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A NOTE ON THE DIOPHANTINE EQUATION $\binom{k}{2} - 1 = q^n + 1$

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In this note we prove that the equation $\binom{k}{2} - 1 = q^n + 1$, $q \geq 2$, $n \geq 3$, has only finitely many positive integer solutions (k, q, n). Moreover, all solutions (k, q, n) satisfy $k < 10^{10^{182}}$, $q < 10^{10^{165}}$ and $n < 2 \cdot 10^{17}$.

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. The solutions (k, q, n) of the equation

$$\binom{k}{2} - 1 = q^n + 1, \quad k, q, n \in \mathbb{N},\ q \geq 2,\ n \geq 3, \tag{1}$$

are connected with some questions in coding theory. In this respect, Alter [1] proved that (1) has no solution (k, q, n) with $q = 8$. Recently, Hering [3] found out that all solutions (k, q, n) of (1) satisfy $3 \mid n$ or q is a prime power with $q < 47$. In this note, we prove a general result as follows.

THEOREM. *The equation* (1) *has only finitely many solutions* (k, q, n). *Moreover*, *all solutions* (k, q, n) *satisfy* $k < 10^{10^{182}}$, $q < 10^{10^{165}}$ *and* $n < 2 \cdot 10^{17}$.

The proof of the Theorem depends on the following lemmas.

Let α be an algebraic number of degree r with conjugates $\sigma_1\alpha, \ldots, \sigma_r\alpha$ and minimal polynomial

$$a_0x^r + a_1x^{r-1} + \ldots + a_r = a_0 \prod_{i=1}^{r} (x - \sigma_i\alpha) \in \mathbb{Z}[x], \quad a_0 > 0.$$

Further, let $\overline{|\alpha|} = \max(|\sigma_1\alpha|, \ldots, |\sigma_r\alpha|)$. Then

$$h(\alpha) = \frac{1}{r}\Big(\log a_0 + \sum_{i=1}^{r} \log\max(1, |\sigma_i\alpha|)\Big)$$

is called *Weil's height* of α.

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LEMMA 1 ([2]). *Let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers, and let $\Lambda = b_1 \log \alpha_1 + \ldots + b_m \log \alpha_m$ for some $b_1, \ldots, b_m \in \mathbb{Z}$. If $\Lambda \neq 0$, then we have*

$$|\Lambda| \geq \exp\Big(-18(m+1)! m^{m+1} (32d)^{m+2} (\log 2md) \Big(\prod_{i=1}^{m} A_i\Big) (\log B)\Big),$$

where d is the degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$,

$$A_i = \max\left(h(\alpha_i), \frac{1}{d}|\log \alpha_i|, \frac{1}{d}\right), \qquad i = 1, \ldots, m,$$

and $B = \max(|b_1|, \ldots, |b_m|, e^{1/d})$.

LEMMA 2 ([4, Notes of Chapter 5]). *Let K be an algebraic number field of degree d, and h_K, R_K, O_K be the class number, the regulator and the algebraic integer ring of K, respectively. Let $\mu \in O_K \setminus \{0\}$, and let $F(X,Y) = a_0 X^n + a_1 X^{n-1} Y + \ldots + a_n Y^n \in O_K[X,Y]$ be a binary form of degree n. If $F(z,1)$ has at least three distinct zeros, then all solutions (x,y) of the equation*

$$f(x,y) = \mu, \qquad x, y \in O_K,$$

satisfy

$$\max(\lceil x \rceil, \lceil y \rceil) \leq \exp(5(d+1)^{50(d+2)} n^6 (h_K R_K)^7 \log \max(e^e, HM)),$$

where $H = \max(\lceil a_0 \rceil, \lceil a_1 \rceil, \ldots, \lceil a_n \rceil)$ and $M = \lceil \mu \rceil$.

Proof of Theorem. Let (k,q,n) be a solution of (1). By [3], we may assume that $q \geq 47$ and $n \geq 4$. From (1) we get

$$(2k-1)^2 - 17 = (2k-1+\sqrt{17})(2k-1-\sqrt{17}) = 8q^n. \tag{2}$$

Let $K = \mathbb{Q}(\sqrt{17})$, and let h_K, R_K, O_K, U_K be the class number, the regulator, the algebraic integer ring and the unit group of K, respectively. It is a well-known fact that $h_K = 1$, $R_K = \log(4+\sqrt{17})$, $O_K = \{(a+b\sqrt{17})/2 \mid a, b \in \mathbb{Z},\ a \equiv b \pmod 2\}$ and $U_K = \{\pm(4+\sqrt{17})^s \mid s \in \mathbb{Z}\}$. Since $5^2 - 17 = 8$ and $(4+\sqrt{17})^2 = 33 + 8\sqrt{17}$, we see from (2) that

$$\frac{2k-1+\sqrt{17}}{2} = \left(\frac{5+\delta_1\sqrt{17}}{2}\right)\left(\frac{X_1+\delta_2 Y_1\sqrt{17}}{2}\right)^n (33+8\sqrt{17})^s, \tag{3}$$
$$\delta_1, \delta_2 \in \{-1, 1\},\ s \in \mathbb{Z},$$

where $X_1, Y_1 \in \mathbb{N}$ satisfy

$$X_1^2 - 17Y_1^2 = 4q, \qquad X_1 \equiv Y_1 \pmod 2, \qquad \gcd(X_1, Y_1) = \begin{cases} 1 & \text{if } 2 \nmid X_1, \\ 2 & \text{if } 2 \mid X_1. \end{cases} \tag{4}$$

For any $u, v \in \mathbb{Z}$ with $u^2 - 17v^2 = 1$, if $X + Y\sqrt{17} = (X_1 \pm Y_1\sqrt{17}) \times (u + v\sqrt{17})$, then $X, Y \in \mathbb{Z}$ satisfy

$$X^2-17Y^2=4q, \quad X\equiv Y \pmod 2), \quad \gcd(X,Y)=\begin{cases}1 & \text{if } 2\nmid X_1,\\ 2 & \text{if } 2\mid X_1,\end{cases}$$

by (4). Therefore, we may assume that X_1 and Y_1 satisfy

$$1<\frac{X_1+Y_1\sqrt{17}}{X_1-Y_1\sqrt{17}}<(33+8\sqrt{17})^2. \tag{5}$$

Notice that $q\ge 47$, $n\ge 4$, $2k-1\ge 6249$,

$$1<\frac{2k-1+\sqrt{17}}{2k-1-\sqrt{17}}<1.02 \quad\text{and}\quad 10.40<\frac{5+\sqrt{17}}{5-\sqrt{17}}<10.41.$$

Since

$$\frac{2k-1-\sqrt{17}}{2}=\left(\frac{5-\delta_1\sqrt{17}}{2}\right)\left(\frac{X_1-\delta_2Y_1\sqrt{17}}{2}\right)^n(33-8\sqrt{17})^s, \tag{6}$$

by (3), we find from (3), (5) and (6) that

$$|s|\le 2n. \tag{7}$$

Let $\eta=(5+\sqrt{17})/2$, $\overline{\eta}=(5-\sqrt{17})/2$, $\varepsilon=(X_1+Y_1\sqrt{17})/2$, $\overline{\varepsilon}=(X_1-Y_1\sqrt{17})/2$, $\varrho=33+8\sqrt{17}$ and $\overline{\varrho}=33-8\sqrt{17}$. Further, let $r=|s|$, $\alpha_1=\eta/\overline{\eta}$, $\alpha_2=\varrho$ and $\alpha_3=\varepsilon/\overline{\varepsilon}$. Then we have

$$h(\alpha_1)=\log(5+\sqrt{17}), \quad h(\alpha_2)=\log(4+\sqrt{17}). \tag{8}$$

Further, by (5), we get

$$h(\alpha_3)=\log(X_1+Y_1\sqrt{17})<\log 2\varrho\sqrt{q}. \tag{9}$$

From (3) and (6), we have

$$\log\frac{2k-1+\sqrt{17}}{2k-1-\sqrt{17}}=\lambda_1\log\alpha_1+2\lambda_2 r\log\alpha_2+\lambda_3 n\log\alpha_3, \qquad \lambda_1,\lambda_2,\lambda_3\in\{-1,1\}. \tag{10}$$

Let $\Lambda=\lambda_1\log\alpha_1+2\lambda_2 r\log\alpha_2+\lambda_3 n\log\alpha_3$. Since $\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)=\mathbb{Q}(\sqrt{17})$, by Lemma 1, we get from (7), (8) and (9) that if $\Lambda\ne 0$, then

$$\begin{aligned}|\Lambda|&\ge \exp(-18(4!)3^4 64^5(\log 12)(\log(5+\sqrt{17}))\\&\quad\times(\log(4+\sqrt{17}))(\log 2(33+8\sqrt{17})\sqrt{q})(\log 4n))\\&>\exp(-5\cdot 10^{14}(5+\log\sqrt{q})(\log 4n)).\end{aligned} \tag{11}$$

On the other hand, from (1) we get

$$\log\frac{2k-1+\sqrt{17}}{2k-1-\sqrt{17}}=\frac{2\sqrt{17}}{2k-1}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{\sqrt{17}}{2k-1}\right)^{2i}<\frac{3\sqrt{17}}{2k-1}<\frac{4.4}{q^{n/2}}. \tag{12}$$

Combination of (9), (11) and (12) yields

$$\log 4.4+5\cdot 10^{14}(5+\log\sqrt{q})(\log 4n)>n\log\sqrt{q},$$

whence we obtain

$$n < 2 \cdot 10^{17}. \tag{13}$$

Let

$$F(X,Y) = \left(\frac{5+\delta_1\sqrt{17}}{2}\right)\varrho^s X^n - \left(\frac{5-\delta_1\sqrt{17}}{2}\right)\overline{\varrho}^s Y^n \in O_K[X,Y].$$

Since $n \geq 3$ and $(5+\sqrt{17})/2$ is a prime in O_K, $F(z,1)$ has at least three distinct zeros. We see from (3) and (6) that $(x,y) = ((X_1+\delta_2 Y_1\sqrt{17})/2, (X_1-\delta_2 Y_1\sqrt{17})/2)$ is a solution of the equation

$$F(x,y) = \sqrt{17}, \quad x,y \in O_K. \tag{14}$$

Therefore, by Lemma 2, from (4), (7) and (14) we get

$$\begin{aligned} \sqrt{q} &< \frac{X_1+Y_1\sqrt{17}}{2} = \max\left(\left|\frac{X_1+\delta_2Y_1\sqrt{17}}{2}\right|, \left|\frac{X_1-\delta_2Y_1\sqrt{17}}{2}\right|\right) \\ &\leq \exp\left(5\cdot 3^{200} n^6 (\log(4+\sqrt{17}))^7 \log\left(\sqrt{17}\left(\frac{5+\sqrt{17}}{2}\right)(33+8\sqrt{17})^n\right)\right). \end{aligned} \tag{15}$$

Substituting (13) into (15), we obtain $q < 10^{10^{165}}$. Finally, from (2) we get $k < 10^{10^{182}}$. The Theorem is proved.

REFERENCES

[1] R. Alter, *On the non-existence of perfect double Hamming-error-correcting codes on $q = 8$ and $q = 9$ symbols*, Inform. and Control 13 (1968), 619–627.

[2] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. Reine Angew. Math. 442 (1993), 19–62.

[3] C. Hering, *A remark on two diophantine equations of Peter Cameron*, in: Groups, Combinatorics and Geometry (Durham, 1990), London Math. Soc. Lecture Note Ser. 165, Cambridge Univ. Press, Cambridge, 1992, 448–458.

[4] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equation*, Cambridge Tracts in Math. 87, Cambridge Univ. Press, Cambridge, 1986.

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