset$, for example $p = (1)_{k=1}^\infty \in \operatorname{Int} K$. Now let $(\alpha_k)_{k=1}^\infty \in l^1$ with $\alpha_k > 0$, $k \in \mathbb{N}$, and define

$$\psi(x) = \sum_{k=1}^\infty \alpha_k x_k + \lim_{k\to\infty} x_k.$$

Then $\psi \in W \cap c^*$.

5. Consider the Banach space l^∞ of all bounded sequences $x = (x_k)_{k=1}^\infty$ with norm $\|x\| = \sup_{k\in\mathbb{N}}|x_k|$, and let $K = \{x \in l^\infty : x_k \ge 0, k \in \mathbb{N}\}$. Then K is normal and $\operatorname{Int} K \ne \emptyset$, for example $p = (1)_{k=1}^\infty \in \operatorname{Int} K$. Again let $(\alpha_k)_{k=1}^\infty \in l^1$ with $\alpha_k > 0$, $k \in \mathbb{N}$, and define

$$\psi(x) = \sum_{k=1}^\infty \alpha_k x_k + \limsup_{k\to\infty} x_k.$$

Then $\psi \in W$. Note that ψ is nonlinear.

6. A possible way to find linear functionals $\psi \in W$ is the following: Let $\psi \in K^*$ and consider a set

$$M \subset \{\varphi \in K^* : \|\varphi\| = 1,\ \exists c \geq 0 : \varphi \leq c\psi\}.$$

If M is weak-$*$ compact and if $\sup\{|\varphi(x)| : \varphi \in M\}$ is an equivalent norm on E, then $\psi \in W$. This is an easy consequence of Dini's Theorem.

7. Condition 4 in Theorem 2 holds if $\|f(t,x)\| \leq A\|x\| + B$, $(t,x) \in [0,T] \times E$, for some constants $A, B \geq 0$.

8. Using Theorem 2 one can prove existence of a solution of problem (1) for right-hand sides which do not satisfy classical existence criteria such as one-sided Lipschitz conditions, conditions formulated with measures of noncompactness, or classical monotonicity conditions.

From Theorem 2 we get the following corollary for the autonomous case:

COROLLARY 1. *Let E and K be as in Theorem* 2 *and let $f : E \to E$ be a continuous function such that*:

1. *f is quasimonotone increasing.*
2. *For every bounded set $M \subset E$, the set $f(M)$ is bounded.*
3. *There exists $\psi \in W \cap E^*$ such that $\psi(f(x)) = 0$, $x \in E$.*
4. *There exist $q \in \operatorname{Int} K$ and $A, B \geq 0$ such that*

$$\|f(sq)\| \leq A|s| + B, \qquad s \in \mathbb{R}.$$

Then the initial value problem $x'(t) = f(x(t))$, $x(0) = x_0$ is uniquely solvable on $[0,\infty)$, and the solution is continuously dependent on the initial value (*in the sense of compact convergence*).

Moreover, *if $x : [0,\infty) \to E$ is a solution of $x'(t) = f(x(t))$ and $t_1 \neq t_2$ then x is periodic for $t \geq \min\{t_1, t_2\}$ if $x(t_1)$ and $x(t_2)$ are comparable.*

To prove the last part of Corollary 1 note that $\psi(x(t)) = \psi(x(0))$, $t \in [0,\infty)$. Hence if for example $x(t_1) \leq x(t_2)$, we have $\psi(x(t_2) - x(t_1)) = 0$, which implies $x(t_1) = x(t_2)$. Thus $x(t)$, $t \geq \min\{t_1, t_2\}$, has $|t_1 - t_2|$ as a period. Note that under the conditions of Corollary 1 we do not have uniqueness to the left. Consider for example $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (-\sqrt[3]{x}, \sqrt[3]{x})$ (K the natural cone and $\psi(x,y) = x + y$).

We will use Theorem 2 to prove the following:

THEOREM 3. *Let E, K be as in Theorem* 2, *let $f : [0,T] \times E \to E$ be continuous*, *let f satisfy conditions* 1 *and* 2 *in Theorem* 2, *and let $u, v \in C^1([0,T], E)$ be such that*

$$u(0) \leq v(0), \qquad u'(t) - f(t, u(t)) \leq v'(t) - f(t, v(t)), \quad t \in [0,T].$$

Then $u(t) \leq v(t)$, $t \in [0,T]$.

This means, in particular, that the solution of problem (1) depends monotonically on the initial value.

2. Approximate solutions. To prove our theorems we will use the following results. Theorem 4 is due to Volkmann [12] and for Theorem 5 see [2], Theorem 1.1.

THEOREM 4. *Let* $D \subset E$, *let* $f : [0,a] \times D \to E$ *be quasimonotone increasing*, *and let* $u, v : [0,a] \to D$ *be differentiable functions with*

$$u(0) \ll v(0), \quad u'(t) - f(t,u(t)) \ll v'(t) - f(t,v(t)), \quad t \in [0,a].$$

Then $u(t) \ll v(t)$, $t \in [0,a]$.

THEOREM 5. *Let* $D = \overline{K(x_1,r)}$, *and let* $f : [0,a] \times D \to E$ *be continuous with* $\|f(t,x)\| \le M$ *on* $[0,a] \times D$. *Let* $\varepsilon > 0$ *and* $a_\varepsilon = \min\{a, r/(M+\varepsilon)\}$. *Then there exists* $x_\varepsilon \in C^1([0,a_\varepsilon], D)$ *such that* $x_\varepsilon(0) = x_1$ *and*

$$\|x'_\varepsilon(t) - f(t,x_\varepsilon(t))\| \le \varepsilon, \quad t \in [0,a_\varepsilon].$$

Next we show the existence of a certain kind of approximate solutions for problem (1) (compare [7] for the case of f bounded and quasimonotone increasing).

PROPOSITION 1. *Let* E, K, x_0 *be as in Theorem* 2, *and let* $f : [0,T] \times E \to E$ *be continuous and quasimonotone increasing. Then there exist* $r > 0$ *and* $\tau \in (0,T]$ *such that for each* $x_1 \in K(x_0,r)$ *and each* σ *with* $|\sigma| \le 1$ *there are sequences* $(u_n)_{n=1}^\infty$, $(v_n)_{n=1}^\infty$ *in* $C^1([0,\tau],E)$ *with the following properties*:

1. $u_m(t) \ll u_{m+1}(t) \ll v_{n+1}(t) \ll v_n(t)$, $t \in [0,\tau]$, $m,n \in \mathbb{N}$.
2. $u_m(0) \ll x_1 \ll v_n(0)$, $m,n \in \mathbb{N}$.
3. *Every solution* $x : [0,\tau] \to E$ *of* $x' = f(t,x) + \sigma p$, $x(0) = x_1$, *satisfies* $u_m(t) \ll x(t) \ll v_n(t)$, $t \in [0,\tau]$, $m,n \in \mathbb{N}$.
4. $\lim_{n\to\infty} u_n(0) = \lim_{n\to\infty} v_n(0) = x_1$.
5. *For* $r_n = u'_n - f(\cdot,u_n) - \sigma p$, $n \in \mathbb{N}$, *and* $s_n = v'_n - f(\cdot,v_n) - \sigma p$, $n \in \mathbb{N}$, *we have* $\lim_{n\to\infty} \max_{t\in[0,\tau]} \|r_n(t)\| = \lim_{n\to\infty} \max_{t\in[0,\tau]} \|s_n(t)\| = 0$.

Proof. Since f is continuous there exists $\delta > 0$ such that

$$\|f(t,x)\| \le 1 + \|f(0,x_0)\|, \quad \max\{t, \|x - x_0\|\} \le \delta.$$

We set $r = \delta/3$ and we consider $x_1 \in K(x_0,r)$. Let $(c_n)_{n=1}^\infty$ be a strictly decreasing sequence of real numbers with limit 0 and let $c_1 \le r$. For $n \in \mathbb{N}$ and $(t,x) \in [0,\delta] \times \overline{K(x_1 \pm c_n p, r)}$ we have $\|x - x_0\| \le \delta$, and therefore

$$\left\| f(t,x) + \sigma p \pm \frac{c_n + c_{n+1}}{2} p \right\| \le M := 2 + \|f(0,x_0)\| + c_1.$$

Now set $\tau := \min\{\delta, r/(M+c_1)\} = r/(M+c_1)$. Then, according to Theorem 5, for $n \in \mathbb{N}$ there exist functions u_n and v_n in $C^1([0,\tau],E)$ with

$u_n(0) = x_1 - c_n p$, $v_n(0) = x_1 + c_n p$ and

$$\left\| u_n'(t) - f(t, u_n(t)) - \sigma p + \frac{c_n + c_{n+1}}{2} p \right\| \leq \frac{c_n - c_{n+1}}{4},$$

$$\left\| v_n'(t) - f(t, v_n(t)) - \sigma p - \frac{c_n + c_{n+1}}{2} p \right\| \leq \frac{c_n - c_{n+1}}{4}.$$

By [7] this implies for $t \in [0, \tau]$ and $m, n \in \mathbb{N}$ that

$$\begin{aligned} -c_m p \ll u_m'(t) - f(t, u_m(t)) - \sigma p \ll -c_{m+1} p \\ \ll c_{n+1} p \ll v_n'(t) - f(t, v_n(t)) - \sigma p \ll c_n p. \end{aligned}$$

Application of Theorem 4 leads to $u_m(t) \ll u_{m+1}(t) \ll v_{n+1}(t) \ll v_n(t)$, $t \in [0, \tau]$, $m, n \in \mathbb{N}$, and $u_m(t) \ll x(t) \ll v_n(t)$, $t \in [0, \tau]$, $m, n \in \mathbb{N}$, for any solution $x : [0, \tau] \to E$ of $x' = f(t, x) + \sigma p$, $x(0) = x_1$. The other properties of u_n and v_n follow immediately from the construction of these functions. ∎

3. Proofs

Proof of Theorem 2. Let conditions 1 and 2 hold. We first prove existence and uniqueness of the solution of $x' = f(t, x) + \sigma p$, $x(0) = x_1$. The parameter σ is needed to prove continuous dependence and is also needed in the proof of Theorem 3.

Let $r > 0$ and $\tau \in (0, T]$ as in Proposition 1. We fix $x_1 \in K(x_0, r)$ and σ with $|\sigma| \leq 1$. Let $(u_n)_{n=1}^\infty$, $(v_n)_{n=1}^\infty$ be the approximate solutions as in Proposition 1 and let r_n, s_n, $n \in \mathbb{N}$, be the corresponding defects. Since $u_n(t) \ll v_n(t)$, $n \in \mathbb{N}$, $t \in [0, \tau]$, we see that for $t \in (0, \tau]$ and for a constant $\lambda > 0$,

$$\begin{aligned} (\psi(v_n - u_n))'_-(t) &= m_{\psi-}[v_n(t) - u_n(t), v_n'(t) - u_n'(t)] \\ &\leq m_{\psi-}[v_n(t) - u_n(t), f(t, v_n(t)) - f(t, u_n(t))] \\ &\quad + \psi(s_n(t) - r_n(t)) \\ &\leq L\psi(v_n(t) - u_n(t)) + \lambda(\|s_n(t)\| + \|r_n(t)\|). \end{aligned}$$

Because $\lim_{n\to\infty} \psi(v_n(0) - u_n(0)) = 0$, application of Gronwall's Lemma leads to $\lim_{n\to\infty} \psi(v_n(t) - u_n(t)) = 0$, $t \in [0, \tau]$, and since $v_n(t) - u_n(t)$ is decreasing we have

$$\lim_{n\to\infty} \|v_n(t) - u_n(t)\| = 0, \qquad t \in [0, \tau].$$

As K is normal, Dini's Theorem implies $\lim_{n\to\infty} \|\!|v_n - u_n\|\!| = 0$ in $C([0, \tau], E)$ (endowed with the maximum norm $\|\!|\cdot\|\!|$). Now from $u_m(t) \ll v_n(t)$, $t \in [0, \tau]$, $m, n \in \mathbb{N}$, we find for $t \in [0, \tau]$ and $m \geq n$ that

$$\|v_n(t) - v_m(t)\| \leq \|v_n(t) - u_n(t)\| \leq \|\!|v_n - u_n\|\!|,$$

and therefore $(v_n)_{n=1}^\infty$ is a Cauchy sequence in $C([0, \tau], E)$. Analogously, $(u_n)_{n=1}^\infty$ is a Cauchy sequence in $C([0, \tau], E)$. The limits of both sequences are equal, and this limit is a solution of $x' = f(t,x) + \sigma p$, $x(0) = x_1$. It is unique, since $u_n(t) \ll x(t) \ll v_n(t)$, $t \in [0,\tau]$, $n \in \mathbb{N}$, for every solution $x : [0,\tau] \to E$ of $x' = f(t,x) + \sigma p$, $x(0) = x_1$ (see Proposition 1).

We prove that the solution of problem (1) is continuously dependent on the initial value $x_1 \in K(x_0, r)$.

Let $(x_{1n})_{n=1}^\infty$ be a sequence in $K(x_0, r)$ with limit $x_1 \in K(x_0, r)$, let $x_n : [0,\tau] \to E$ be the solution of $x_n'(t) = f(t, x_n(t))$, $x_n(0) = x_{1n}$, $n \in \mathbb{N}$, and let $x : [0,\tau] \to E$ be the solution of problem (1). Now assume that there exists $\varepsilon > 0$ with $\|x_n - x\| \geq \varepsilon$, $n \in \mathbb{N}$.

There exist strictly decreasing sequences $(\lambda_n)_{n=1}^\infty$ and $(\mu_n)_{n=1}^\infty$ of positive numbers, both with limit 0 and with

$$x_1 \text{ (and } x_{1n}) \gg u_{1n} := x_{1n} - \mu_n p, \qquad n \in \mathbb{N},$$
$$x_1 \text{ (and } x_{1n}) \ll v_{1n} := x_{1n} + \lambda_n p, \qquad n \in \mathbb{N}.$$

There exists $n_0 \in \mathbb{N}$ such that the initial value problems $u_n'(t) = f(t, u_n(t)) - \mu_n p$, $u_n(0) = u_{1n}$, and $v_n'(t) = f(t, v_n(t)) + \lambda_n p$, $v_n(0) = v_{1n}$, have solutions $u_n, v_n : [0,\tau] \to E$ for each $n \geq n_0$. There is a subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=n_0}^\infty$ with

$$u_{1n_{k+1}} \gg u_{1n_k}, \quad v_{1n_{k+1}} \ll v_{1n_k}, \qquad k \in \mathbb{N}.$$

Since λ_n and μ_n are strictly decreasing, Theorem 4 shows for $t \in [0,\tau]$ and $k \in \mathbb{N}$ that

$$u_{n_k}(t) \ll u_{n_{k+1}}(t) \ll x(t) \text{ (and } x_{n_{k+1}}(t)) \ \ll v_{n_{k+1}}(t) \ll v_{n_k}(t).$$

Therefore for $t \in [0,\tau]$ and $k \in \mathbb{N}$ we have

$$u_{n_k}(t) - x(t) \ll x_{n_k}(t) - x(t) \ll v_{n_k}(t) - x(t).$$

Hence

$$\|x_{n_k}(t) - x(t)\| \leq \max\{\|x(t) - u_{n_k}(t)\|, \|v_{n_k}(t) - x(t)\|\},$$

which implies

$$\|x_{n_k} - x\| \leq \max\{\|x - u_{n_k}\|, \|v_{n_k} - x\|\}.$$

Moreover, for $t \in [0,\tau]$ and $k \in \mathbb{N}$ we have

$$0 \ll x(t) - u_{n_{k+1}}(t) \ll x(t) - u_{n_k}(t), \qquad 0 \ll v_{n_{k+1}}(t) - x(t) \ll v_{n_k}(t) - x(t).$$

Now for $t \in (0,\tau]$,

$$(\psi(x - u_{n_k}))'_-(t) \leq L\psi(x(t) - u_{n_k}(t)) + \mu_{n_k}\psi(p),$$

and $\lim_{k\to\infty} \psi(x(0) - u_{n_k}(0)) = 0$. Thus $\lim_{k\to\infty} \psi(x(t) - u_{n_k}(t)) = 0$, $t \in [0,\tau]$, which implies $\lim_{k\to\infty} \|x(t) - u_{n_k}(t)\| = 0$, $t \in [0,\tau]$, since $\psi \in W$. Again by Dini's Theorem we have $\lim_{k\to\infty} \|x - u_{n_k}\| = 0$. Analogously we get $\lim_{k\to\infty} \|v_{n_k} - x\| = 0$. Therefore $\lim_{k\to\infty} \|x_{n_k} - x\| = 0$, which is a contradiction.

We now add Conditions 3 and 4 and prove existence of the solution on $[0,T]$.

We have

$$\|f(t,sq(t))\| \leq A|s| + B, \quad t \in [0,T],\ s \in \mathbb{R},$$

Therefore

$$-(A|s|+B)p \leq f(t,sq(t)) \leq (A|s|+B)p, \quad t \in [0,T],\ s \in \mathbb{R}.$$

Let $c > 0$. For $\lambda, \mu > 0$ we consider the functions

$$\begin{aligned} u_0(t) &= -\lambda \exp(\mu t) q(t), \quad && t \in [0,T], \\ v_0(t) &= \lambda \exp(\mu t) q(t), && t \in [0,T]. \end{aligned}$$

Now

$$u_0'(t) - f(t,u_0(t)) \leq -\lambda \exp(\mu t)(\mu q(t) + q'(t) - Ap) + Bp,$$

and

$$v_0'(t) - f(t,v_0(t)) \geq \lambda \exp(\mu t)(\mu q(t) + q'(t) - Ap) - Bp.$$

Since $q([0,T])$ is a compact subset of $\operatorname{Int} K$,

$$\mu q([0,T]) + q'([0,T]) - Ap$$

is a compact subset of $\operatorname{Int} K$ if μ is sufficiently large. Then for λ sufficiently large

$$u_0'(t) - f(t,u_0(t)) \ll -cp \ll cp \ll v_0'(t) - f(t,v_0(t)), \quad t \in [0,T],$$

and

$$u_0(0) \ll x_0 \ll v_0(0).$$

Let $x : [0,\omega) \to E$ be a solution of problem (1). Theorem 4 gives

$$u_0(t) \ll x(t) \ll v_0(t), \quad t \in [0,\omega).$$

Hence $\|x(t)\| \leq \max\{\|u_0(t)\|, \|v_0(t)\|\}$. Now $x(t)$ is bounded on $[0,\omega)$, and therefore $x'(t)$ is bounded on $[0,\omega)$. Thus $\lim_{t\to\omega-} x(t)$ exists, and problem (1) is uniquely solvable on $[0,T]$. ∎

Proof of Theorem 3. We set $w = v - u$, and we define $g : [0,T] \times E \to E$ by

$$\begin{aligned} g(t,x) := {} & f(t,u(t)+x) - f(t,u(t)) \\ & + v'(t) - f(t,v(t)) - (u'(t) - f(t,u(t))). \end{aligned}$$

Then $w'(t) = g(t,w(t))$, $t \in [0,T]$, and $g(t,0) \geq 0$, $t \in [0,T]$; compare [2], p. 71. Assume that $w(t) \geq 0$ does not hold for $t \in [0,T]$ and consider $t_0 := \inf\{t \in [0,T] : w(t) \notin K\}$. Note that $w(t_0) \geq 0$. The function f and hence g satisfies Conditions 1 and 2 in Theorem 2. Therefore there exists

$\varepsilon > 0$ such that the initial value problems

$$w_n'(t) = g(t, w_n(t)) + \frac{p}{n}, \quad w_n(t_0) = w(t_0) + \frac{p}{n}, \quad n \in \mathbb{N},$$

have solutions $w_n : [t_0, t_0 + \varepsilon] \to E$. We have $0 \ll w_n(t_0)$ and $w(t_0) \ll w_n(t_0)$. For $t \in [t_0, t_0 + \varepsilon]$

$$-g(t,0) \le 0 \ll w_n'(t) - g(t, w_n(t)),$$
$$w'(t) - g(t, w(t)) \ll w_n'(t) - g(t, w_n(t)).$$

Theorem 4 gives $0 \ll w_n(t)$ and $w(t) \ll w_n(t)$, $t \in [t_0, t_0 + \varepsilon]$. Once again using condition 2 of Theorem 2, we find that $(w_n)_{n=1}^{\infty}$ tends uniformly to w on $[t_0, t_0 + \varepsilon]$. Thus $w(t) \ge 0$ on $[t_0, t_0 + \varepsilon]$, which contradicts the definition of t_0. ∎

4. Examples. We illustrate our results by examples. Let the spaces c and l^∞ be normed and ordered as in Section 1.

1. Let $E = c$. We consider the linear functional $\psi \in W$ defined by

$$\psi(x) = \sum_{k=1}^{\infty} \frac{x_k}{k^2} + \lim_{k\to\infty} x_k.$$

Now consider the function

$$f(x) = \left(\sqrt[3]{x_2}, \sqrt[3]{x_3} - 4\sqrt[3]{x_2}, \sqrt[3]{x_4} - \frac{9}{4}\sqrt[3]{x_3}, \ldots, \sqrt[3]{x_{k+1}} - \frac{k^2}{(k-1)^2}\sqrt[3]{x_k}, \ldots\right).$$

The function $f : c \to c$ is continuous, quasimonotone increasing, $\psi(f(x)) = 0$, $x \in c$, and $\|f(x)\| \le 5\|x\| + 5$, $x \in c$. By Corollary 1 the initial value problem $x'(t) = f(x(t))$, $x(0) = x_0$, is uniquely solvable on $[0, \infty)$.

2. Let $E = c$ and let

$$\psi(x) = \sum_{k=1}^{\infty} \frac{x_k}{2^k} + \lim_{k\to\infty} x_k.$$

Again $\psi \in W$. Now consider

$$f(t,x) = (2\lim_{k\to\infty} x_k^3 + x_2^3 - 3(1+t^2)x_1^3,\ x_3^3 - 2x_2^3 + (1+t^2)x_1^3,$$
$$x_4^3 - 2x_3^3 + (1+t^2)x_1^3, \ldots, x_{k+1}^3 - 2x_k^3 + (1+t^2)x_1^3, \ldots).$$

For every $T > 0$, the function $f : [0,T] \times c \to c$ is continuous, quasimonotone increasing, $\psi(f(t,x)) = 0$, $(t,x) \in [0,T] \times c$, and for $q(t) = ((1+t^2)^{-1/3}, 1, 1, \ldots) \in \operatorname{Int} K$, $t \in [0,T]$, we have $f(t, sq(t)) = 0$, $t \in [0,T]$, $s \in \mathbb{R}$. By Theorem 2 problem (1) is uniquely solvable on $[0,T]$.

3. Next let $E = l^\infty$, and consider the function $\psi \in W$ defined by

$$\psi(x) = \sum_{k=1}^{\infty} \frac{kx_k}{2^k} + \limsup_{k\to\infty} x_k.$$

Consider

$$f(t,x) = x + t\Big(\sqrt[3]{x_2}, \frac{\sqrt[3]{x_3} - 2\sqrt[3]{x_2}}{2}, \frac{\sqrt[3]{x_4} - 2\sqrt[3]{x_3}}{3}, \dots, \frac{\sqrt[3]{x_{k+1}} - 2\sqrt[3]{x_k}}{k}, \dots\Big).$$

For every $T > 0$, the function $f : [0,T] \times l^\infty \to l^\infty$ is continuous, quasimonotone increasing, $\psi(f(t,y) - f(t,x)) \le \psi(y - x)$, $t \in [0,T]$, $x \le y$, and $\|f(t,x)\| \le \big(\frac{3}{2}T + 1\big)\|x\| + \frac{3}{2}T$, $(t,x) \in [0,T] \times l^\infty$. Hence, by Theorem 2 problem (1) is uniquely solvable on $[0,T]$.

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